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ON THE MATRIX EQUATION $XA - AX = \tau(X)$

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We study the matrix equation $XA - AX = \tau(X)$ in $M_n(K)$, where τ is an automorphism of a field K of finite order k. A criterion under which this equation has a nontrivial solution is given. In case when $k = 1$ that criterion boils down to an already known result.

1. INTRODUCTION

The main purpose of this paper is to develop the connection between the eigenvalues of a class of pseudo-linear transformation over a field K and the eigenvalues of a certain linear transformation. The use of linear transformations enables us to use Cayley- Hamilton theorem which in pseudo-linear setting does not hold.

This work was directly inspired by the paper [**2**] for $p = 1$. In this case we get linear matrix equation $XA - AX = X$. We went one step further by introducing an automorphism τ of a field K of finite order k, $XA - AX = \tau(X)$. Since it does not remain linear matrix equation anymore, the classical methods can not be used. By equivalent transformations this equation can be viewed in another form $\tau^{-1}(X)\tau^{-1}(A) - \tau^{-1}(X)\tau^{-1}(A) = X$. The left hand side of the equation is a pseudo-linear transformation of $M_n(K)$, $T(X) = \tau^{-1}(X)\tau^{-1}(A) - \tau^{-1}(A)\tau^{-1}(X)$. In fact, in order to find out if the equation has nontrivial solutions we will investigate whether $\lambda = 1$ is the eigenvalue of T or, equivalently, of linear transformation T^k. In case $k = 1$ we get an already known criterion.

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2. RECAPITULATION

Let K be a field and $\sigma \in \mathrm{Aut}\,(K)$. A skew polynomial ring (also called Ore extension) $K[t;\sigma]$ consists of polynomials $\sum\limits_{i=0}^{n} a_i t^i$, $a_i \in K$ which are added in the usual way but are multiplied according to the following rule

$$ta = \sigma(a)t, \qquad a \in K.$$

The evaluation $f(a)$ of a polynomial $f(t) \in K[t;\sigma]$ at some element $a \in K$ is the remainder one gets when $f(t) = \sum\limits_{i=0}^{n} a_i t^i$ is divided on the right by $t - a$. It is easy to show by induction that

$$f(a) = \sum_{i=0}^{n} a_i N_i(a)$$

where the maps N_i are defined by induction in the following way. For any $a \in K$

$$N_0(a) = 1 \text{ and } N_{i+1}(a) = \sigma\big(N_i(a)\big)a,$$

which leads to

$$N_k(a) = \sigma^{k-1}(a)\sigma^{k-2}(a)\cdots\sigma(a)a \qquad (k \in \mathbb{N}).$$

We define $f(A)$ for $A \in M_n(K)$ similarly:

$$f(A) = \sum_{i=0}^{n} a_i N_i(A)$$

where σ has been extended to $M_n(K)$ in the natural way.

Let V be a vector space over K. A $\sigma-$pseudo-linear transformation of V is an additive map $T : V \to V$ such that

$$T(\alpha v) = \sigma(\alpha)T(v), \quad \alpha \in K.$$

We will use the abbreviation σ-PLT for a pseudo-linear transformation with respect to the automorphism σ. A vector $v \in V \setminus \{0\}$ is an eigenvector of the $\sigma - PLT\ T$ with the corresponding eigenvalue $\lambda \in K$ if and only if

$$T(v) = \lambda v.$$

An important feature of σ-PLT is the absence of a Cayley-Hamilton theorem. In addition to that, unlike the classical linear transformations of a finite dimensional vector space over a commutative field, a pseudo-linear transformation need not be algebraic.

If V is finite-dimensional and $e = [e_1, \dots, e_n]$ is a basis of V, let us write $T(e_i) = \sum_{j=1}^{n} a_{ij} e_j$, $a_{ij} \in K$ or, in the matrix notation $Te = Ae$, where $A = [a_{ij}] \in M_n(K)$. The matrix A will be denoted by $[T]_e$. The equality

$$[f(T)]_e = f([T]_e)$$

holds for any polynomial $f(t) \in K[t, \sigma]$ as well. If v is an eigenvector of the σ-PLT T with an eigenvalue $\lambda \in K$ then

$$\sigma(v_e)[T]_e = \lambda v_e$$

where v_e denotes coordinates of the vector v with respect to the basis e ([**6**]).

If T is an algebraic σ-PLT on V and $\mu_T \in K[t; \sigma]$ is its minimal polynomial than $\lambda \in K$ is an eigenvalue for T if and only if $t - \lambda$ divides on the right (left) the polynomial μ_T in $K[t; \sigma]$ (Proposition 4.5. [**6**]).

We will also use the notion of a Wedderburn polynomial. For $f \in K[t; \sigma]$, let

$$V(f) := \{a \in K \mid f(a) = 0\}.$$

A (monic) polynomial is said to be Wedderburn if $f = \mu_{V(f)}$ i.e. f is equal to the minimal polynomial of $V(f)$-set of its roots ([**5**]).

3. GENERAL RESULTS

Let K be a field, $\sigma \in \mathrm{Aut}\,(K)$ of order k, i.e. $\sigma \neq id_K$ and k is the least nonnegative integer such that $\sigma^k = id_K$. If T is σ-PLT on a vector space V over K then T^k is a linear transformation of V since it is additive and

$$T^k(\alpha v) = \sigma^k(\alpha) T^k(v) = \alpha T^k(v), \quad \alpha \in K.$$

Therefore, if V is a finite-dimensional vector space, there exist $m \in \mathbb{N}$, $a_0, \dots, a_m \in K$, $a_m \neq 0$, such that

$$a_m (T^k)^m + \cdots + a_1 T^k + a_0 I = 0,$$

which means that σ-PLT T is algebraic. We will denote its minimal polynomial by μ_T. This polynomial is invariant in $K[t; \sigma]$ and it is also the right factor of the polynomial $\varphi_{T^k}(t^k)$, where φ_{T^k} denotes the characteristic polynomial of T^k. What we want is to find relations between eigenvalues of the linear transformation T^k and σ-PLT T.

Theorem 1. *Let T be σ-PLT on a finite dimensional vector space V over a field K and $\sigma \in \mathrm{Aut}\,(K)$ of order k. An element $\lambda \in K$ is the eigenvalue of T if and only if $N_k(\lambda)$ is an eigenvalue of T^k.*

Proof. Let $v \in V \setminus \{0\}$ be such that $T(v) = \lambda v$. Then

$$\begin{aligned} T^k(v) &= T^{k-1}(\lambda v) = \sigma^{k-1}(\lambda)T^{k-1}(v) \\ &\vdots \\ &= \sigma^{k-1}(\lambda)\cdots\sigma(\lambda)\lambda v = N_k(\lambda)v. \end{aligned}$$

The polynomial $h(t) = t^k - N_k(\lambda)$ is a WEDDERBURN polynomial, since it is the minimal polynomial of the set

$$\Gamma = \{\sigma(c)\lambda c^{-1} \mid c \in K^*\}.$$

For any $c \in K^*$, we have

$$N_k(\sigma(c)\lambda c^{-1}) = \sigma^k(c)N_k(\lambda)c^{-1} = N_k(\lambda).$$

The above shows that h vanishes on Γ. Let $f(t) = \sum_{i=1}^{m} a_i t^i$ be the monic minimal polynomial of Γ. Then $m = \deg f \leqslant k$, and the constant term $a_0 \neq 0$. Let $d \in K^*$. For any $e \in \Gamma$, we have $0 = \sum_{i=0}^{m} a_i\sigma^i(d)N_i(e)d^{-1}$. Thus, Γ satisfies the polynomial $\sum_{i=0}^{m} a_i\sigma^i(d)t^i$. By the uniqueness of the minimal polynomial, we must have $\sigma^m(d)a_i = a_i\sigma^i(d)$ for every i. Since $a_0 \neq 0$, this implies that $\sigma^m = id_K$. Therefore, we have $m = k$ and $f(t) = t^k - N_k(\lambda)$.

We can write $t^k - N_k(\lambda) = (t-\lambda_k)(t-\lambda_{k-1})\cdots(t-\lambda_1)$ where $\lambda_1, \dots, \lambda_k$ are $\sigma-$conjugated to λ (Theorem 5.1. [**5**]). This gives us

$$T^k - N_k(\lambda)id_K = (T - \lambda_k id_K)(T - \lambda_{k-1}id_K)\cdots(T - \lambda_1 id_K).$$

Now it is easy to conclude that if there exists $0 \neq v \in V$ such that $(T^k - N_k(\lambda)id_K)(v) = 0$, then there exist $l \in \{1, \dots, k\}$ and $0 \neq u \in V$ such that $(T - \lambda_l id_K)(u) = 0$. Since λ_l is $\sigma-$conjugated to λ, there exists $a \in K^*$ such that $\lambda_l = \sigma(a)\lambda a^{-1}$. Then for $u_0 = a^{-1}u$ we obtain

$$T(u_0) = T(a^{-1}u) = \sigma(a^{-1})T(u) = \sigma(a^{-1})\sigma(a)\lambda a^{-1}u = \lambda u_0$$

i.e. λ is an eigenvalue for T, as desired. □

4. APPLICATIONS

Let K be a field, $\tau \in \mathrm{Aut}\,(K)$ of order k and $A \in M_n(K)$. What we want is to find all solutions of the matrix equation

$$XA - AX = \tau(X). \tag{4.1}$$

Instead of this equation we will consider the equivalent equation

$$\sigma(X)B - B\sigma(X) = X \tag{4.2}$$

where $\sigma = \tau^{-1}$ and $B = \tau^{-1}(A)$. This equation always has a solution, for any given B, namely $X = 0$. The mapping $T : M_n(K) \to M_n(K)$,

$$T(X) = \sigma(X)B - B\sigma(X)$$

is σ-PLT. Relative to the basis $e = [E_{ij}, 1 \leqslant i, j \leqslant n]$ of $M_n(K)$ T has the matrix:

$$\mathcal{B} = [T]_e = E \times B - B^T \times E$$

where $\times$ denotes KRONECKER product of the matrices. The matrix equation (4.2) has a nontrivial solution if and only if σ-PLT T has the eigenvalue $\lambda = 1$. By Theorem 1 this is equivalent to the fact that linear transformation T^k also has the eigenvalue $N_k(1) = 1$. Since $[T^k]_e = N_k(\mathcal{B})$, in order to find out if the equation (4.2) has nontrivial solutions or not we will examine if 1 is a zero of the characteristic polynomial φ_{T^k} of linear operator T^k or not.

We will assume in the majority of cases that $k \geqslant 2$. If $k = 1$ we obtain the linear matrix equation $XA - AX = X$ which is a special case of the SYLVESTER matrix equation $AX + XB = C$. Let $L : M_n(K) \to M_n(K)$, with $L(X) = AX + XB$ be the SYLVESTER operator. It is well known that when K is an algebraically closed field the linear operator L is singular if and only if A and $-B$ have a common eigenvalue. For $B = E - A$ we obtain the following result.

Proposition 2. *The matrix equation $XA - AX = X$ has a nonzero solution if and only if A and $A - E$ have a common eigenvalue.*

This proposition is equivalent to the fact that the matrix equation $XA - AX = X$ has nonzero solutions if and only if 1 is an eigenvalue of the matrix $E \times A - A^T \times E$. Since the eigenvalues of $C \times E + E \times D$ are all of the form $\lambda + \mu$ where λ and μ are eigenvalues of C and D respectively, 1 is the eigenvalue of $E \times A - A^T \times E$ if and only if $1 = \lambda - \mu$ for some eigenvalues λ and μ of A. This means that λ and $\lambda - 1$ are two different eigenvalues of A which is equivalent to the fact that A and $A - E$ have a common eigenvalue.

EXAMPLE 1. Let

$$A = \begin{bmatrix} -i+1 & 1 \\ -1 & i \end{bmatrix} \in M_2(\mathbb{C})$$

and $\sigma \in \operatorname{Aut}(\mathbb{C})$, $\tau(x) = \bar{x}$, the complex conjugation. We are looking for all nonzero solutions of the equation

$$XA - AX = \bar{X}, \tag{4.3}$$

or the equivalent equation

$$\bar{X}\bar{A} - \bar{A}\bar{X} = X. \tag{4.4}$$

In this case, τ is the automorphism of $\mathbb{C}$ of order $k = 2$. Therefore $\tau^{-1} = \tau$. First, for $B = \bar{A}$, we determine the matrix $P = E \times B - B^T \times E$,

$$P = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & -2i-1 & 0 & 1 \\ -1 & 0 & 2i+1 & 1 \\ 0 & -1 & -1 & 0 \end{bmatrix},$$

then the matrix

$$N_2(P) = \bar{P}P = \begin{bmatrix} -2 & -2i-1 & 2i+1 & 2 \\ -2i+1 & 3 & -2 & 2i-1 \\ 2i-1 & -2 & 3 & -2i+1 \\ 2 & 2i+1 & -2i-1 & -2 \end{bmatrix}.$$

The matrix $N_2(P)$ can be calculated using the following formula as well:

$$N_2(P) = E \times N_2(\bar{A}) - A^T \times \bar{A} - \bar{A}^T \times A + N_2(\bar{A}^T) \times E.$$

Next, we calculate the characteristic polynomial $\varphi_{N_2(P)}$ and check whether 1 is its root or not. In this case we have

$$\varphi_{N_2(P)}(t) = t^2(t-1)^2.$$

Since $\varphi_{N_2(P)}(1) = 0$, we can conclude that the our matrix equation has nonzero solutions.

In this case, we go one step further. We are going to determine all non zero solutions of the equation (4.3). Since $\mu_{N_2(P)}(t) = t(t-1)$,

$$M_2(\mathbb{C}) = \ker T^2 \oplus \ker(T^2 - id_K),$$

where $T : M_2(\mathbb{C}) \to M_2(\mathbb{C}), \quad T(X) = \bar{X}\bar{A} - \bar{A}\bar{X}$.

All solutions of the equation (4.3) belong to the set $U = \ker(T^2 - id_K)$ which has the basis $[C, D]$, where

$$C = \begin{bmatrix} -1 & -1-2i \\ 0 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

System $[D, T(D)]$ is one basis of U as well, since $T(D) \neq 0$. So, if $X \in M_2(\mathbb{C})$ satisfies (4.4), then $X = \alpha D + \beta T(D)$ for uniquely determined $\alpha, \beta \in \mathbb{C}$. From $T(X) = X$ it follows

$$\bar{\alpha}T(D) + \bar{\beta}D = \alpha D + \beta T(D),$$

which is valid for any $\alpha \in \mathbb{C}$ and $\beta = \bar{\alpha}$. Finally,

$$X = \alpha D + \bar{\alpha}T(D) = \alpha \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \bar{\alpha} \begin{bmatrix} -2 & -1-2i \\ 1+2i & 2 \end{bmatrix}$$

i.e.

$$X = \begin{bmatrix} -2\bar{\alpha} & \alpha - \bar{\alpha}(1+2i) \\ \alpha + \bar{\alpha}(1+2i) & 2\bar{\alpha} \end{bmatrix}, \alpha \in \mathbb{C}.$$

So, the set of solutions is

$$\left\{ \begin{bmatrix} -2\bar{\alpha} & \alpha - \bar{\alpha}(1+2i) \\ \alpha + \bar{\alpha}(1+2i) & 2\bar{\alpha} \end{bmatrix} \mid \alpha \in \mathbb{C} \right\}.$$

In general for

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{C})$$

the characteristic polynomial of the matrix $N_2(P)$ is

$$\varphi_{N_2(P)} = t^2 \left(t - (|a-d|^2 + 2(\bar{b}c + b\bar{c})\right)^2.$$

So, the equation $XA - AX = \bar{X}$ has a nontrivial solution if and only if

$$|a-d|^2 + 2(\bar{b}c + b\bar{c}) = 1.$$

Example 2. Let

$$A = J(n, \lambda) = \begin{bmatrix} \lambda & 1 & \cdots & 0 & 0 \\ 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & \cdots & 0 & \lambda \end{bmatrix} \in M_n(\mathbb{C})$$

and take $\tau \in \operatorname{Aut}(\mathbb{C})$, $\tau(x) = \bar{x}$ to be the complex conjugation. The equation

$$XA - AX = \bar{X},$$

has only the trivial solution $X = 0$, since in this case

$$\operatorname{rank}\big(N_2(P) - E\big) = n^2.$$

In the end, we state some basic properties of the solutions of (4.1).

1. If X is a solution then $\operatorname{tr} X = 0$.

2. If X is a solution then so is cX for any $c \in K_0$, where

$$K_0 = \{a \in K \mid \tau(a) = a\}$$

i.e. the set of all solutions is one K_0 vector subspace of $M_n(K)$.

3. *Let* $A, X \in M_n(K)$ *and* $A_1 = SAS^{-1}, X_1 = SXS^{-1}$, $S \in Gl_n(K_0)$. *Then*

$$XA - AX = \tau(X) \quad \Leftrightarrow \quad X_1A_1 - A_1X_1 = \tau(X_1).$$

Proof. The equation $\tau(X) = XA - AX$ is equivalent to

$$\begin{aligned}\tau(X_1) = \tau(SXS^{-1}) &= S\tau(X)S^{-1} = S(XA - AX)S^{-1} \\ &= (SXS^{-1})(SAS^{-1}) - (SAS^{-1})(SXS^{-1}) \\ &= X_1A_1 - A_1X_1.\end{aligned}$$ □

Having applied the previous property with $A_1 = SAS^{-1} = A$ where $S \in Gl_n(K_0)$ we obtain the following.

4. *If* X_0 *is a matrix solution of* $XA - AX = \tau(X)$ *then so is* $X = SX_0S^{-1}$, *for any* $S \in C(A) \cap Gl_n(K_0)$, *where* $C(A) = \{S \in M_n(K) \mid SA = AS\}$ *is the centralizer of* A. □

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