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INTEGRABLE BOEHMIANS, FOURIER TRANSFORMS, AND POISSON'S SUMMATION FORMULA

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The space of integrable Boehmians $\beta_\ell(\mathbb{R})$ contains a subspace which can be identified with $L^1(\mathbb{R})$. The Fourier transform can be defined for each element of $\beta_\ell(\mathbb{R})$. The Fourier transform of an integrable Boehmian is a continuous function which satisfies a growth condition. We investigate the Fourier transform on $\beta_\ell(\mathbb{R})$, and as an application, we extend Poisson's summation formula to the space $\beta_\ell(\mathbb{R})$.

1. INTRODUCTION

Boehmians are classes of generalized functions whose construction is algebraic. The first construction appeared in a paper that was published in 1981 [**6**].

In [**8**], P. Mikusiński constructs a space of Boehmians, $\beta_{L_1}(\mathbb{R})$, in which each element has a Fourier transform. Mikusiński shows that the Fourier transform of a Boehmian satisfies some basic properties, and he also proves an inversion theorem. However, the range of the Fourier transform is not investigated. Also, Mikusiński states that $\beta_{L_1}(\mathbb{R})$ contains some elements which are not Schwartz distributions, but no examples are given. We will address these problems in this paper.

In this note, we will construct a space of Boehmians $\beta_\ell(\mathbb{R})$. The space of integrable functions on the real line can be identified with a proper subspace of $\beta_\ell(\mathbb{R})$. Each element of $\beta_\ell(\mathbb{R})$ has a Fourier transform which is a continuous function and satisfies a growth condition at infinity. Conditions are given which ensure that a given function is the Fourier transform of an element of $\beta_\ell(\mathbb{R})$.

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The space $\beta_\ell(\mathbb{R})$ is slightly less general than the space MIKUSIŃSKI constructs. However, each element of $\beta_\ell(\mathbb{R})$ has local properties similar to those of a continuous function. For example, each Boehmian has a support. Also, as we will see, each element of $\beta_\ell(\mathbb{R})$ satisfies a version of POISSON's summation formula.

This article is organized as follows. Section 2 contains notation and the construction of the space of Boehmians. In Section 3, we construct and investigate the space of integrable Boehmians $\beta_\ell(\mathbb{R})$. Section 4 contains the construction and some known facts about the space of periodic Boehmians. In Section 5, as an application, we prove the POISSON summation formula for integrable Boehmians.

2. PRELIMINARIES

Let $L^1_{loc}(\mathbb{R})$ denote the space of all locally integrable functions on the real line $\mathbb{R}$, and let $\mathrm{D}(\mathbb{R})$ be the subspace of $L^1_{loc}(\mathbb{R})$ of all infinitely differentiable functions with compact support.

For $f \in L^1_{loc}(\mathbb{R})$, let

$$\gamma_n(f) = \int_{|x|\le n} |f(x)|\,\mathrm{d}x, \qquad \text{for } n = 1, 2, \dots .$$

The separating countable family of seminorms $\{\gamma_n\}$ generate a topology for $L^1_{loc}(\mathbb{R})$. A sequence of locally integrable functions $\{f_n\}$ converges in $L^1_{loc}(\mathbb{R})$ to $f \in L^1_{loc}(\mathbb{R})$ provided that for each p, $\gamma_p(f_n - f) \to 0$ as $n \to \infty$.

A sequence $\varphi_n \in \mathrm{D}(\mathbb{R})$ is called a *delta sequence* provided:

(i) $\int_{-\infty}^{\infty} \varphi_n(x)\,\mathrm{d}x = 1$ for all $n \in \mathbb{N}$,

(ii) $\int_{-\infty}^{\infty} |\varphi_n(x)|\,\mathrm{d}x \le M$ for some constant M and all $n \in \mathbb{N}$,

(iii) $\operatorname{supp}\varphi_n \to \{0\}$ as $n \to \infty$.

A pair of sequences (f_n, φ_n) is called a *quotient of sequences* if $f_n \in L^1_{loc}(\mathbb{R})$ for $n \in \mathbb{N}$, $\{\varphi_n\}$ is a delta sequence, and $f_k * \varphi_m = f_m * \varphi_k$ for all $k, m \in \mathbb{N}$, where $*$ denotes convolution:

$$(f * \varphi)(x) = \int_{-\infty}^{\infty} f(x-u)\varphi(u)\,\mathrm{d}u.$$

Two quotients of sequences (f_n, φ_n) and (g_n, ψ_n) are said to be equivalent if $f_k * \psi_m = g_m * \varphi_k$ for all $k, m \in \mathbb{N}$. A straightforward calculation shows that this is an equivalence relation. The equivalence classes are called Boehmians. The space of all Boehmians will be denoted by $\beta(\mathbb{R})$ and a typical element of $\beta(\mathbb{R})$ will be written as $F = \left[\frac{f_n}{\varphi_n}\right]$.

The operations of addition, scalar multiplication, and differentiation are defined as follows:

$$\left[\frac{f_n}{\varphi_n}\right] + \left[\frac{g_n}{\psi_n}\right] = \left[\frac{f_n * \psi_n + g_n * \varphi_n}{\varphi_n * \psi_n}\right], \tag{2.1}$$

$$\alpha\left[\frac{f_n}{\varphi_n}\right] = \left[\frac{\alpha f_n}{\varphi_n}\right], \text{where } \alpha \in \mathbb{C}, \tag{2.2}$$

$$D^k\left[\frac{f_n}{\varphi_n}\right] = \left[\frac{f_n * D^k\varphi_n}{\varphi_n * \varphi_n}\right]. \tag{2.3}$$

If f is a locally integrable function on $\mathbb{R}$, then it can be identified with the Boehmian $\left[\frac{f * \varphi_n}{\varphi_n}\right]$. Thus, we may view $L^1_{loc}(\mathbb{R})$ as a subspace of $\beta(\mathbb{R})$. Likewise, the space of Schwartz distributions [**14**] can be identified with a proper subspace of $\beta(\mathbb{R})$.

For $\psi \in \mathcal{D}(\mathbb{R})$ and $F = \left[\frac{f_n}{\varphi_n}\right] \in \beta(\mathbb{R})$, $F * \psi$ is defined as $F * \psi = \left[\frac{f_n * \psi}{\varphi_n}\right]$.

Definition 2.1. *A sequence of Boehmians $\{F_n\}$ is said to be δ-convergent to a Boehmian F, denoted $\delta\text{-}\lim_{n\to\infty} F_n = F$, if there exists a delta sequence $\{\varphi_n\}$ such that $F_n * \varphi_k, F * \varphi_k \in L^1_{loc}(\mathbb{R})$ for all $k, n \in \mathbb{N}$, and for each $k \in \mathbb{N}, F_n * \varphi_k \to F * \varphi_k$ in $L^1_{loc}(\mathbb{R})$ as $n \to \infty$.*

For more on δ-convergence, see [**7**].

3. INTEGRABLE BOEHMIANS

Denote by $L^1(\mathbb{R})$ the space of complex-valued Lebesgue integrable functions on the real line $\mathbb{R}$. The space of integrable Boehmians will be denoted by $\beta_\ell(\mathbb{R})$. Thus, $F = \left[\frac{f_n}{\varphi_n}\right] \in \beta_\ell(\mathbb{R})$ provided that $F \in \beta(\mathbb{R})$ and $f_n \in L^1(\mathbb{R}), n \in \mathbb{N}$.

Since each $f \in L^1(\mathbb{R})$ can be identified with $\left[\frac{f * \varphi_n}{\varphi_n}\right] \in \beta_\ell(\mathbb{R})$, we may consider $L^1(\mathbb{R})$ a subspace of $\beta_\ell(\mathbb{R})$. Theorems 3.4 and 3.5 show that the space $\beta_\ell(\mathbb{R})$ is considerably larger than $L^1(\mathbb{R})$. Moreover, Theorem 3.5 may be used to construct an integrable Boehmian which is not a Schwartz distribution.

Remark. The name integrable Boehmians is usually associated with the space constructed in [**8**]. Since $\beta_\ell(\mathbb{R})$ can be identified with a subspace of this space, we will call elements of $\beta_\ell(\mathbb{R})$, integrable Boehmians.

The Fourier transform of an $L^1(\mathbb{R})$ function is given by

$$\widehat{f}(x) = \int_{-\infty}^{\infty} f(t)e^{-ixt}\,\mathrm{d}t. \tag{3.1}$$

The Fourier transform can be extended to the space $\beta_\ell(\mathbb{R})$ as follows.

Definition 3.1. *Let* $F = \left[\frac{f_n}{\varphi_n}\right] \in \beta_\ell(\mathbb{R})$*. The Fourier transform of F, denoted by* $\widehat{F}$*, is the function defined for each* $x \in \mathbb{R}$ *by*

$$\widehat{F}(x) = \lim_{n\to\infty} \widehat{f}_n(x). \tag{3.2}$$

The above limit exists, and is independent of the representative. Moreover, the FOURIER transform of a Boehmian satisfies the same basic properties as the classical FOURIER transform of an L^1 function (see [**8**]).

It is not difficult to show that $\widehat{F}$ is continuous on $\mathbb{R}$. That is, $\widehat{F} \in C(\mathbb{R})$. Moreover, as the next theorem will show, $\widehat{F}$ satisfies a growth condition.

Theorem 3.2. *Let* $\theta(x)$ *be a positive increasing function such that*

$$\int_1^\infty \frac{\theta(x)}{x^2}\,dx = \infty.$$

If $F \in \beta_\ell(\mathbb{R})$*, then* $\liminf_{x\to\infty} e^{-\theta(x)}|\widehat{F}(x)| = 0$.

Proof. Let $F = \left[\frac{f_n}{\varphi_n}\right] \in \beta_\ell(\mathbb{R})$. Thus, $F * \varphi_n = f_n \quad (n \in \mathbb{N})$, and hence,

$$\widehat{F}(x)\widehat{\varphi}_n(x) = \widehat{f}_n(x), \tag{3.3}$$

for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

Now, suppose that there exist constants $\epsilon > 0$ and $x_0 \in \mathbb{R}$ such that

$$|\widehat{F}(x)| \geq \epsilon e^{\theta(x)}, \tag{3.4}$$

for all $x \geq x_0$.

Thus, by (3.3) and (3.4), for each $n \in \mathbb{N}$,

$$\widehat{\varphi}_n(x) = \mathrm{O}(e^{-\theta(x)}) \quad \text{as } x \to \infty.$$

Since φ_n has compact support, Theorem XXII in [**5**] implies that $\varphi_n \equiv 0$, for all $n \in \mathbb{N}$.

This contradiction completes the proof of the theorem. □

In the previous theorem, the growth condition for $\widehat{F}$ at infinity can be replaced by an equivalent condition at negative infinity.

The proof of the following lemma is left to the reader.

Lemma 3.3. *Let* $g \in C^{(2)}(\mathbb{R})$ *such that* $g^{(j)} \in L^1(\mathbb{R}) \cap C_0(\mathbb{R})$ *for* $j = 0, 1, 2$*. Then there exists* $f \in L^1(\mathbb{R})$ *such that* $\widehat{f}(x) = g(x)$ *for all* $x \in \mathbb{R}$.

In the above lemma, $C_0(\mathbb{R})$ denotes the space of all continuous functions which vanish at infinity.

Also, a function f is in $C^{(2)}(\mathbb{R})$ provided that f is twice differentiable and $f'' \in C(\mathbb{R})$.

In the next theorem, w is a continuous real-valued function on $\mathbb{R}$ such that

(i) $0 = w(0) \leq w(x+y) \leq w(x) + w(y)$ for all $x, y \in \mathbb{R}$,

(ii) $\int\limits_{-\infty}^{\infty} \frac{w(x)}{1+x^2}\,\mathrm{d}x < \infty$,

(iii) $w(x) \geq a + b\ln(1+|x|)$, for some real a and positive b and all $x \in \mathbb{R}$.

Theorem 3.4. *Let $g \in C^{(2)}(\mathbb{R})$ such that $g^{(j)}(x) = O(e^{w(x)})$ as $|x| \to \infty$ for $j = 0, 1, 2$. Then there exists $F \in \beta_\ell(\mathbb{R})$ such that $\widehat{F}(x) = g(x)$, $x \in \mathbb{R}$.*

Proof. By Theorem 1.4.1 in [**2**], there exists $\psi \in D(\mathbb{R})$ such that $\int\limits_{-\infty}^{\infty} \psi(x)\,\mathrm{d}x = 1$ and, for each $n \in \mathbb{R}$, there exists a constant $M_n > 0$ such that

$$|\widehat{\psi}(x)| \leq M_n e^{-2nw(x)}, \quad x \in \mathbb{R}.$$

For $n \in \mathbb{N}$, define $\psi_n(x) = n\psi(nx)$, $\quad x \in \mathbb{R}$. Then, $\{\psi_n\}$ is a delta sequence and, for each $n \in \mathbb{N}$,

$$|\widehat{\psi_n}(x)| \leq M_n e^{-2w(x)}, \quad x \in \mathbb{R}.$$

Now, let $\varphi_n = \psi_n * \psi_n * \psi_n$, $\quad n \in \mathbb{N}$. Thus, $\{\varphi_n\}$ is a delta sequence. Moreover, for $j = 0, 1, 2$ and $n \in \mathbb{N}$,

$$g\widehat{\varphi}_n \in C^{(2)}(\mathbb{R}) \text{ and } (g\widehat{\varphi}_n)^{(j)} \in L^1(\mathbb{R}) \cap C_0(\mathbb{R}).$$

Thus, by Lemma 3.3, for each $n \in \mathbb{N}$, there exists $f_n \in L^1(\mathbb{R})$ such that $\widehat{f_n} = g\widehat{\varphi}_n$. Now,

$$(f_n * \varphi_k)^\wedge = \widehat{f_n}\widehat{\varphi}_k = (g\widehat{\varphi}_n)\widehat{\varphi}_k = (g\widehat{\varphi}_k)\widehat{\varphi}_n = \widehat{f_k}\widehat{\varphi}_n = (f_k * \varphi_n)^\wedge.$$

Thus, $f_n * \varphi_k = f_k * \varphi_n$, for all $n, k \in \mathbb{N}$. Therefore, $F = \left[\frac{f_n}{\varphi_n}\right] \in \beta_\ell(\mathbb{R})$.

Moreover,

$$\widehat{F}(x) = \lim_{n\to\infty} \widehat{f_n}(x) = \lim_{n\to\infty} g(x)\widehat{\varphi}_n(x) = g(x), \quad x \in \mathbb{R}. \qquad \square$$

A Boehmian F is said to vanish on an open interval (a, b) provided that there exists a delta sequence $\{\varphi_n\}$ such that $F * \varphi_n \in C(\mathbb{R}), n \in \mathbb{N}$, and $F * \varphi_n \to 0$ uniformly on compact subsets of (a, b) as $n \to \infty$. The support of F is the complement of the largest open set on which F vanishes. Every Boehmian with bounded support is an element of $\beta_\ell(\mathbb{R})$.

J. Burzyk [**3**] proved the following Paley-Wiener type theorem.

Theorem 3.5. *Suppose F is a Boehmian such that $\operatorname{supp} F \subseteq [-\sigma, \sigma]$ for some $\sigma \geq 0$. Then $\widehat{F}$ is an entire function. Moreover,*

(i) *For every* $\epsilon > 0$, *there exists a constant* A_ϵ *such that*

$$|\widehat{F}(z)| < A_\epsilon e^{(\sigma+\epsilon)|z|} \tag{3.5}$$

for $z \in \mathbb{C}$, *and*

(ii)

$$\int_{-\infty}^{\infty} \frac{\ln_+ |\widehat{F}(x)|}{1+x^2}\,\mathrm{d}x < \infty. \tag{3.6}$$

Conversely, if an entire function g satisfies conditions (3.5) and (3.6), then it is the Fourier transform of a Boehmian F whose support is contained in $[-\sigma, \sigma]$.

An Inversion Theorem is given in [**8**]. The next theorem gives another inversion formula, which has the form of the classical inversion formula for L^1 functions.

Theorem 3.6. *Let* $F \in \beta_\ell(\mathbb{R})$. *Then,* $F = \int_{-\infty}^{\infty} e^{ixt}\widehat{F}(t)\,\mathrm{d}t$.

$$(\textit{That is, } F = \delta\text{-}\lim_{n\to\infty} \int_{|t|\le n} e^{ixt}\widehat{F}(t)\,\mathrm{d}t.)$$

Proof. Let $F = \left[\frac{f_n}{\varphi_n}\right] \in \beta_\ell(\mathbb{R})$. We may assume that for each $n \in \mathbb{N}$, $f_n, \widehat{f_n} \in L^1(\mathbb{R}) \cap C(\mathbb{R})$. For, if not, notice that $F = \left[\frac{f_n * \varphi_n}{\varphi_n * \varphi_n}\right]$ and $f_n * \varphi_n, (f_n * \varphi_n)^\wedge \in L^1(\mathbb{R}) \cap C(\mathbb{R})$.

Now, for each $n \in \mathbb{N}$, let

$$F_n(x) = \int_{|t|\le n} e^{ixt}\widehat{F}(t)\,\mathrm{d}t, \quad x \in \mathbb{R}.$$

Thus,

$$\begin{aligned}(F_n * \varphi_k)(x) &= \int_{|t|\le n} e^{ixt}\widehat{F}(t)\widehat{\varphi}_k(t)\,\mathrm{d}t \\ &= \int_{|t|\le n} e^{ixt}\widehat{f_k}(t)\,\mathrm{d}t, \text{ for all } n, k \in \mathbb{N} \text{ and } x \in \mathbb{R}.\end{aligned}$$

Therefore, for each k,

$$F_n * \varphi_k \to f_k \quad \text{uniformly as } n \to \infty.$$

That is,

$$\delta\text{-}\lim_{n\to\infty} \int_{|t|\le n} e^{ixt}\widehat{F}(t)\,\mathrm{d}t = F.$$ □

Remarks. (i) The delta sequences used in [**8**] are more general than the delta sequences used in this paper. The space of integrable Boehmians in [**8**] is larger than $\beta_\ell(\mathbb{R})$. It can be shown that if $g \in C^{(2)}(\mathbb{R})$, then there exists an $F \in \beta_{L_1}(\mathbb{R})$ such that $\widehat{F}(x) = g(x)$, $x \in \mathbb{R}$. However, unlike the space $\beta_{L_1}(\mathbb{R})$ in [**8**], each element of $\beta_\ell(\mathbb{R})$ has local properties such as a support.

(ii) It would be of interest to find necessary and sufficient conditions for a given continuous function to be the FOURIER transform of some integrable Boehmian. Since there is no nice necessary and sufficient condition which can be used to determine whether a given continuous function (which vanishes at infinity) is the FOURIER transform of an $L^1(\mathbb{R})$ function, this is most likely a difficult problem.

4. PERIODIC BOEHMIANS

Let T denote the unit circle. We make no distinction between a function on T and a 2π-periodic function on $\mathbb{R}$.

In this section, we give a brief introduction to the space of periodic Boehmians $\beta(T)$. The space $\beta(T)$ is quite large. It contains a subspace which can be identified with the space of periodic SCHWARTZ distributions, as well as some elements which can be identified with a subspace of periodic hyperfunctions.

The material in this section will be needed in Section 5. For the proofs of the theorems and for more results on $\beta(T)$, see [**9**, **10**, **11**].

For $f \in L^1_{loc}(\mathbb{R})$, let $\tau_a f(x) = f(x+a)$, $a \in \mathbb{R}$.

The translation operator τ_a can be extended to the space $\beta(\mathbb{R})$.

For $F = \left[\dfrac{f_n}{\varphi_n}\right] \in \beta(\mathbb{R})$, define $\tau_a F = \left[\dfrac{\tau_a f_n}{\varphi_n}\right], a \in \mathbb{R}$. It is routine to show that $\left[\dfrac{\tau_a f_n}{\varphi_n}\right] \in \beta(\mathbb{R})$.

The space of periodic Boehmians will be denoted by $\beta(T)$. That is, $F \in \beta(T)$ provided $F \in \beta(\mathbb{R})$ and $\tau_{2\pi} F = F$.

Lemma 4.1. *Let* $F = \left[\dfrac{f_n}{\varphi_n}\right] \in \beta(\mathbb{R})$. *Then,* $F \in \beta(T)$ *if and only if* $f_n \in L^1(T)$, *for all* $n \in \mathbb{N}$.

For $f \in L^1(T)$, the k^{th} FOURIER coefficient is given by

$$c_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx}\, \mathrm{d}x, \quad k \in \mathbb{Z}. \tag{4.1}$$

Definition 4.2. *Let* $F = \left[\dfrac{f_n}{\varphi_n}\right] \in \beta(T)$. *The* k^{th} *Fourier coefficient of* F *is given by*

$$c_k(F) = \lim_{n\to\infty} c_k(f_n). \tag{4.2}$$

The above limit exists, and is independent of the representative.

Theorem 4.3. *Let w be a real-valued even function defined on the integers $\mathbb{Z}$ such that $0 = w(0) \leq w(n+m) \leq w(n) + w(m)$ for all $n, m \in \mathbb{Z}$ and $\sum_{n=1}^{\infty} \frac{w(n)}{n^2} < \infty$. Suppose that the set of positive integers is partitioned into two disjoint sets $\{t_n\}$ and $\{s_n\}$ such that $\sum_{n=1}^{\infty} \frac{1}{t_n} < \infty$. If $\{\xi_n\}$ is a sequence of complex numbers such that $\xi_{\pm s_n} = O(e^{w(s_n)})$ as $n \to \infty$, then there exists a periodic Boehmian F such that $c_n(F) = \xi_n$, $n \in \mathbb{Z}$.*

The next theorem is a stronger version of Theorem 3.5 in [**11**]. Since the proof is similar to that of Theorem 3.5, it is omitted.

Theorem 4.4. *Let $\theta(x)$ be an increasing function such that*

$$\int_1^{\infty} \frac{\theta(x)}{x^2} \, \mathrm{d}x = \infty.$$

Let $\{\lambda_n\}$ be an increasing sequence of positive integers such that $\lim_{n\to\infty} \frac{n}{\lambda_n} = D > 0$. Then, for each $F \in \beta(T)$, $\liminf_{n\to\infty} e^{-\theta(\lambda_n)} |c_{\lambda_n}(F)| = 0$.

By making the appropriate changes, Theorem 4.4 is also valid for a sequence of negative integers $\{\lambda_n\}$.

In the next section, Theorem 4.4 will be used to strengthen Theorem 3.2.

Theorem 4.5. *Let $F \in \beta(T)$. Then, $F = \sum_{k=-\infty}^{\infty} c_k(F) e^{ikx}$.*

$$(\textit{That is, } F = \delta\text{-}\lim_{n\to\infty} \sum_{k=-n}^{n} c_k(F) e^{ikx}.)$$

Remark. By using Theorem 4.3 it is clear that $\beta(T)$ contains a proper subspace which can be identified with the space of periodic Schwartz distributions. Theorem 4.3 also shows that there are Boehmians which are not hyperfunctions. Conversely, by using Theorem 4.4, we see that there are hyperfunctions which are not Boehmians.

5. THE POISSON SUMMATION FORMULA

The importance of the Poisson summation formula is well-known. It has been found to be useful in many areas of mathematics, such as, number theory, differential equations, and signal analysis. For a nice introduction to some applications of the Poisson summation formula, see [**12**].

One form of Poisson's summation formula, for a well-behaved function f, is given by

$$2\pi \sum_{k=-\infty}^{\infty} f(x + 2\pi k) = \sum_{k=-\infty}^{\infty} \widehat{f}(k) e^{ikx}, \tag{5.1}$$

where $\widehat{f}(\xi) = \int\limits_{-\infty}^{\infty} f(x)e^{-i\xi x}\,\mathrm{d}x$.

In this section, see Theorem 5.7, we will present a version of the POISSON summation formula for $\beta_\ell(\mathbb{R})$.

An integrable function does not necessarily satisfy POISSON's summation formula (see [**4**]). However, recall that $L^1(\mathbb{R})$ can be identified with a subspace of $\beta_\ell(\mathbb{R})$. Thus, POISSON's summation formula for integrable Boehmians, Theorem 5.7, is valid for any $L^1(\mathbb{R})$ function.

The *periodization operator* $^\# : L^1(\mathbb{R}) \to L^1(T)$ is given by

$$f^\#(x) = \sum_{k=-\infty}^{\infty} f(x + 2\pi k), \ \text{ for } f \in L^1(\mathbb{R}). \tag{5.2}$$

We will see that the mapping $^\#$ can be extended onto the space $\beta_\ell(\mathbb{R})$ by

$$F^\# = \left[\frac{f_n^\#}{\varphi_n}\right], \ \text{where } F = \left[\frac{f_n}{\varphi_n}\right] \in \beta_\ell(\mathbb{R}). \tag{5.3}$$

Hence, $^\# : \beta_\ell(\mathbb{R}) \to \beta(T)$.

The proof of the following lemma may be found in [**1**].

Lemma 5.1. *Let $f \in L^1(\mathbb{R})$ and $\{\varphi_n\}$ be a delta sequence. Then*

(i) $2\pi c_p(f^\#) = \widehat{f}(p)$, *for all* $p \in \mathbb{Z}$;

(ii) $2\pi c_p(f^\# * \varphi_n) = \widehat{f}(p)\widehat{\varphi}_n(p)$, *for all $p \in \mathbb{Z}$ and $n \in \mathbb{N}$.*

Lemma 5.2. *Let $\left[\frac{f_n}{\varphi_n}\right] \in \beta_\ell(\mathbb{R})$. Then, $\left[\frac{f_n^\#}{\varphi_n}\right] \in \beta(T)$.*

Proof. $2\pi c_p(f_n^\# * \varphi_k) = \widehat{f}_n(p)\widehat{\varphi}_k(p) = (f_n * \varphi_k)^\wedge(p) = (f_k * \varphi_n)^\wedge(p) = \widehat{f}_k(p)\widehat{\varphi}_n(p)$ $= 2\pi c_p(f_k^\# * \varphi_n)$. Thus, $f_n^\# * \varphi_k = f_k^\# * \varphi_n$, for all $k, n \in \mathbb{N}$.

Therefore, $\left[\frac{f_n^\#}{\varphi_n}\right] \in \beta(T)$. ☐

Since the proof of the following lemma is similar to the proof of Lemma 5.2, it is omitted.

Lemma 5.3. *Let $\left[\frac{f_n}{\varphi_n}\right], \left[\frac{g_n}{\psi_n}\right] \in \beta_\ell(\mathbb{R})$ such that $\left[\frac{f_n}{\varphi_n}\right] = \left[\frac{g_n}{\psi_n}\right]$. Then, $\left[\frac{f_n^\#}{\varphi_n}\right] = \left[\frac{g_n^\#}{\psi_n}\right]$.*

By Lemmas 5.2 and 5.3, the mapping $^\#$ is well-defined and maps $\beta_\ell(\mathbb{R})$ into $\beta(T)$.

Lemma 5.4. *Let $F \in \beta_\ell(\mathbb{R})$. Then, $2\pi c_k(F^\#) = \widehat{F}(k)$, $k \in \mathbb{Z}$.*

Proof. Let $F = \left[\frac{f_n}{\varphi_n}\right] \in \beta_\ell(\mathbb{R})$. Then,

$$2\pi c_k(F^{\#}) = 2\pi \lim_{n\to\infty} c_k(f_n^{\#}) = \lim_{n\to\infty} \widehat{f}_n(k) = \widehat{F}(k). \qquad \square$$

By applying Lemma 5.4 to Theorem 4.4, an improvement of Theorem 3.2 is obtained.

Theorem 5.5. *Let $\theta(x)$ be an increasing function such that*

$$\int_1^\infty \frac{\theta(x)}{x^2}\,\mathrm{d}x = \infty.$$

Let $\{\lambda_n\}$ be an increasing sequence of positive integers such that $\lim\limits_{n\to\infty} \frac{n}{\lambda_n} = D > 0$. Then, for each $F \in \beta_\ell(\mathbb{R})$, $\liminf\limits_{n\to\infty} e^{-\theta(\lambda_n)}|\widehat{F}(\lambda_n)| = 0$.

Lemma 5.6. *Let $F \in \beta_\ell(\mathbb{R})$. Then, $F^{\#} = \sum\limits_{k=-\infty}^{\infty} F(x + 2\pi k)$.*

(That is, $F^{\#} = \delta\text{-}\lim\limits_{n\to\infty} \sum\limits_{|k|\le n} \tau_{2\pi k}F$.)

Proof. Let $F = \left[\frac{f_n}{\varphi_n}\right] \in \beta_\ell(\mathbb{R})$. Then, for each $p \in \mathbb{N}$, $\varphi_p * \sum\limits_{|k|\le n} \tau_{2\pi k}F = \sum\limits_{|k|\le n} \tau_{2\pi k} f_p \to f_p^{\#}$ in $L^1_{loc}(\mathbb{R})$ as $n \to \infty$ (see [**1**], Lemma 1). That is, $\delta\text{-}\lim\limits_{n\to\infty} \sum\limits_{|k|\le n} \tau_{2\pi k}F = F^{\#}$. $\square$

The following is the Poisson summation formula for integrable Boehmians.

Theorem 5.7. *Let $F \in \beta_\ell(\mathbb{R})$. Then,*

$$2\pi \sum_{k=-\infty}^{\infty} F(x + 2\pi k) = \sum_{k=-\infty}^{\infty} \widehat{F}(k) e^{ikx}. \tag{5.4}$$

Proof.

$$\sum_{k=-\infty}^{\infty} \widehat{F}(k)e^{ikx} = \sum_{k=-\infty}^{\infty} 2\pi c_k(F^{\#})e^{ikx} = 2\pi F^{\#} = 2\pi \sum_{k=-\infty}^{\infty} F(x + 2\pi k). \qquad \square$$

Corollary 5.8. *Let $f \in L^1(\mathbb{R})$ and $\{\varphi_n\}$ be a delta sequence. Then,*

$$2\pi \sum_{k=-\infty}^{\infty} f(x + 2\pi k) = \lim_{n\to\infty} \sum_{k=-\infty}^{\infty} \widehat{\varphi}_n(k)\widehat{f}(k)e^{ikx} \tag{5.5}$$

in $L^1_{loc}(\mathbb{R})$.

It can be shown that if $\delta\text{-}\lim\limits_{n\to\infty} F_n = F$ and $F_n = 0$ on (a, b) for all $n \in \mathbb{N}$, then $F = 0$ on (a, b). Combining this with Poisson's summation formula and Theorem 3.5, we obtain the following.

Let g be an entire function satisfying the following conditions.

(i) For each $\epsilon > 0$, there exists a constant A_ϵ such that $|g(z)| < A_\epsilon e^{(\sigma+\epsilon)|z|}$, for all $z \in \mathbb{C}$ (for some $0 \le \sigma < \pi$).

(ii) $\int\limits_{-\infty}^{\infty} \frac{\ln_+ |g(x)|}{1+x^2}\, dx < \infty$.

Then, $\sum\limits_{n=-\infty}^{\infty} g(n)e^{inx} \in \beta(T)$. Moreover, $\sum\limits_{n=-\infty}^{\infty} g(n)e^{inx} = 0$ on $\sigma < |x| < 2\pi - \sigma$.

For example, the Mittag-Leffler function $E_\alpha(z) = \sum\limits_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}$ (where $\alpha > 0$ and $\Gamma(x) = \int\limits_0^\infty e^{-t}t^{x-1}\, dt$) is an entire function of order $1/\alpha$.

Thus, for $\alpha > 1$, $\sum\limits_{n=-\infty}^{\infty} E_\alpha(n)e^{inx} \in \beta(T)$ and $\sum\limits_{n=-\infty}^{\infty} E_\alpha(n)e^{inx} = 0$ on $0 < |x| < 2\pi$.

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