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NONLOCAL SYMMETRIES PAST, PRESENT AND FUTURE

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Nonlocal symmetries entered the literature in the Eighties of the last century largely through the work of Peter Olver. It was observed that there could be gain of symmetry in the reduction of order of an ordinary differential equation. Subsequently the reverse process was also observed. In each case the source of the 'new' symmetry was a nonlocal symmetry, *ie* a symmetry with one or more of the coefficient functions containing an integral. A considerable number of different examples and occurrences were reported by Abraham-Shrauner and Guo in the early Nineties. The role of nonlocal symmetries in the integration, indeed integrability, of differential equations was excellently illustrated by Abraham-Shrauner, Govinder and Leach with the equation $yy'' - y'^2 + f'(x)y^{p+2} + pf(x)y'y^{p+1} = 0$ which had been touted as a trivially integrable equation devoid of any point symmetry. Further theoretical contributions were made by Govinder, Feix, Bouquet, Géronimi and others in the second half of the Nineties. This included their role in reduction of order using the nonnormal subgroup. The importance of nonlocal symmetries was enhanced by the work of Krause on the Complete Symmetry Group of the Kepler Problem. Krause's work was furthered by Nucci and there has been considerable development of the use of nonlocal symmetries by Nucci, Andriopoulos, Cotsakis and Leach. The determination of the Complete Symmetry Group for integrable systems such as the simplest version of the Ermakov equation, $y'' = y^{-3}$, which possesses the algebra $sl(2, R)$ has proven to be highly nontrivial and requires some nonintuitive nonlocal symmetries. The determination of the nonlocal symmetries required to specify completely the differential equations of nonintegrable and/or chaotic systems remains largely an open question.

1. INTRODUCTION

New ideas, concepts and objects tend to originate in esoteric contexts. The commonplace does not invite deep thinking since the solution of problems there proceeds via methods upon which the experienced practitioner need not dwell for

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their execution. This does not mean that new ideas, concepts and objects cannot be found in the commonplace. One is sometimes pushed to think uncommonly in the context of the commonplace to see that the new has been under our noses since the beginnings of time if not earlier. In a pedagogical context the elimination of the esoteric origin is essential to lead the neophyte to understanding. So it is with nonlocal symmetries.

Barbara Abraham-Shrauner firstly heard of nonlocal symmetries from Peter Olver around 1990. She found them of interest and produced a series of papers with her student, Ann Guo, and some others [**1, 2, 3, 4, 5, 23**] chronicling their occurrence and characteristics. The collaboration spread and a number of papers [**6, 7, 19, 20, 21, 22, 42, 55, 56, 57, 58, 39, 41**] devoted to the subject has appeared over the years. During that time there has been some change in the relative importance of the elements of the subject of nonlocal symmetries. This has been a simple consequence of the movement from the esoteric to the commonplace.

In this paper we give an indication of the evolution of the subject of nonlocal symmetries. Firstly we look to an esoteric example and then to the removal of 'esoteric' with the presentation of some classic examples for which commonplace is too grand a word. Next we consider a standard result of the Lie theory and show that it is ill-based in its implication. Our third point is the necessity of nonlocal symmetries in a relatively new area which is the complete specification of a differential equation by means of symmetries.

That more or less covers the Past and the Present. What of the Future? Perhaps it would be better to reveal a little so that together we may seek that which is to come.

2. FROM ESOTERICA TO BANALITY

The differential equation

$$2yy'''' + 5y'y''' = 0 \tag{1}$$

arises in the study of the symmetries of the Emden-Fowler equation [**28, 30, 29, 44, 14**] and is easily shown to possess only the three obvious Lie point symmetries

$$\Gamma_1 = \partial_x, \quad \Gamma_2 = x\partial_x \quad \text{and} \quad \Gamma_3 = y\partial_y. \tag{2}$$

We use a symmetry, Γ_1 with invariants $u = y$ and $v = y'$, to reduce the order by one and obtain

$$2u\left(v^2v''' + 4vv'v'' + v'^3\right) + 5\left(v^2v'' + vv'^2\right) = 0 \tag{3}$$

which inherits

$$\Sigma_2 = v\partial_v \quad \text{and} \quad \Sigma_3 = u\partial_u + v\partial_v \tag{4}$$

from Γ_2 and Γ_3, respectively. Note that the second term of Σ_3 is redundant due to Σ_2. However, when we calculate the Lie point symmetries of (3), we find also

$$\Sigma_4 = 2u^2\partial_u + uv\partial_v \tag{5}$$

as an unexpected but very pleasant surprise. If we continue the process of reduction using Σ_2 and Σ_3, the result is an ABEL's equation of the second kind of the most hideous aspect. The invariants of Σ_4 are $r = vu^{-1/2}$ and $s = \frac{1}{2}\left(v'u^{3/2} - \frac{1}{2}vu^{1/2}\right)^2$ and the reduced equation,

$$s'' + 3s' + 2s = 0, \tag{6}$$

is pleasingly linear [**30**].

Evidently Σ_4 cannot have its origin in a point symmetry of (1) since Γ_1 was used for the reduction and Γ_2 and Γ_3 lead to Σ_2 and Σ_3. Under the reduction of (1) a symmetry

$$\Gamma_4 = \xi\partial_x + \eta\partial_y \quad \longrightarrow \quad \Sigma_4 = \eta\partial_u + (\eta' - y'\xi')\,\partial_v$$

so that

$$\eta = 2y^2 \quad \text{and} \quad \eta' - y'\xi' = yy'$$

whence $\xi = 3\int y\,\mathrm{d}x$ and we have the nonlocal symmetry

$$\Gamma_4 = 3\left(\int y\,\mathrm{d}x\right)\partial_x + 2y^2\partial_y. \tag{7}$$

The symmetry, Γ_4, is termed an 'hidden symmetry of Type II' since it appears as a point symmetry on reduction of order. Likewise an 'hidden symmetry of Type I' arises when a point symmetry becomes nonlocal on increase of order [**2, 3, 4, 23**].

The nonlinear second-order ordinary differential equation

$$y'' = \frac{y'^2}{y} + f'(x)y^{p+1} + pf(x)y'y^p \tag{8}$$

is devoid of LIE point symmetries for general f and p and yet is trivially integrable. As such it was presented as a counterexample to the need for the presence of LIE symmetries for an equation to be integrable [**18, 60**]. However, on the nonlocal changes of variable [**6**]

$$\begin{aligned} x &= x & y &= -\frac{w'}{pf(x)w} \\ X &= x & W &= \log w' \end{aligned} \tag{9}$$

the nonlinear (8) becomes

$$\frac{\mathrm{d}^2W}{\mathrm{d}X^2} = 0 \tag{10}$$

which is not only trivially integrable but also possesses eight LIE point symmetries which translate to a collection of nonlocal symmetries of (8). Unfortunately these nonlocal symmetries are complicated expressions and one could easily think it unlikely that anyone would ever essay the solution of (8) using a suitable pair of them. However, the unlikely was done recently by NUCCI [**52**].

In the two examples presented the ordinary differential equations inspiring the work were somewhat special. The large number of nonlocal symmetries found for (8) suggested that 'hidden' symmetries could be of common occurrence. Several studies of both differential equations and associated first integrals/invariants [**19, 15, 16, 31, 33**] revealed that this indeed be the case. To give a flavour of the result of the studies we list the connections between the LIE point symmetries of the two equations $Y'' = 0$ and $y''' = 0$ which are related by the nonlocal transformation $X = x$ and $Y = y'$.

The standard symmetries of

$y''' = 0$	and $Y'' = 0$.
$\Gamma_1 = \partial_y$	$\Sigma_1 = \partial_Y$
$\Gamma_2 = x\partial_y$	$\Sigma_2 = X\partial_Y$
$\Gamma_3 = x^2\partial_y$	
$\Gamma_4 = \partial_x$	$\Sigma_3 = \partial_X$
$\Gamma_5 = x\partial_x + y\partial_y$	$\Sigma_4 = X\partial_X + \frac{1}{2}Y\partial_Y$
$\Gamma_6 = x^2\partial_x + 2xy\partial_y$	$\Sigma_5 = X^2\partial_X + XY\partial_Y$
$\Gamma_7 = y\partial_y$	$\Sigma_6 = Y\partial_Y$
$\Gamma_8 = y'\partial_x + \frac{1}{2}y'^2\partial_y$	$\Sigma_7 = Y\partial_X$
$\Gamma_9 = 2(xy' - y)\partial_x + xy'^2\partial_y$	$\Sigma_8 = XY\partial_X + Y^2\partial_Y$
$\Gamma_{10} = (x^2y' - 2xy)\partial_x$	
$\quad + (\frac{1}{2}x^2y'^2 - 2y^2)\partial_y$	

$y''' = 0$	Fate	$Y'' = 0$	Source
Γ_1	annihilated	Σ_1	Γ_2
Γ_2	Σ_1	Σ_2	Γ_3
Γ_3	$(2)\Sigma_2$	Σ_3	$(\frac{1}{2})\Gamma_4$
Γ_4	Σ_3	Σ_4	$\Gamma_5 - \frac{1}{2}\Gamma_7$
Γ_5	$\Sigma_4 - \frac{1}{2}\Sigma_6$	Σ_5	$x^2\partial_x + 3(xy - \int y\mathrm{d}x)\partial_y$
Γ_6	$X^2\partial_X + (2\int Y\mathrm{d}X)\partial_Y$	Σ_6	Γ_7
Γ_7	Σ_6	Σ_7	Γ_8
Γ_8	Σ_7	Σ_8	$xy'\partial_x + \frac{1}{2}(xy'^2 + 3\int y'^2\mathrm{d}x)\partial_y$
Γ_9	$-2(\int Y\mathrm{d}X - XY)\partial_X + Y^2\partial_Y$		
Γ_{10}	$-2(X\int Y\mathrm{d}X - X^2Y)\partial_X$ $-(2Y\int Y\mathrm{d}X - XY^2)\partial_Y$		

Table 1: Fates of the symmetries of $y''' = 0$ and sources of the symmetries of $Y'' = 0$. The numerical factors in parentheses indicate the precise relationship between each pair of symmetries.

Just because the two equations are simple and are simply connected, it does not mean that their symmetries are equally so!

So far we had been looking at nonlocal symmetries in the sense of their manifestation through hidden symmetries. Theo Pillay, a thoughtful student given to Physics, made a nice job of unifying symmetry in a very direct fashion [**55**].

Everyone knows that the Lie point symmetries of

$$y'' = 0 \tag{11}$$

are eight in number and possess the Lie algebra $sl(3, R)$. How do we find them? We assume that a symmetry of (11) has the form

$$\Gamma = \xi(x, y)\partial_x + \eta(x, y)\partial_y, \tag{12}$$

apply the second extension,

$$\Gamma^{[2]} = \xi\partial_x + \eta\partial_y + (\eta' - y'\xi')\,\partial_{y'} + (\eta'' - 2y''\xi' - y'\xi'')\,\partial_{y''}, \tag{13}$$

to (11) and determine ξ and η by separating

$$\Gamma^{[2]} y''_{|y''=0} = 0 \tag{14}$$

by powers of y'. If instead of (12) one writes [**54**] [24ff]

$$\Gamma = \xi\partial_x + \eta\partial_y \tag{15}$$

without any specification of the variable dependence in ξ and η, (13) and (14) still apply, but we can no longer apply the simple rules that $\xi' = \partial\xi/\partial x + y'\partial\eta/\partial y$ *etc.* The application of (13) to (14) gives

$$\eta'' = y'\xi'' \tag{16}$$

which we may integrate by parts to obtain

$$\eta' = B + y'\xi', \tag{17}$$

$$\eta = A + Bx + \int y'\xi'\mathrm{d}x = A + Bx + y'\xi \tag{18}$$

after we take (11) into account for both integrations by parts. Alternately we could write $\xi'' = \eta''/y'$ which leads to

$$\xi = C + Dx + \int \frac{\eta'}{y'}\,\mathrm{d}x. \tag{19}$$

Relations (18) and (19) yield point symmetries only for quite specific choices of ξ and η. Obviously, if we put $\xi = 0$ in (18) and $\eta = 0$ in (19), we obtain

$$\begin{array}{ll} \Lambda_1 = \partial_y & \Lambda_3 = \partial_x \\ \Lambda_2 = x\partial_y & \Lambda_4 = x\partial_x, \end{array}$$

but that is just four of the required eight. Bear in mind that in (18) ξ must be just a function of x and y (resp (19) and η).

Two more symmetries follow easily. If in (16) we put $\xi = 0$, respectively $\eta = 0$, we obtain an equation of the same appearance as (11) for which y is a solution. Thus we have

$$\Lambda_5 = y\partial_y \quad \text{and} \quad \Lambda_6 = y\partial_x.$$

The two remaining symmetries are

$$\Lambda_7 = x^2\partial_x + xy\partial_y \quad \text{and} \quad \Lambda_8 = xy\partial_x + y^2\partial_y.$$

That these symmetries fit into the general form (18) is not obvious. We examine Λ_8; Λ_7 is treated in the same way. If ξ and η are given by

$$\xi = xy \quad \text{and} \quad \eta = y^2,$$

then (16) is automatically satisfied. In the case of (17) we obtain

$$2yy' = y'\left(xy' + y\right) + B \quad \Leftrightarrow \quad B = y'\left(y - xy'\right).$$

We recall that $I_1 = y'$ and $I_2 = y - xy'$ are first integrals of $y'' = 0$. Hence (17) is satisfied. In the case of (18) the substitution of ξ and η gives

$$y^2 = y'xy + Bx + A$$

which, when we take the integrals into account, becomes

$$y = \frac{1}{I_2}\left(Bx + A\right)$$

which is the solution of (11).

We see that, once the integration procedure is commenced, the first integrals and solutions of the equation, which are consequences of integration of the original equation, need to be taken into account. This is a case of integral consequences, as opposed to the more familiar differential consequences.

We observe that a little hoop-jumping has to be done to obtain the Lie point symmetries of (11) using (18/19). In general, given a function ξ (resp η), (18) (resp (19)) gives a nonlocal symmetry of (11). From a casual point of view the celebrated eight Lie point symmetries are lost in a sea of nonlocal symmetries.

It takes little to realise that every differential equation, indeed every differential function, possesses an infinite number of nonlocal symmetries of which a subset may be related to hidden symmetries by some nonlocal transformation. It is an interesting prospect, although scarcely conceivable of realisation, to determine the coordinate system in which the maximal number of hidden symmetries is revealed.

However, there is a subset of ordinary differential equations for which the question may be realistic. Equation (8) belongs to the PAINLEVÉ 50 for $p = 1$ and is integrable in terms of analytic functions apart from polelike singularities. We saw that it could be 'easily' transformed to a second-order differential equation of maximal point symmetry. If one examines the PAINLEVÉ 50 for symmetry, the results are somewhat mixed in that the number of Lie point symmetries ranges from eight – the 'beloved' equation[2], $y'' + 3yy' + y^3 = 0$ – to zero as for the six PAINLEVÉ transcendents. Yet they are all integrable. This suggests that somewhere there is an ordinary differential equation related one by one with a nonlocal transformation to the PAINLEVÉ 50 and that this ordinary differential equation has the requisite number of LIE point symmetries, if not more, for solution by quadrature. The resolution of this question presents something of a challenge!

3. GOING DOWN THE WRONG WAY

One of the purposes for determining the LIE point symmetries of the differential equation is to use the symmetries to reduce the order of the equation with the ultimate aim to achieve a performable quadrature. Given a set of LIE point symmetries, $\Gamma_i,\ i = 1, n$, the algebra is determined by the LIE Brackets

$$[\Gamma_i, \Gamma_j]_{LB} = C_k^{ij}\Gamma_k, \quad i, j, k = 1, n, \tag{20}$$

where the C_k^{ij} are the structure constants. To reduce the order of the equation for which the Γ_i are the set of LIE point symmetries one selects some symmetry, determines its zeroth- and first-order invariants and expresses the differential equation in terms of these invariants. The result is a differential equation of order one lower. The symmetry used for the reduction is obviously not relevant to the reduced equation. The fates of the other symmetries depend upon their LIE Brackets with the reducing symmetry. If Γ_r is the reducing symmetry and Γ_a some symmetry, one has that, if

$$[\Gamma_a, \Gamma_r]_{LB} = \lambda\Gamma_r, \tag{21}$$

λ a constant which may be zero, Γ_a becomes a point symmetry of the reduced equation. Otherwise Γ_a becomes a nonlocal symmetry of the reduced equation.

Conventional wisdom is that one wants to keep the symmetries as point symmetries and the choice for Γ_r should be such that the number of symmetries of which (21) applies is optimal. However, a closer look [**25, 17**] at the unfavoured

[2]So termed by the unfortunately late MARC FEIX who grew to appreciate the fascinating properties of this equation.

option reveals a greater delicacy in the situation[3]. Suppose that

$$[\Gamma_a,\,\Gamma_r]_{LB}=\Gamma_a \tag{22}$$

(any constant multiplier is absorbed into Γ_r). Without loss of generality we may write Γ_r in the canonical form, ∂_x. If we take $\Gamma_a=\xi(x,y)\partial_x+\eta(x,y)\partial_y$, it follows from (22) that

$$\xi=\mathrm{e}^x f(y) \quad \text{and} \quad \eta=\mathrm{e}^x g(y), \tag{23}$$

where f and g are arbitrary functions. Under the reduction of order using Γ_r the invariants are y and y' so that Γ_a becomes

$$\Sigma_a=\mathrm{e}^x\left\{g(u)\partial_u+\left[g(u)+v\left(g'(u)-f(u)\right)-v^2f'(u)\right]\partial_v\right\} \tag{24}$$

and it becomes necessary to express x in terms of u and v as $x=\int \mathrm{d}u/v$ so that we have the nonlocal symmetry

$$\Sigma_a=\exp\left[\int\frac{\mathrm{d}u}{v}\right]\left\{g(u)\partial_u+\left[g(u)+v(g'(u)-f(u))-v^2f'(u)\right]\partial_v\right\}. \tag{25}$$

The nonlocality in Σ_a occurs in the common exponential multiplier and so it is called an exponential nonlocal symmetry.

If instead of (22) one has

$$[\Gamma_a,\,\Gamma_r]_{LB}=\Gamma_b, \tag{26}$$

where Γ_b is anything but Γ_a or Γ_r, reduction by Γ_r produces a nonlocal symmetry in which the nonlocality is not conveniently separated as in the exponential nonlocal symmetry of (25)[4]. Despite the conventional wisdom of reduction by the normal subgroup, reduction using Γ_r when (22) applies is still feasible since, for the second reduction using Σ_a in the associated Lagrange's system for the invariants of Σ_a, the exponential terms cancel and one is left with

$$\begin{aligned}\frac{\mathrm{d}u}{g(u)}&=\frac{\mathrm{d}v}{g(u)+v\left(g'(u)-f(u)\right)-v^2f'(u)}\\&=\frac{\mathrm{d}v'}{g'(u)+v\left(g''(u)-f'(u)\right)-v^2f''(u)-v'f'-2vv'f'}\end{aligned} \tag{27}$$

[3]One notes that in the related matter of the existence of an integrating factor nonlocal symmetries often play a pivotal role [**35**].

[4]One must emphasise that the discussion here relates to scalar ordinary differential equations. In the case of systems of ordinary differential equations a nonlocal symmetry not of exponential form may not present a hindrance to reduction of order. An example of this occurs in one of the three integrable cases of the Hénon-Heilles Hamiltonian. The nonlocal component of the symmetry does not play a role in the reduction. This phenomenon has been observed in studies using the last multiplier of Jacobi. In general a system of ordinary differential equations invariant under time translation can have symmetries with the time-coefficient nonlocal without any adverse effect upon reducibility.

and this gives a properly defined pair of equations for the zeroth-order and first-order invariants of Σ_a.

We conclude this Section with an example which illustrates the irony of doing the wrong thing thrice.

The Chazy equation

$$y''' + yy'' - \tfrac{3}{2}\,y'^2 = 0 \tag{28}$$

has the three Lie point symmetries [**17**]

$$\Gamma_1 = \partial_x, \quad \Gamma_2 = x\partial_x - y\partial_y \quad \text{and} \quad \Gamma_3 = x^2\partial_x + (12 - 2xy)\partial_y \tag{29}$$

which constitute a representation of the nonsolvable algebra $sl(2,R)$. Since

$$[\Gamma_1,\,\Gamma_2]_{LB} = \Gamma_1,\ [\Gamma_1,\,\Gamma_3]_{LB} = 2\Gamma_2 \text{ and } [\Gamma_2,\,\Gamma_3]_{LB} = \Gamma_3, \tag{30}$$

the conventional approach would have us reduce the order of the equation by either Γ_1 or Γ_3 and certainly not Γ_2.

However, we take the unconventional approach. The invariants of Γ_2 are $u = xy$ and $v = x^2y'$ and Γ_1 and Γ_3 become, respectively,

$$\begin{aligned} \Sigma_1 &= \exp\left[-\int \frac{\mathrm{d}u}{u+v}\right]\{u\partial_u + 2v\partial_v\} \\ \Sigma_3 &= \exp\left[\int \frac{\mathrm{d}u}{u+v}\right]\big\{(12-u)\partial_u - 2(u+v)\partial_v\big\}. \end{aligned} \tag{31}$$

As both Σ_1 and Σ_3 are exponential nonlocal, both are available for reduction of order. If we take Σ_1, its invariants are

$$p = \frac{v}{u^2} \quad \text{and} \quad q = \frac{(u+v')v' - 2v}{u^3} \tag{32}$$

and Σ_3 becomes

$$\Delta_3 = -2\exp\left[\int \frac{p\,\mathrm{d}p}{q - 2p^2}\right]\big[(12p+1)\partial_p + 3(6q+p)\partial_q\big]. \tag{33}$$

Since Δ_3 is also exponential nonlocal, we may use it for a final reduction of order. The invariants are

$$r = \frac{72q + 3p + 2}{72\zeta} \text{ and } s = \big[q'(\zeta^2 - 72\zeta r + 1) - 18r\zeta^3 - 54r\zeta + 1\big]/\zeta^2, \tag{34}$$

where $\zeta = 12p + 1$.

Under these successive reductions of order the Chazy equation becomes the simple algebraic equation

$$4s + 3 = 0. \tag{35}$$

Perhaps this is a rare instance of three wrongs making a right!

4. COMPLETE SYMMETRY GROUPS

The concept of a Complete Symmetry Group was introduced by JORGE KRAUSE in 1994 [**26, 27**] in the context of the KEPLER Problem. Essentially he sought the minimum number of symmetries, $\Gamma_i,\ i = 1, N$, such that

$$\Gamma_i^{[2]} \left\{\ddot{\mathbf{x}} - \mathbf{f}(x, \dot{x}, t)\right\}_{|\ddot{\mathbf{x}}=\mathbf{f}} = 0 \tag{36}$$

required that $\mathbf{f}$ be the Newtonian force. In the case of the KEPLER Problem it was necessary to introduce nonlocal symmetries of the form

$$\mathbf{\Gamma} = \left(2\textstyle\int \mathbf{r}\mathrm{d}t\right)\partial_t + \mathbf{r}\mathbf{r}.\partial_{\mathbf{r}} \tag{37}$$

to complete the specification. Curiously the symmetries reflecting the conservation of angular momentum were not part of the Complete Symmetry Group of the KEPLER Problem. Subsequently NUCCI [**47**] showed that the nonlocal symmetries in (37) were a natural consequence of the LIE point symmetries of a related system. The story was completed by systematic account of the method of reduction of order [**48**] which is a group theoretic approach to the classical method [**61**] [p 78] to reduce the KEPLER Problem to an harmonic oscillator with a forcing term. A similar line of thinking showed that many integrable orbit problems related to the KEPLER Problem were essentially the same problem as far as the underlying algebraic basis is concerned [**49**]. A number of other problems [**36, 40, 50, 43, 51, 53**] also yielded to the same procedure.

In this sense the KEPLER Problem belongs to the class of problems we mentioned above. The task is to find the coordinate system in which its essential symmetries are point. One must observe that invariance under time translation makes the transition to the new coordinate system possible. Time is not a variable in the new coordinate system which means that in the reversion from the transformed system to the original KEPLER Problem the symmetry of invariance under time translation must be included as an element of the Complete Symmetry Group. Although one may hesitate to term the KEPLER Problem as an esoteric problem since it has been with us for some four centuries and its natural resolution in terms of the ERMANNO-BERNOULLI constants is about to celebrate its tercentenary [**13, 24, 11, 36, 37, 38, 49, 51**], nevertheless its defining differential equation is nonlinear and we all know the pitfalls associated with nonlinear ordinary differential equations.

In a manner of speaking Complete Symmetry Groups were the province of exotic differential equations which needed nonlocal symmetries to provide a complete specification. Fortunately the simple-minded came to provide some basic theory about complete symmetry groups which did not involve nonlocal symmetries [**9**]. The investigation of the KEPLER Problem led to the simple harmonic oscillator. The simple harmonic oscillator is related to the free particle by an easy point transformation and so the completeness of our celestial Kosmos could be explained by the analysis of a particle moving in its own universe.

These theoretical studies tended to exchange nonlocal symmetries for point symmetries by showing that the system considered could be transformed to systems possessing the appropriate number of Lie point symmetries for reduction to quadrature.

In the case of nonintegrable systems the evidence is somewhat thinner[5]. Leach *et al* [**34**] considered the third-order differential equation

$$y''' + y'' + yy' = 0 \tag{38}$$

which arises in general relativity and showed that it was completely specified by the nonlocal symmetries

$$\begin{aligned}\Lambda_1 &= \partial_x \\ \Lambda_2 &= \left\{\int \frac{\mathrm{d}x}{y'^2\mathrm{e}^x}\right\}\partial_x \\ \Lambda_3 &= \left\{\int \frac{\mathrm{d}x}{y'^2}\left(y - \mathrm{e}^{-x}\int y\mathrm{e}^x\mathrm{d}x\right)\right\}\partial_x \\ \Lambda_4 &= \left\{\int \frac{1}{y'^2\mathrm{e}^x}\left[\int yy'\mathrm{e}^x\mathrm{d}x\right]\mathrm{d}x\right\}\partial_x + \partial_y.\end{aligned} \tag{39}$$

For general values of the initial conditions (38) is nonintegrable. Indeed a study [**59**] of its Lyapunov exponents suggested that it exhibited chaotic behaviour away from the surface in its three-dimensional space of initial conditions on which it is demonstrably integrable in terms of analytic functions, but subsequent advice was that the solution was simply very badly behaved. The distinction between chaotic behaviour and nonintegrability can at times be visually difficult to discern. The time-dependent oscillator

$$\ddot{q} + \omega^2(t)q = 0 \tag{40}$$

is a case in point.

It has been established that the number of symmetries necessary to specify an ordinary differential equation completely is $n+1$, where n is the order of the equation [**10**].

5. THE FUTURE

We mention three recent developments.

5.1. PARTIAL DIFFERENTIAL EQUATIONS (A)

A partial differential equation with a sufficient number of Lie point symmetries can be specified completely by these symmetries [**45**]. The heat equation

[5] Although we have not detailed the results, one can expect to find the Complete Symmetry Group for an integrable equation rather more easily than for a nonintegrable equation.

and a number of equations arising in Financial Mathematics, such as the BLACK-SCHOLES equation, are so specified. In the absence of a sufficient number of suitable LIE point symmetries one must look to nonlocal symmetries to complete the specification. SENZO MYENI [**6**] has recently devised a method to deal with the problem of determining the nonlocal symmetries required. The class of partial differential equations we consider comprise the general second-order evolution partial differential equation,

$$F(x, u, u_x, u_t, u_{xx}) = 0. \tag{41}$$

The most important step in this type of analysis for the symmetry group is to identify at what point in the analysis a nonlocal symmetry is required. The guideline is at a point where the arbitrary function found after the application of a particular point symmetry still depends on the variable that one is trying to remove. We illustrate this by an example drawn from the Mathematics of Finance. The equation we consider is a nonlinear partial differential equation for volatility [**12**]

$$u^2 u_{xx} + (r-q)xu_x + u_t - (r-q)u = 0. \tag{42}$$

The economic model assumes frictionless markets, no arbitrage and that the underlying stock price process is a one-dimensional diffusion starting from a positive value. It also assumes a proportional risk-neutral drift of $r-q$, where $r \geq 0$ is the constant risk-free rate and $q \geq 0$ is the constant dividend yield. The absolute volatility rate is a positive $C^{2,1}$ function $u(x,t)$ of the stock price $x \in (0, \infty)$ and time $t \in (0, T)$, where T is some distant horizon exceeding the longest maturity of the option to be priced.

We rescale the variables to achieve an equation simpler in appearance, *videlicet*

$$u^2 u_{xx} + xu_x + u_t - u = 0, \tag{43}$$

and it is for this equation that we find the complete symmetry group.

The LIE point symmetries of (43) are

$$\begin{aligned}
\Sigma_1 &= \partial_t \\
\Sigma_2 &= e^t\,\partial_x \\
\Sigma_3 &= \partial_t + x\,\partial_x + u\,\partial_u \\
\Sigma_4 &= t\,\partial_t + tx\,\partial_x + \left(t - \tfrac{1}{2}\right) u\,\partial_u.
\end{aligned} \tag{44}$$

We write (41) as

$$u_t = f(x, t, u, u_x, u_{xx}). \tag{45}$$

Application of $\Sigma_1 = \partial_t$ gives

$$u_t = f(x, u, u_x, u_{xx}). \tag{46}$$

The second extension of $\Sigma_2 = e^t\partial_x$ is

$$\Sigma_2^{[2]} = e^t\partial_x + (0)\partial_{u_x} - e^t u_x\,\partial_{u_t} + (0)\partial_{u_{xx}}$$

and its application to (16) yields

$$-u_x = \frac{\partial f}{\partial x} \Rightarrow f = -x\,u_x + h(u_x, u_{xx}, u).$$

This is not good since h still depends explicitly upon u_x. Before applying Σ_2 we become proactive and require that

$$u_t = f(u, xu_x, u_{xx}).$$

Obviously there is a nonlocal symmetry which allows the above constraint. We find it as follows.

The characteristics would be

$$u_t, u, u_{xx}, xu_x$$

which come from the associated Lagrange's system

$$\frac{\mathrm{d}u_x}{-u_x} = \frac{\mathrm{d}u}{0} = \frac{\mathrm{d}u_{xx}}{0} = \frac{\mathrm{d}u_t}{0} = \frac{\mathrm{d}x}{x}.$$

This suggests that the second extension of the nonlocal symmetry, say $\Sigma_5 = \xi\partial_x + \tau\partial_t + \eta\partial_u$, is

$$\Sigma_5^{[2]} = \xi\,\partial_x + \tau\,\partial_t + \eta\,\partial_u + \zeta_x\,\partial_{u_x} + \zeta_t\,\partial_{u_t} + \zeta_{xx}\,\partial_{u_{xx}},$$

where

$$\xi = x, \quad \eta = 0,$$

ζ_x, ζ_t and ζ_{xx} are the extensions of the operator Σ_5 relevant to the derivatives indicated. Specifically they are given by

$$\zeta_x = \frac{\partial\eta}{\partial x} + \left[\frac{\partial\eta}{\partial u} - \frac{\partial\xi}{\partial x}\right]u_x - \frac{\partial\tau}{\partial x}u_t \tag{47}$$

$$\zeta_t = \frac{\partial\eta}{\partial t} + \left[\frac{\partial\eta}{\partial u} - \frac{\partial\tau}{\partial t}\right]u_t - \frac{\partial\xi}{\partial t}u_x \tag{48}$$

$$\zeta_{xx} = \frac{\partial^2\eta}{\partial x^2} + \left[2\frac{\partial^2\eta}{\partial x\partial u} - \frac{\partial^2\xi}{\partial x^2}\right]u_x - \frac{\partial^2\tau}{\partial x^2}u_t - 2\frac{\partial\xi}{\partial x}u_{xx}. \tag{49}$$

The symmetry-generating system is

$$\zeta_{xx} = 0, \quad \zeta_t = 0, \quad \zeta_x = -u_x$$

with solution

$$\tau(x,t) = 2\int\int \frac{u_{xx}}{u_t}\,\mathrm{d}x\,\mathrm{d}t, \tag{50}$$

where arbitrary functions and constants of integration have been omitted.

The nonlocal symmetry is

$$\Sigma_5 = x\,\partial_x + \tau\,\partial_t,$$

where $\tau(x,t)$ is given by (50).

Hence we have the desired result that

$$u_t = f(u, x\,u_x, u_{xx}).$$

We further proceed with the application of the remaining Lie point symmetries. The application of the second extension of $\Sigma_2 = e^t\partial_x$ gives

$$u_t + x\,u_x = h(u, u_{xx}). \tag{51}$$

The application of the second extension of Σ_4 is

$$\Sigma_4^{[2]} = t\,\partial_t + tx\,\partial_x + \left(t - \tfrac{1}{2}\right)u\,\partial_u - \tfrac{1}{2}\,u_x\,\partial_{u_x} + \left[u + \left(t - \tfrac{3}{2}\right)u_t - x\,u_x\right]\partial_{u_t} - \left(t + \tfrac{1}{2}\right)u_{xx}\,\partial u_{xx} \tag{52}$$

which leads to

$$u_t + xu_x = u + \gamma u^2 u_{xx},$$

where γ is an arbitrary constant. The use of the nonlocal symmetry[6]

$$\Sigma_6 = \tau\,\partial_t$$

with τ given by

$$\tau(x,t) = 2\int\int \frac{u_{xx}}{u_t u_x}\,\mathrm{d}x\,\mathrm{d}t \tag{53}$$

requires γ to be -1.

The implicit and quasi-implicit complete symmetry group approach not only provides us with the sufficient number of symmetries to form a complete symmetry group but also provides a more direct way to find nonlocal symmetries.

[6]The calculation of which parallels the calculation given in detail above.

5.2. PARTIAL DIFFERENTIAL EQUATIONS (B)

Abraham-Shrauner and Govinder [**8**] have recently shown a new potential source of hidden symmetries for partial differential equations. The symmetries do not come from nonlocal symmetries, but are a result of the possibility that several partial differential equations could lead to the same partial differential equation on reduction of order. We illustrate their method with a simple example ([**8**], equation (2.1)),

$$u_{xxx} + u\left(u_t + cu_x\right) = 0, \tag{54}$$

which possesses the Lie point symmetries

$$\begin{aligned} \Gamma_1 &= \partial_t \\ \Gamma_2 &= \partial_x \\ \Gamma_3 &= 3t\partial_t + (x + 2ct)\partial_x \\ \Gamma_4 &= t\partial_t + ct\partial_x + u\partial_u. \end{aligned} \tag{55}$$

We reduce (54) to an ordinary differential equation using the symmetry $c\Gamma_2 + \Gamma_1$ for which the invariants are $w = u$ and $y = x - ct$, *ie* we seek a travelling-wave solution. Note that this is not an invertible point transformation and so preservation of point symmetries is not guaranteed. The reduced equation is simply

$$w_{yyy} = 0 \tag{56}$$

which has the seven Lie point symmetries

$$\begin{aligned} &\Upsilon_1 = \partial_y && \Upsilon_5 = y\partial_w \\ &\Upsilon_2 = \partial_w && \Upsilon_6 = w\partial_w \\ &\Upsilon_3 = y^2\partial_w && \Upsilon_7 = \tfrac{1}{2}y^2\partial_y + yw\partial_w. \\ &\Upsilon_4 = y\partial_y && \end{aligned}$$

Equation (54) is not the only source of (56) under reduction. Equally it can be obtained from

$$u_{xxx} = 0, \quad u_{ttt} = 0, \quad u_{xxt} = 0 \quad \text{and} \quad u_{xtt} = 0, \tag{57}$$

where u is still a function of t and x, by means of the same invariants. For example the first of (57) has an eightfold infinity of Lie point symmetries. They are

$$\begin{aligned} &\Delta_1 = F_1(t)\partial_x && \Delta_5 = F_5(t)x\partial_x \\ &\Delta_2 = F_2(t)\partial_u && \Delta_6 = F_1(t)x\partial_u \\ &\Delta_3 = F_3(t)\partial_t && \Delta_7 = F_7(t)u\partial_u \\ &\Delta_4 = F_4(t)x^2\partial_u && \Delta_8 = F_8(t)\left(\tfrac{1}{2}x^2\partial_x + xu\partial_u\right), \end{aligned}$$

where the $F_i(t)$, $i = 1, 8$, are arbitrary functions. A subset of these symmetries is obtained by making specific choices for the arbitrary functions and in suitable combinations we have

$$\begin{array}{ll} \Sigma_1 = \partial_x & \Sigma_5 = (x-ct)\partial_x \\ \Sigma_2 = \partial_u & \Sigma_6 = (x-ct)\partial_u \\ \Sigma_3 = \partial_t & \Sigma_7 = u\partial_u \\ \Sigma_4 = (x-ct)^2\partial_u & \Sigma_8 = \frac{1}{2}(x-ct)^2\partial_x + (x-ct)u\partial_u \end{array}$$

which reduce to the seven Lie point symmetries of (56).

A similar result applies for the second equation in (57). However, for the third and fourth members of (57) the symmetry Υ_7 is not obtained.

In this example the invariants used for the reduction of order were the same. There is no requirement for this to be the case and Abraham-Shrauner and Govinder discuss the procedure to be used in this more general case.

5.3. WILL IT WORK FOR ORDINARY DIFFERENTIAL EQUATIONS?

We conclude with a very underdeveloped example of the application of the idea of Abraham-Shrauner and Govinder to the area of ordinary differential equations. The third-order equations

$$y''' = 0 \tag{58}$$

$$2y'y''' - 3y''^2 = 0 \tag{59}$$

are reduced to the second-order equation

$$Y'' = 0 \tag{60}$$

(now the prime denotes differentiation with respect to the transformed independent variable, X, which happens to be the same as the original independent variable in this case) by means of the transformations

$$X = x \qquad Y = y' \quad \text{and} \tag{61}$$

$$X = x \qquad Y = y'^{-1/2}, \tag{62}$$

respectively.

For (58) the symmetry generating the transformation (61) is $\Gamma_1 = \partial_y$. The remaining six Lie point symmetries are transformed as

$$\begin{array}{ll} \Gamma_2 = x\partial_y & \Lambda_2 = \partial_Y \\ \Gamma_3 = \frac{1}{2}x^2\partial_y & \Lambda_3 = X\partial_Y \\ \Gamma_4 = y\partial_y & \Lambda_4 = Y\partial_Y \\ \Gamma_5 = \partial_x & \Lambda_5 = \partial_X \\ \Gamma_6 = x\partial_x + y\partial_y & \Lambda_6 = \partial_X \\ \Gamma_7 = x^2\partial_x + 2xy\partial_y & \Lambda_7 = X^2\partial_X + \left(2\int Y\mathrm{d}X\right)\partial_Y \end{array}$$

from which it is evident that we are missing three of the Lie point symmetries of (60). The missing three are

$$\begin{aligned}
\Sigma_1 &= X^2\partial_X + XY\partial_Y \\
\Sigma_2 &= Y\partial_X \\
\Sigma_3 &= XY\partial_X + Y^2\partial_Y
\end{aligned}$$

and it is a simple calculation to show that they have their origins from the symmetries

$$\begin{aligned}
\Delta_1 &= x^2\partial_x + 3\left[\tfrac{1}{2}x^2y' - \tfrac{1}{6}x^3y''\right]\partial_y \\
\Delta_2 &= y'\partial_x + \tfrac{1}{2}y'^2\partial_y \\
\Delta_3 &= xy'\partial_x + \left[2xy'^2 - \tfrac{3}{2}x^2y'y'' + \tfrac{1}{2}x^3y''^2\right]\partial_y
\end{aligned}$$

of (58). The symmetry Δ_2 is one of the contact symmetries of (58). The other two are generalised symmetries and have been written as such instead of the nonlocal version since the integration of (58) is trivial.

In the case of (59) and the reduction (62) the Lie point symmetries of the former and their expression as symmetries of (60) are

$$\begin{aligned}
\Gamma_1 &= \partial_x & \Lambda_1 &= \partial_X \\
\Gamma_2 &= x\partial_x & \Lambda_2 &= X\partial_X + \tfrac{1}{2}Y\partial_Y \\
\Gamma_3 &= x^2\partial_x & \Lambda_3 &= X^2\partial_X + XY\partial_Y \\
\Gamma_5 &= y\partial_y & \Lambda_5 &= Y\partial_Y \\
\Gamma_6 &= y^2\partial_y & \Lambda_6 &= Y\textstyle\int Y^2\mathrm{d}X\partial_Y.
\end{aligned}$$

The symmetry $\Gamma_4 = \partial_y$ is the symmetry used for the transformation (62).

The missing Lie point symmetries are

$$\begin{aligned}
\Sigma_1 &= \partial_Y \\
\Sigma_2 &= X\partial_Y \\
\Sigma_3 &= Y\partial_X \\
\Sigma_4 &= XY\partial_X + Y^2\partial_Y.
\end{aligned}$$

We note that in both cases the noncartan symmetries of (60) are absent in the reduction of the point symmetries of the third-order equations.

What we do wish to emphasise is that the two reductions gave us a different selection of the Lie point symmetries of (60) which is precisely the same effect reported by Abraham-Shrauner and Govinder for partial differential equations.

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