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ON DIFFERENTIAL EQUATIONS WITH NONSTANDARD COEFFICIENTS

Biljana Jolevska-Tuneska, Arpad Takači, Emin Özçağ̄

In the frame of Colombeau generalized functions we prove the existence and uniqueness of the solution of a system of linear differential equations with given initial data. The obtained result is applied to the Riccati equation.

1. INTRODUCTION

The classical distribution theory turns out to be insufficient for treating certain differential equations involving nonlinear operations and distributions. This kind of problems appear, e.g., in a rather simple dynamical system which describes the evolution of the population densities of predators and preys.

In the theory of distributions there are two complementary points of view:

(1) A distribution $f \in \mathcal{D}'(\mathbb{R}^n)$ is a continuous linear functional on the space $\mathcal{D}(\mathbb{R}^n)$ of compactly supported smooth functions with an appropriate convergence, see, [**8**]. Here we have a linear action

$$\varphi \to \langle f, \varphi \rangle$$

of f on a test function φ.

(2) If $\{\varphi_n\}$ is a sequence of smooth functions converging to the Dirac δ function, a family of regularizations $\{f_n\}$ can be produced by the convolution,

$$f_n(x) = f * \varphi_n = \langle f(y), \varphi_n(x-y) \rangle,$$

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since it converges weakly to the original distribution $f \in \mathcal{D}'(\mathbb{R}^n)$. Identifying two sequences $\{f_n\}$ and $\{\overline{f}_n\}$ if they have the same limit, we obtain a sequential representation of the space of distributions. Other authors use the equivalence classes of nets of regularization. The delta-net $\{\varphi_\epsilon\}_{\epsilon>0}$ is defined by $\varphi_\epsilon(x) = \epsilon^{-n}\varphi(x/\epsilon)$. This point of view is due to P. Antosik, J. Mikusiński and R. Sikorski, see [**1**]. But, when working with regularizations, the nonlinear structure is lost by identifying sequences (nets) with the same limit. Furthermore, in associative algebras of generalized functions multiplication and differentiation cannot simultaneously extend the corresponding classical operations unrestrictedly. One, therefore, has to reduce requirements on the multiplication.

The actual construction of differential algebras enjoying these optimal properties is due to J. F. Colombeau, see [**2**]. Colombeau theory of algebras of generalized functions offers the possibility of applying large classes of nonlinear operations to distributional objects. Colombeau algebra, denoted by $\mathcal{G}$, is an associative differential algebra with distributions linearly embedded in it. In Section 2, we give only few notions from Colombeau's theory, in fact those that will be used in the paper. Besides [**2**], on the theory of Colombeau algebras one can see also the monograph [**6**], and the papers [**5**], [**3**], [**4**] and [**7**].

In Section 3 we analyze the following system of differential equations

$$z_1'(t) = z_2(t),\ z_2'(t) = z_3(t), \dots, z_n'(t) = f(t)\, z_1(t), \tag{1}$$

with initial conditions

$$z_1(0) = z_{10},\ z_2(0) = z_{20}, \dots, z_n(0) = z_{n0}, \tag{2}$$

where f and z_i, $i = 1, 2, \dots, n$ are elements from the Colombeau algebra $\mathcal{G}(\mathbb{R})$, and z_{10}, z_{20}, $\dots$, z_{n0} are given elements from the Colombeau algebra $\overline{\mathcal{C}}$ of generalized complex numbers. Note that products like fz_1, appearing in (1), have sense when both f and z_1 are in $\mathcal{G}$.

We prove that under certain conditions the system (1) has a unique solution in $\mathcal{G}(\mathbb{R})$. Note that the system (1) corresponds to $z^{(n)} = f(t)z$ (with the corresponding initial conditions), where f is an arbitrary Colombeau generalized function with an appropriate condition given for a representative f_ε of f.

Next we turn to the case $n = 2$ and introduce a function $x(t)$ by $x(t) = z'(t)/z(t)$. It turns out that x is the solution of the Riccati differential equation

$$x'(t) + x^2(t) = f(x), \quad x(0) = x_0,$$

where x_0 is a Colombeau generalized number. In fact, using the relation between x_0 and the data $(z_1(0), z_2(0))$, we solve the system (1) and thus obtain the solution of the Ricati equation.

2. COLOMBEAU ALGEBRAS

We start with the basic notions of COLOMBEAU theory. The main idea of this theory in its simplest form is that of embedding the space of distributions into a factor algebra. We use the following spaces.

$$\begin{aligned}\mathcal{A}_0(\mathbb{R}^n) &= \left\{\varphi \in \mathcal{D}(\mathbb{R}^n) \;:\; \int \varphi(x)dx = 1\right\}, \text{ and}\\ \mathcal{A}_q(\mathbb{R}^n) &= \left\{\varphi \in \mathcal{A}_0(\mathbb{R}^n) \;:\; \int x^\alpha \varphi(x)dx = 0 \text{ for } 1 \le |\alpha| \le q\right\}.\end{aligned}$$

Let $\mathcal{E}(\mathbb{R}^n)$ be the algebra of functions

$$u(\varphi, x) : \mathcal{A}_0(\mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{C},$$

where $u(\varphi, x)$ is required to be infinitely differentiable by a fixed "parameter" φ. Thus $\mathcal{E}(\mathbb{R}^n)$ denotes an algebra of complex valued functions having appropriate smoothness properties on a suitable domain.

Now let $\varphi_\epsilon(x) = \epsilon^{-n}\varphi(x/\epsilon)$ for $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$. The sequence $(u * \varphi_\epsilon)_{\epsilon>0}$ converges to u in $\mathcal{D}'(\mathbb{R}^n)$. Taking this sequence as a representative of u, we obtain an embedding of $\mathcal{D}'(\mathbb{R}^n)$ into the algebra $\mathcal{E}(\mathbb{R}^n)$. However, embedding $\mathcal{C}^\infty(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$ into this algebra via convolution as above will not yield a subalgebra because, in general,

$$(f * \varphi_\epsilon)(g * \varphi_\epsilon) - (fg) * \varphi_\epsilon \neq 0. \tag{3}$$

The idea, therefore, is to find an ideal $\mathcal{N}(\mathbb{R}^n)$ such that the difference on the left-hand side of (3) vanishes in the resulting quotient. In fact, for the construction of $\mathcal{N}(\mathbb{R}^n)$ it is sufficient to find an ideal containing all the differences of the form $((f * \varphi_\epsilon) - f)_{\epsilon>0}$.

The TAYLOR expansion of $(f * \varphi_\epsilon) - f$ shows that this term will vanish faster that any power of ϵ, uniformly on compact sets, in all derivatives. The set of all such sequences is not an ideal in $\mathcal{E}(\mathbb{R}^n)$, so we consider the set of moderate sequences $\mathcal{E}_M(\mathbb{R}^n)$ whose every derivative is bounded uniformly on compact sets by some negative power of ϵ. Then the generalized functions of COLOMBEAU are elements of the quotient algebra

$$\mathcal{G} \equiv \mathcal{G}(\mathbb{R}^n) = \mathcal{E}_M(\mathbb{R}^n)/\mathcal{N}(\mathbb{R}^n).$$

Here the moderate functionals $\mathcal{E}_M(\mathbb{R}^n)$ are defined by the property:

$$\begin{aligned}&\forall K \subset\subset \mathbb{R}^n \ \forall \alpha \in \mathbb{N}_0^n \ \exists p \ge 0 \text{ such that } \forall \varphi \in \mathcal{A}_q(\mathbb{R}^n),\\ &\qquad \sup_{x\in K} |\partial^\alpha u(\varphi_\epsilon, x)| = \mathcal{O}(\epsilon^{-p}) \text{ as } \epsilon \to 0,\end{aligned}$$

and the null functionals $\mathbb{N}(\mathbb{R}^n)$ are defined by the property:

$$\begin{aligned}&\forall K \subset\subset \mathbb{R}^n \ \forall \alpha \in \mathbb{N}_0^n \ \exists p \ge 0 \text{ such that } \forall q \ge p \text{ and } \forall \varphi \in \mathcal{A}_q(\mathbb{R}^n),\\ &\qquad \sup_{x\in K} |\partial^\alpha u(\varphi_\epsilon, x)| = \mathcal{O}(\epsilon^{q-p}) \text{ as } \epsilon \to 0.\end{aligned}$$

In words, moderate functionals satisfy a locally uniform polynomial estimate as $\epsilon \to 0$ when acting on φ_ϵ, together with all derivatives, while null functionals vanish faster than any power of ϵ in the same situation. The null functionals form a differential ideal in the collection of moderate functionals.

We define the space of functions $u : \mathcal{A}_0 \to \mathcal{C}$ and denote it by $\mathcal{E}_0(\mathbb{R}^n)$. It is a subalgebra in $\mathcal{E}(\mathbb{R}^n)$. Moderate functionals, denoted by $\mathcal{E}_{0M}(\mathbb{R}^n)$, are defined by the property:

$$\exists p \geq 0, \text{ such that } \forall \varphi \in \mathcal{A}_p(\mathbb{R}^n), \quad |u\,(\varphi_\epsilon)\,| = \mathcal{O}(\epsilon^{-p}) \text{ as } \epsilon \to 0.$$

Null functionals, denoted by $\mathcal{N}_0(\mathbb{R}^n)$, are defined by the property:

$$\exists p \geq 0 \text{ such that } \forall q \geq p \text{ and } \forall \varphi \in \mathcal{A}_q(\mathbb{R}^n), \quad |u\,(\varphi_\epsilon)\,| = \mathcal{O}(\epsilon^{q-p}) \text{ as } \epsilon \to 0.$$

Now the space of generalized complex numbers and of generalized real numbers is respectively the factor algebra defined as

$$\overline{\mathbb{C}} = \mathcal{E}_{0M}(\mathbb{C})/\mathcal{N}_0(\mathbb{C}), \quad \overline{\mathbb{R}} = \mathcal{E}_{0M}(\mathbb{R})/\mathcal{N}_0(\mathbb{R}).$$

The algebra $\mathcal{G}$ contains the space of distributions $\mathcal{D}'(\mathbb{R}^n)$ on $\mathbb{R}^n$ embedded by the map

$$i \ : \ \mathcal{D}'(\mathbb{R}^n) \to \mathcal{G}(\mathbb{R}^n), \ i(u) = \tilde{u} = \text{class}\left[(u * \varphi)(x) : \varphi \in \mathcal{A}_q(\mathbb{R}^n)\right].$$

Equivalence classes of sequences $(u_\epsilon)_{\epsilon>0}$ in $\mathcal{G}(\mathbb{R}^n)$ will be denoted by

$$U = \text{class}\left[(u_\epsilon)_{\epsilon>0}\right].$$

Definition 1. *A generalized function $F \in \mathcal{G}(\mathbb{R}^n)$ is said to admit some $u \in \mathcal{D}'(\mathbb{R}^n)$ as “associated distribution”, denoted $F \approx u$, if for some representative $f(\varphi_\epsilon, x)$ of F and any $\psi(x) \in D(\mathbb{R}^n)$ there is a $q \in \mathbb{N}_0$ such that, for any $\varphi(x) \in \mathcal{A}_q(\mathbb{R}^n)$,*

$$\lim_{\epsilon \to 0} \int f(\varphi_\epsilon, x)\psi(x)\,\mathrm{d}x = \langle u, \psi \rangle.$$

This definition is independent of the representatives and the association is a faithful generalization of the equality of distributions.

Definition 2. *Let $F, G \in \mathcal{G}(\mathbb{R}^n)$. Then they are associated generalized functions, denoted $G \approx F$, if there exist representatives $g(\varphi_\epsilon, x)$ and $f(\varphi_\epsilon, x)$ of G and F respectively, and for any $\psi(x) \in D(\mathbb{R}^n)$ there is a $q \in \mathbb{N}_0$ such that, for any $\varphi(x) \in \mathcal{A}_q(\mathbb{R}^n)$*

$$\lim_{\epsilon \to 0} \int \big(g(\varphi_\epsilon, x) - f(\varphi_\epsilon, x)\big)\psi(x)\,\mathrm{d}x = 0\,.$$

3. COLOMBEAU ALGEBRA AND ORDINARY DIFFERENTIAL EQUATIONS

Now we will use the COLOMBEAU algebra for solving the system of ordinary differential equations from (1). with initial conditions (2). As noted in the introduction, f and $z_i,\ i=1,2,\dots,n$ are elements from COLOMBEAU algebra $\mathcal{G}(\mathbb{R})$, while $z_{10},\ z_{20},\dots,z_{n0}$ are given elements from COLOMBEAU algebra $\overline{\mathcal{C}}$ of generalized complex numbers.

The system (1) has its equivalent matrix form:

$$Z'(t)=A(t)Z(t)\,,\quad Z(0)=Z_0, \tag{4}$$

where $Z=(z_1,z_2,\dots,z_n)\,,\ Z_0=(z_{10},z_{20},\dots,z_{n0})$ and

$$A(t)=(A_{ij}(t))=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & & \dots & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ f(t) & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Now, let K be a compact set in $\mathbb{R}$, $A(t)=\text{class}\left[a\left(\varphi_\epsilon,t\right)_{\epsilon>0}\right]$ an element from $\mathcal{G}^{n\times n}(\mathbb{R})$, and $\|a\left(\varphi_\epsilon,t\right)\|_K=\sup_{t\in K}\|a\left(\varphi_\epsilon,t\right)\|$. We will prove the following theorem.

Theorem 1. *Let $A\in\mathcal{G}^{n\times n}(\mathbb{R})$. Assume that there exists a representative $a\left(\varphi_\epsilon,t\right)$ of A such that for every compact set K in $\mathbb{R}$, and for every $\varphi\in\mathcal{A}_q$ ($q\in\mathbb{N}$ large enough) there is a constant $c>0$ satisfying the following condition:*

$$\exp\left\{\int_0^t\|a(\varphi_\epsilon,x)\|_K\,\mathrm{d}x\right\}\le\frac{c}{\epsilon^q}.$$

Then the matrix differential equation $Z'(t)=A(t)Z(t)$, with the initial condition $Z_{|t=0}=Z_0=(z_{10},z_{20},\cdots,z_{n0})\in\overline{\mathcal{C}}^n$, has a unique solution $Z=(z_1,z_2,\dots,z_n)\in\mathcal{G}^n(\mathbb{R})$.

Proof. The proof consists of two parts. First, we will build up a solution that belongs to the COLOMBEAU algebra. Second, existence and uniqueness in $\mathcal{G}^n(\mathbb{R})$ will be proved by showing that every solution of the above equation is in the same class of the COLOMBEAU algebra.

We have that $A(t)=(A_{ij}(t))\,,\ i,j=1,2,\dots,n$, where $A_{ij}\in\mathcal{G}(\mathbb{R})$ for $i,j=1,2,\dots,n$. Now we are examining the problem

$$\begin{pmatrix} z_1\\ z_2\\ \vdots\\ z_n\end{pmatrix}'(t)=\begin{pmatrix} 0 & 1(\varphi_\epsilon,t) & 0 & \cdots & 0\\ 0 & 0 & 1(\varphi_\epsilon,t) & \cdots & 0\\ \vdots & \vdots & & \dots & \vdots\\ 0 & 0 & 0 & \cdots & 1(\varphi_\epsilon,t)\\ f(\varphi_\epsilon,t) & 0 & 0 & \cdots & 0\end{pmatrix}\begin{pmatrix} z_1\\ z_2\\ \vdots\\ z_n\end{pmatrix}(t)\,.$$

Integrating with respect to t and inserting initial conditions, we get an equivalent system of integral equations:

$$\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}(t) = \begin{pmatrix} z_{10} \\ z_{20} \\ \vdots \\ z_{n0} \end{pmatrix} + \int\limits_0^t \begin{pmatrix} 0 & 1(\varphi_\epsilon,\tau) & 0 & \cdots & 0 \\ 0 & 0 & 1(\varphi_\epsilon,\tau) & \cdots & 0 \\ \vdots & \vdots & & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1(\varphi_\epsilon,\tau) \\ f(\varphi_\epsilon,\tau) & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} z_1(\tau) \\ z_2(\tau) \\ \vdots \\ z_n(\tau) \end{pmatrix} \mathrm{d}\tau.$$

Let $Z = (z_1, z_2, \ldots, z_n)$ and $\|Z(\varphi_\epsilon, t)\|_K = \max_{j=1,2,\ldots,n} \sup_{t\in K} |z_j(\varphi_\epsilon, t)|$. Using Gronwall's inequality, we have

$$\|Z(\varphi_\epsilon,t)\|_K \le \|Z_0(\varphi_\epsilon,t_0)\| \exp\left\{ \int\limits_0^t \|a(\varphi_\epsilon,x)\|_K \,\mathrm{d}x \right\} \le \frac{c*}{\epsilon^q},$$

for $\epsilon > 0$ and $\varphi \in \mathcal{A}_q$. Also, there is a sufficiently large $q_r \in \mathbb{N}$, such that for $\varphi \in \mathcal{A}_{q_r}$ and $\epsilon > 0$ we have

$$\|\partial^r \left(Z(\varphi_\epsilon,t)\right)\|_K \le \frac{c_r}{\epsilon^{q_r}}.$$

The last relation implies that $Z(\varphi_\epsilon, t) \in \mathcal{E}^n_M(\mathbb{R})$, so we have the solution Z of (4) in $\mathcal{G}^n$ and that solution is the class in $\mathcal{G}^n$ containing the element $Z(\varphi_\epsilon, t)$.

Concerning uniqueness, let us suppose that W =class$[w(\varphi_\epsilon, t)]$ is another solution of (4) in $\mathcal{G}^n(\mathbb{R})$ different from Z. This means that for a representative $w(\varphi_\epsilon, t)$ of W we have $w'(\varphi_\epsilon,t) = a(\varphi_\epsilon,t)w(\varphi_\epsilon,t) + n(\varphi_\epsilon,t)$ where $n(\varphi_\epsilon, t)$ belongs in $\mathcal{N}^n(\mathbb{R})$. Similarly as before, now we have:

$$\begin{aligned} \|Z(\varphi_\epsilon,t) - W(\varphi_\epsilon,t)\|_K &\le \|Z(\varphi_\epsilon,t_0) - W(\varphi_\epsilon,t_0)\| \\ &\quad + \|n(\varphi_\epsilon,t)\|_K \exp\left\{ \int\limits_0^t \|a(\varphi_\epsilon,x)\|_K \,\mathrm{d}x \right\} \le O(\epsilon^q) \end{aligned}$$

and we get that $Z = W$ in $\mathcal{G}^n(\mathbb{R})$, which completes the proof of the theorem.

Now we will show that the system (1) is equivalent to a nonlinear differential equation. In fact, differentiating the first equation (1), we obtain $z_1''(t) = z_2'(t) = z_3(t)$. Repeating this method we finally obtain

$$z_1^{(n)}(t) = z_n'(t) = f(t)z_1(t).$$

Now if z_1 is a solution of the equation

$$z_1'(t) = x(t)z_1(t), \tag{5}$$

where x is an element from Colombeau algebra, then using the Leibniz formula, we have

$$z_1^{(n)}(t) = f(t)z_1(t)\,, \quad (x(t)z_1(t))^{(n-1)} = f(t)z_1(t)\,,$$
$$\sum_{k_1=0}^{n-1}\binom{n-1}{k_1}x^{(n-k_1-1)}(t)z_1^{(k_1)}(t) = f(t)z_1(t)\,, \quad \text{and}$$
$$\sum_{k_1=0}^{n-1}\binom{n-1}{k_1}x^{(n-k_1-1)}(t)\left[\sum_{k_2=0}^{k_1-1}\binom{k_1-1}{k_2}x^{(k_1-k_2-1)}(t)\left(\sum\dots z_1^{(k_n)}(t)\right)\right] = f(t)z_1(t).$$

From the fact that $0 \le k_1 \le n-1,\ 0 \le k_2 \le n-2, \dots, 0 \le k_{n-1} \le 1$, we have that $k_n = 0$, i. e.,

$$\sum_{k_1=0}^{n-1}\binom{n-1}{k_1}x^{(n-k_1-1)}(t)\left[\sum_{k_2=0}^{k_1-1}\binom{k_1-1}{k_2}x^{(k_1-k_2-1)}(t)\cdots z_1(t)\right] = f(t)z_1(t),$$

which is a nonlinear differential equation of the form:

$$x^{(n-1)}(t) + nx(t)x^{(n-2)}(t) + \dots + x^n(t) = f(t). \tag{6}$$

Putting $n = 2$ in (6) we get the Riccati nonlinear differential equation

$$x'(t) + x^2(t) = f(t), \tag{7}$$

where f and x are elements from Colombeau algebra $\mathcal{G}(\mathbb{R})$. Let, moreover

$$x(0) = x_0, \tag{8}$$

where x_0 is a known element from $\overline{\mathbb{C}}$. In view of the previous analysis, the solution of the problem (8) is equivalent to the solution of the following system of differential equations:

$$z_1'(t) = z_2(t), \qquad z_2'(t) = f(t)\cdot z_1(t),$$

where z_1 is solution of the equation (5), satisfying the conditions $z_1(0) = 1$ and $z_1'(0) = z_2(0) = x_0$.

The equivalent matrix form of this system is

$$Z'(t) = A(t)Z(t), \quad Z(0) = Z_0\,,$$

where $Z = (z_1, z_2)$, $Z_0 = (z_1(0), z_2(0)) = (1, x_0)$, and $A(t) = \begin{pmatrix} 0 & 1 \\ f(t) & 0 \end{pmatrix}$.

As a result we have the following corollary:

Corolary 1. *Let $A \in \mathcal{G}^{2\times 2}(\mathbb{R})$. Assume that there exists a representative $a(\varphi_\epsilon, t)$ of A exists such that for every K compact set in $\mathbb{R}$, and for every $\varphi \in \mathcal{A}_q$ ($q \in \mathbb{N}$ large enough) there is a constant $c > 0$ satisfying the following condition:*

$$\exp\left\{\int_0^t \|a(\varphi_\epsilon, x)\|_K \,\mathrm{d}x\right\} \le \frac{c}{\epsilon^q}.$$

Then the Ricati differential equation (7) *with the initial condition* (8) *has a unique solution in* $\mathcal{G}(\mathbb{R})$.

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B. Jolevska-Tuneska
Faculty of Electrical Engineering, Karpos II bb,
Skopje, Republic of Macedonia
E–mail: biljanaj@etf.ukim.edu.mk

A. Takači
Department of Mathematics and Informatics,
Faculty of Science, University of Novi Sad
Trg Dositeja Obradovića 4,
21000 Novi Sad, Serbia
E–mail: takacsi@eunet.yu

E. Özçağ
Department of Mathematics, University of Hacettepe,
06532 Beytepe, Ankara, Turkey
E–mail: ozcag1@hacettepe.edu.tr