Applicable Analysis and Discrete Mathematics, **1** (2007), 251–256.
Available electronically at **http://pefmath.etf.bg.ac.yu**

Presented at the conference: *Topics in Mathematical Analysis and Graph Theory*, Belgrade, September 1–4, 2006.

ON A CORRELATION BETWEEN DIFFERENTIAL EQUATIONS AND THEIR CHARACTERISTIC EQUATIONS

Boro M. Piperevski

Abstract: The aim of this paper is to derive the dependence of the nature of a solution of a class of differential equations of n-th order with polynomial coefficients on the solutions of the corresponding characteristic algebraic equation of n-th degree.

1. INTRODUCTION

Many theoretical and practical problems from theory and practice acquire resolving trough plenty of concrete process characteristics. Problems of this type are eigenvalues problems, boundary value problems, optimal values at variation calculation, of polynomials existence as special functions with determined characteristic.

Classical theory of partial differential equations from mathematical physics is related with the functions of Lame, Mathieu, classical orthogonal polynomials, polynomials of Appell, polynomials of Stieltjes, etc.

Heine problem for the number of linear differential equations with polynomial coefficients that have polynomial solution, connected with the practical problem of equilibrium (Stieltjes [**10**]), is known [**6**, **7**].

The connection between roots of characteristic algebraic equation with the form of solutions of homogenous linear differential equation with constant coefficients, is already known in classical theory of differential equations.

The similar result for existence of polynomial solution of linear homogenous differential equation with polynomial coefficients is obtained [**8**].

1991 Mathematics Subject Classification. 34-A05.
Key Words and Phrases. Differential equations, polynomial solutions, root of characteristic equation.

2. NOTIONS, DEFINITIONS AND KNOWN RESULTS

Here we will try to generalize already known results.

Let us consider differential equation:

$$Ay'' + By' + Cy = 0,$$

where

$$A = a_2x^2 + a_1x + a_0, \quad B = b_1x + b_0, \quad C = c_0, \quad a_2 \neq 0.$$

Its characteristic equation is

$$\frac{t(t-1)}{2}\,A'' + tB' + C = 0.$$

It is known that if one root of characteristic equation is positive integer, then the solution of the differential equation is polynomial function. This solution is given by known RODRIGUES's formula:

$$y = A\exp\Big(-\int\frac{B}{A}\,\mathrm{d}x\Big)\frac{\mathrm{d}^n}{\mathrm{d}x^n}\bigg(A^{n-1}\exp\Big(\int\frac{B}{A}\mathrm{d}x\Big)\bigg).$$

This problem for existence of polynomial solutions of differential equation is closely related with operators eigenvalues and eigenfunctions. LEGENDRE's and other classical orthogonal polynomials are well known solution of differential equations [**2, 3, 4**].

We can show that differential equation

$$(x^2-1)y'' + 2xy' - n(n+1)y = 0,$$

$n-$positive integer, having LEGENDRE polynomial solution given by

$$y = \frac{\mathrm{d}^n}{\mathrm{d}x^n}\big((x^2-1)^n\big),$$

has characteristic equation whose roots n and $-(n+1)$ are integer numbers.

We also know that if characteristic algebraic equation

$$\binom{t}{0}P_0(x) + \binom{t}{1}P_1'(x) + \binom{t}{2}P_2''(x) + \cdots + \binom{t}{m}P_m^{(m)}(x) = 0, \tag{1}$$

of degree m with respect to t of differential equation

$$P_m(x)y^{(m)} + P_{m-1}(x)y^{(m-1)} + \cdots + P_0(x)y = 0, \tag{2}$$

where $P_i(x), i = \overline{0,m}$ are polynomial of i-th degree, has positive integer root, then this equation has polynomial solution [**8**].

When the roots of characteristic equation (1) are successive positive integers, then the result is known [**1,9**].

Here, we will try to find a solution of differential equation (2) in case when the roots of characteristic equation (1) are successive negative integers.

3. MAIN RESULT

Theorem. *The linear differential equation*

$$P_m(x)y^{(m)} + P_{m-1}(x)y^{(m-1)} + \cdots + P_0(x)y = 0,$$

where $P_i(x), i = \overline{0,m}$, *are polynomials of* i*-th degree and* $P_m^{(m)}(x) \neq 0$, *is solvable if the following conditions are satisfied*:

a) *The roots of characteristic equation* (1) *are a* $m-1$ *negative integer* $t_1 = -(n+1), t_2 = -(n+2), \ldots, t_{m-1} = -(n+m-1), n$*-positive integer. If* t_m *is negative integer then* $|t_m| > |t_i|, i = \overline{1, m-1}$.

b) *The polynomial coefficients are satisfying the following conditions*:

$$P_{k-1}(x) - \binom{n+m-1}{1}P'_k(x) + \binom{n+m}{2}P''_{k+1}(x) + \cdots + (-1)^{m-k+1}\binom{n+2m-k-1}{m}P_m^{(m-k+1)}(x) = 0, \quad k = \overline{1, m-1}. \tag{3}$$

In that case, the general solution can be given by

$$y = \frac{\mathrm{d}^n}{\mathrm{d}x^n}\Bigg(P_m^{n+m-1}\exp\Big(-\int\frac{P_{m-1}}{P_m}\,\mathrm{d}x\Big)\Big(C_1 + \int(C_2 + C_3x + \cdots + C_mx^{m-2})P_m^{-(n+m)}\exp\Big(\int\frac{P_{m-1}}{P_m}\,\mathrm{d}x\Big)\mathrm{d}x\Big)\Bigg), \tag{4}$$

where $C_1, C_2, \ldots, C_m$ *are arbitrary constants.*

Proof. Let suppose that the conditions of theorem are satisfied. We consider linear differential equation of the first order

$$P_m(x)z' + \big(P_{m-1}(x) - (n+m-1)P'_m(x)\big)z = C_2 + C_3x + C_4x^2 + \ldots + C_mx^{m-2},$$

where $C_2, \ldots, C_m$ are arbitrary constants.

Differentiating it $n+m-1$ times, we obtain the equation (2) where $z^{(n)} = y$ and condition (3) is used.

With this differentiation procedure, we get a formula for general solution (4) using formula for the general solution of the linear differential equation of the first order .

4. ANALYSIS OF THE NATURE OF THE SOLUTION OF THE DIFFERENTIAL EQUATION (2)

We consider special case, when m-th root

$$t_m = (m+n)(m-1) - \frac{P_{m-1}^{(m-1)}}{(m-1)!}$$

is positive integer. Then, equation (2) has unique polynomial solution of degree $(m+n)(m-1) - \dfrac{P_{m-1}^{(m-1)}}{(m-1)!}$.

Example 1. For differential equation

$$x(x-1)(x-2)y''' + (x+1)(x+2)y'' + (-52x+72)y' - 108y = 0$$

we get $t_1 = -3, t_2 = -4, t_3 = 9, n = 2$, and general solution is obtained by formula

$$y = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\bigg(\frac{x^3(x-1)^{10}}{(x-2)^2}\bigg(C_1 + \int (C_2 + C_3 x)\frac{x-2}{x^4(x-1)^{11}}\,\mathrm{d}x\bigg)\bigg).$$

Since $t_3 = 9$, equation has only one polynomial solution of *nine*-th degree.

Particularly, if $P_{m-1}(x) = kP'_m(x)$, and if the m-th root $t_m = (m+n)(m-1) - km$ -positive integer, then there exists a polynomial solution, given by the formula

$$y = \frac{\mathrm{d}^n}{\mathrm{d}x^n}\big(P_m^{n+m-1-k}(x)\big).$$

Remark 1. For $m = 3$, equation (2) has a form

$$Ay''' + By'' + Cy' + Dy = 0,$$

where

$$A = \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0, B = \beta_2 x^2 + \beta_1 x + \beta_0, C = \gamma_1 x + \gamma_0, D = \delta, \alpha_3 \neq 0.$$

If conditions of the theorem are fulfilled, the general solution is

$$y = \frac{\mathrm{d}^n}{\mathrm{d}x^n}\bigg(A^{n+2}\exp\Big(-\int\frac{B}{A}\,\mathrm{d}x\Big)\bigg(C_1 + \int (C_2 + C_3 x)A^{-(n+3)}\exp\Big(\int\frac{B}{A}\,\mathrm{d}x\Big)\mathrm{d}x\bigg)\bigg),$$

and the equation has a form

$$Ay''' + By'' + (n+2)\Big(B' - \frac{n+3}{2}A''\Big)y' + (n+1)(n+2)\Big(\frac{1}{2}B'' - \frac{n+3}{3}A'''\Big)y = 0,$$

$n-$ positive integer .

Example 2. For the differential equation

$$x(x-1)(x-2)\,y''' + 5(x+1)(x+2)\,y'' + (-6x+81)\,y' - 18y = 0$$

we get $t_1 = -2, t_2 = -3, t_3 = 3, n = 1$ and polynomial solution of third degree is obtained by

$$y = x^3 + \frac{63}{4}\, x^2 + 129x + 598.$$

Particularly, if $B = kA'$, than differential equation has a form

$$Ay''' + kA'y'' + (n+2)\Big(k - \frac{n+3}{2}\Big)A''y' + (n+2)\Big(\frac{n+1}{2}\, k - \frac{(n+1)(n+3)}{3}\Big)A'''y = 0,$$

and the general solution is given by the formula

$$y = \frac{\mathrm{d}^n}{\mathrm{d}x^n}\bigg(A^{n-k+2}\Big(C_1 + \textstyle\int (C_2 + C_3 x)A^{k-n-3}\,\mathrm{d}x\Big)\bigg).$$

If $t_3 = 2n - 3k + 6$ is positive integer, then the polynomial solution of degree $2n - 3k + 6$ is given by the formula

$$L_{n,k} = \frac{\mathrm{d}^n}{\mathrm{d}x^n}(A^{n-k+2}).$$

For $k = 2$ and $A = x(1 - x^2)$ we have special differential equation

$$x(1 - x^2)\, y''' + 2(1 - 3x^2)\, y'' - 3(n-1)(n+2)xy' + 2n(n+1)(n+2)y = 0$$

and the polynomial solution of $2n$-th degree is

$$L_{2n} = \frac{\mathrm{d}^n}{\mathrm{d}x^n}\big(x^n(1 - x^2)^n\big).$$

These polynomials are Appell's polynomials or generalized Legendre polynomials [**5**].

Remark 2. For $m = 4$ the equation (2) with conditions given above, has a form

$$\begin{aligned} P_4(x)y^{(IV)} &+ P_3(x)y''' + (n+3)\Big(P_3'(x) - \frac{n+4}{2}P_4''(x)\Big)y'' \\ &+ (n+2)(n+3)\Big(\frac{1}{2}\, P_3''(x) - \frac{n+4}{3}\, P_4'''(x)\Big)y' \\ &+ \quad (n+1)(n+2)(n+3)\Big(\frac{1}{6}\, P_3'''(x) - \frac{n+4}{8}\, P_4^{(IV)}(x)\Big)y = 0, \end{aligned}$$

$n-$ positive integer .

If $3n + 12 - \dfrac{P_3'''(x)}{3!}$ is positive integer then there will be only one polynomial solution for the equation.

Particularly, for $P_3(x) = kP_4'(x)$, if $3n + 12 - 4k$ is positive integer than the polynomial solution will be obtained.

For $k = 3$, we have polynomial solution of $3n$-th degree given by the formula

$$P_{3n}(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n}\big(P_4^n(x)\big)$$

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Faculty of Electrical Engineering,
Ss Ciril and Methodius University,
P.O.Box 574, 1000 Skopje,
Republic of Macedonia
E–mail: borom@etf.ukim.edu.mk

(Received October 27, 2006)