

SPECIAL FUNCTIONS: APPROXIMATIONS AND BOUNDS

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The STEFFENSEN inequality and bounds for the ČEBYŠEV functional are utilised to obtain bounds for some classical special functions. The technique relies on determining bounds on integrals of products of functions. The above techniques are used to obtain novel and useful bounds for the BESSEL function of the first kind, the Beta function, and the Zeta function.

1. INTRODUCTION AND REVIEW OF SOME RECENT RESULTS

There are a number of results that provide bounds for integrals of products of functions. The main techniques that shall be employed in the current article involve the STEFFENSEN inequality and a variety of bounds related to the ČEBYŠEV functional. There have been some developments in both of these in the recent past with which the current author has been involved. These have been put to fruitful use in a variety of areas of applied mathematics including quadrature rules, in the approximation of integral transforms, as well as in applied probability problems (see [31], [22] and [11]). This article is a review of these developments and some new results are also presented.

It is intended that in the current article the techniques will be utilised to obtain useful bounds for special functions. The methodologies will be demonstrated through obtaining bounds for the BESSEL function of the first kind, the Beta function and the Zeta function.

It is instructive to introduce some techniques for approximating and bounding integrals of the product of functions. We first present inequalities due to STEFFENSEN and then review bounds for the ČEBYŠEV functional.

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The following theorem is due to sc Steffensen [45] (see also [11] and [16]).

Theorem 1. Let $h : [a, b] \rightarrow \mathbb{R}$ be a nonincreasing mapping on $[a, b]$ and $g : [a, b] \rightarrow \mathbb{R}$ be an integrable mapping on $[a, b]$ with

$$-\infty < \phi \leq g(t) \leq \Phi < \infty \quad \text{for all } x \in [a, b],$$

then

$$(1.1) \quad \phi \int_a^{b-\lambda} h(x) \, dx + \Phi \int_{b-\lambda}^b h(x) \, dx \leq \int_a^b h(x) g(x) \, dx \\ \leq \Phi \int_a^{a+\lambda} h(x) \, dx + \phi \int_{a+\lambda}^b h(x) \, dx,$$

where

$$(1.2) \quad \lambda = \int_a^b G(x) \, dx, \quad G(x) = \frac{g(x) - \phi}{\Phi - \phi}, \quad \Phi \neq \phi.$$

REMARK 1. We note that the result (1.1) may be rearranged to give STEFFENSEN'S better known result that

$$\int_{b-\lambda}^b h(x) \, dx \leq \int_a^b h(x) G(x) \, dx \leq \int_a^{a+\lambda} h(x) \, dx,$$

where λ is as given by (1.2) and $0 \leq G(x) \leq 1$.

Equation (1.3) has a very pleasant interpretation, as observed by STEFFENSEN, that if we divide by λ then

$$(1.4) \quad \frac{1}{\lambda} \int_{b-\lambda}^b h(x) \, dx \leq \frac{\int_a^b G(x) h(x) \, dx}{\int_a^b G(x) \, dx} \leq \frac{1}{\lambda} \int_a^{a+\lambda} h(x) \, dx.$$

Thus, the weighted integral mean of $h(x)$ is bounded by the integral means over the end intervals of length λ , the total weight.

Now, for two measurable functions $f, g : [a, b] \rightarrow \mathbb{R}$, define the functional, which is known in the literature as Čebyšev's functional, by

$$(1.5) \quad T(f, g) := \mathcal{M}(fg) - \mathcal{M}(f)\mathcal{M}(g),$$

where the integral mean is given by

$$(1.6) \quad \mathcal{M}(f) := \frac{1}{b-a} \int_a^b f(x) \, dx.$$

The integrals in (1.5) are assumed to exist.

The weighted ČEBYŠEV functional is defined by

$$(1.7) \quad T(f, g; p) := \mathcal{M}(fg; p) - \mathcal{M}(f; p) \mathcal{M}(g; p),$$

where the weighted integral mean $\mathcal{M}(f; p)$ is given by

$$(1.8) \quad P \cdot \mathcal{M}(f; p) = \int_a^b p(x) f(x) dx, \quad P = \int_a^b p(x) dx$$

with the weight P satisfying $0 < P < \infty$.

We note that

$$T(f, g; 1) \equiv T(f, g) \quad \text{and} \quad \mathcal{M}(f; 1) \equiv \mathcal{M}(f).$$

We further note that bounds for (1.5) and (1.7) may be looked upon as approximating the integral mean of the product of functions in terms of the product of integral means which are more easily calculated explicitly. Bounds are perhaps best procured from identities. It is worthwhile noting that a number of identities relating to the ČEBYŠEV functional already exist. (The reader is referred to [40] Chapters IX and X.) KORKINE's identity is well known, see [40, p. 296] and is given by

$$(1.9) \quad T(f, g) = \frac{1}{2(b-a)^2} \int_a^b \int_a^b (f(x) - f(y))(g(x) - g(y)) dx dy.$$

It is identity (1.9) that is often used to prove an inequality due to GRÜSS for functions bounded above and below, [40].

The GRÜSS inequality [35] is given by

$$(1.10) \quad |T(f, g)| \leq \frac{1}{4} (\Phi_f - \phi_f) (\Phi_g - \phi_g),$$

where $\phi_f \leq f(x) \leq \Phi_f$ for $x \in [a, b]$, with ϕ_f, Φ_f constants and similarly for $g(x)$.

The interested reader is also referred to DRAGOMIR [30] and FINK [34] for extensive treatments of the GRÜSS and related inequalities.

Identity (1.9) may also be used to prove the ČEBYŠEV inequality which states that for $f(\cdot)$ and $g(\cdot)$ synchronous, namely $(f(x) - f(y))(g(x) - g(y)) \geq 0$, a.e. $x, y \in [a, b]$, then

$$(1.11) \quad T(f, g) \geq 0.$$

As mentioned earlier, there are many identities involving the ČEBYŠEV functional (1.5) or more generally (1.7). Recently, CERONE [11] obtained, for $f, g : [a, b] \rightarrow \mathbb{R}$ where f is of bounded variation and g continuous on $[a, b]$, the identity

$$(1.12) \quad T(f, g) = \frac{1}{(b-a)^2} \int_a^b \psi(t) df(t),$$

where

$$(1.13) \quad \psi(t) = (t-a)G(t,b) - (b-t)G(a,t)$$

with

$$(1.14) \quad G(c,d) = \int_c^d g(x) dx.$$

The following theorem was proved in [11].

Theorem 2. *Let $f, g : [a, b] \rightarrow \mathbb{R}$, where f is of bounded variation and g is continuous on $[a, b]$. Then*

$$(1.15) \quad (b-a)^2 |T(f, g)| \leq \begin{cases} \sup_{t \in [a, b]} |\psi(t)| \bigvee_a^b(f), \\ L \int_a^b |\psi(t)| dt, & \text{for } f \text{ } L\text{-Lipschitzian,} \\ \int_a^b |\psi(t)| df(t), & \text{for } f \text{ monotonic nondecreasing,} \end{cases}$$

where $\bigvee_a^b(f)$ is the total variation of f on $[a, b]$.

The bounds for the ČEBYŠEV functional were utilised to procure approximations to moments and moment generating functions in [11] and [24].

The reader is referred to [31] and the references therein for applications to numerical quadrature of trapezoidal and OSTROWSKI functionals, which were shown to be related to the ČEBYŠEV functional in [15].

For other GRÜSS type inequalities, see the books [9] and [40], and the papers [19], [23], [26], [29], [30], where further references are given.

Recently, CERONE and DRAGOMIR [19]–[23] have pointed out generalisations of the above results for integrals defined on two different intervals and more generally in a measurable space setting (see also, [8] and [14]).

The functional $T(f, g; p)$ defined in (1.7) satisfies a number of identities including that due to SONIN [42]

$$(1.16) \quad P \cdot |T(f, g; p)| = \left| \int_a^b p(x) (f(x) - \gamma)(g(x) - \mathcal{M}(g; p)) dx \right|$$

from which the following bounds may be procured. Namely,

$$(1.17) \quad P \cdot |T(f, g; p)| \leq \begin{cases} \inf_{\gamma \in \mathbb{R}} \|f(\cdot) - \gamma\| \int_a^b p(x) |g(x) - \mathcal{M}(g; p)| dx, \\ \left(\int_a^b p(x) (f(x) - \mathcal{M}(f; p))^2 dx \right)^{1/2} \\ \quad \times \left(\int_a^b p(x) (g(x) - \mathcal{M}(g; p))^2 dx \right)^{1/2}, \end{cases}$$

where

$$(1.18) \quad \int_a^b p(x) (h(x) - \mathcal{M}(h; p))^2 dx = \int_a^b p(x) h^2(x) dx - P \cdot \mathcal{M}^2(h; p)$$

and P is as defined in (1.8). Further, it may be easily shown by direct calculation that,

$$(1.19) \quad \inf_{\gamma \in \mathbb{R}} \left[\int_a^b p(x) (f(x) - \gamma)^2 dx \right] = \int_a^b p(x) (f(x) - \mathcal{M}(f; p))^2 dx.$$

Some of the above results are used to find bounds for the BESSEL function (Section 2), the Beta function (Section 3), the Zeta function (Section 4) (see also [9] for further details).

2. BOUNDING THE BESSEL FUNCTION

In this section we investigate techniques for determining bounds on the BESSEL function of the first kind (see also [12], [13]).

In ABRAMOWITZ and STEGUN [1] equation (9.1.21) defines the BESSEL of the first kind

$$(2.1) \quad J_\nu(z) = \gamma_\nu(z) \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} \cos(zt) dt, \quad \operatorname{Re}(\nu) > -\frac{1}{2},$$

where

$$(2.2) \quad \gamma_\nu(z) = \frac{2 \left(\frac{z}{2}\right)^\nu}{\sqrt{\pi} \Gamma\left(\nu + \frac{1}{2}\right)}.$$

For the current work the interest is in both z and ν real.

Theorem 3. *For z real then*

$$(2.3) \quad \begin{aligned} & \frac{1}{2} B\left(\frac{1}{2}, \nu + \frac{1}{2}\right) - B\left(\frac{1}{2}, \nu + \frac{1}{2}; (1-\lambda)^2\right) \\ & \leq \frac{J_\nu(z)}{\gamma_\nu(z)} \\ & \leq B\left(\frac{1}{2}, \nu + \frac{1}{2}; \lambda^2\right) - \frac{1}{2} B\left(\frac{1}{2}, \nu + \frac{1}{2}\right), \quad \nu > \frac{1}{2} \end{aligned}$$

and

$$(2.4) \quad B\left(\frac{1}{2}, \nu + \frac{1}{2}; \lambda^2\right) - \frac{1}{2} B\left(\frac{1}{2}, \nu + \frac{1}{2}\right)$$

$$\begin{aligned} &\leq \frac{J_\nu(z)}{\gamma_\nu(z)} \\ &\leq \frac{1}{2}B\left(\frac{1}{2}, \nu + \frac{1}{2}\right) - B\left(\frac{1}{2}, \nu + \frac{1}{2}; (1-\lambda)^2\right), \quad -\frac{1}{2} < \nu < \frac{1}{2}, \end{aligned}$$

where

$$(2.5) \quad B(\alpha, \beta; x) = \int_0^x u^{\alpha-1} (1-u)^{\beta-1} du, \quad \text{the incomplete Beta function,}$$

$$(2.6) \quad B(\alpha, \beta) = B(\alpha, \beta; 1) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad \text{the Beta function,}$$

and

$$(2.7) \quad 2\lambda - 1 = \frac{\sin z}{z}.$$

Taking $\nu = \frac{1}{2}$ produces equality in (2.3) and (2.4), namely, $J_{\frac{1}{2}}(z) = \gamma_{\frac{1}{2}}(z) \frac{\sin z}{z}$.

Proof. Consider the case $\nu > \frac{1}{2}$ then $h(t) = (1-t^2)^{\nu-\frac{1}{2}}$ is nonincreasing for $t \in [0, 1]$. Further, taking $g(t) = \cos zt$ we have that $-1 \leq g(t) \leq 1$ for $t \in [0, 1]$ and, from (1.2)

$$\lambda = \frac{1}{2} \int_0^1 (\cos zt + 1) dt = \frac{1}{2} \left(1 + \frac{\sin z}{z}\right).$$

Utilising Theorem 1 and after some algebra, the above results are procured. \square

REMARK. We note from (2.1) that we may obtain a classical bound (see [1, p. 362]) for $J_\nu(z)$, namely

$$|J_\nu(z)| \leq \frac{2 \left(\frac{|z|}{2}\right)^\nu}{\sqrt{\pi}\Gamma\left(\nu + \frac{1}{2}\right)} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} dt,$$

where from (2.5) and (2.6)

$$(2.8) \quad \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} dt = \frac{1}{2}B\left(\frac{1}{2}, \nu + \frac{1}{2}\right) = \frac{1}{2} \cdot \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\nu + \frac{1}{2}\right)}{\Gamma(\nu+1)}$$

to give

$$(2.9) \quad |J_\nu(z)| \leq \left|\frac{z}{2}\right|^\nu \frac{1}{\Gamma(\nu+1)}.$$

The following theorem gives a bound on the deviation of the BESSEL function from an approximant (see also [17]). This is accomplished via bounds on the ČEBYŠEV functional for which there are numerous results.

Theorem 4. *The following result holds for the Bessel function of the first kind $J_\nu(z)$. Namely,*

$$(2.10) \quad \left| J_\nu(z) - \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)} \cdot \frac{\sin z}{z} \right| \leq \left(\frac{|z|}{2}\right)^\nu \left[\frac{2}{\sqrt{\pi}} \cdot \frac{\Gamma(2\nu)}{\Gamma^2\left(\nu + \frac{1}{2}\right)\Gamma\left(2\nu + \frac{1}{2}\right)} - \frac{1}{\Gamma^2(\nu+1)} \right]^{1/2} \times \left[\left(\frac{\cos z}{4}\right)^2 + \frac{1}{2} - \left(\frac{\sin z}{z} - \frac{\cos z}{4}\right)^2 \right]^{1/2}.$$

Proof. (Sketch) We use the 2–norm result for the ČEBYŠEV functional. From (2.1) and (2.2) consider,

$$(2.11) \quad Q_\nu(z) = \frac{J_\nu(z)}{\gamma_\nu(z)} = \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} \cos(z t) dt.$$

Let $f(t) = (1-t^2)^{\nu-\frac{1}{2}}$ and $g(t) = \cos z t$. □

3. BOUNDING THE BETA FUNCTION

The incomplete beta function is defined by

$$(3.1) \quad B(x, y; z) = \int_0^z t^{x-1} (1-t)^{y-1} dt, \quad 0 < z \leq 1.$$

We shall restrict our attention to $x > 1$ and $y > 1$.

In this region we observe that

$$(3.2) \quad 0 \leq t^{x-1} \leq z^{x-1} \quad \text{and} \quad (1-z)^{y-1} \leq (1-t)^{y-1} \leq 1$$

with t^{x-1} , an increasing function and $(1-t)^{y-1}$, a decreasing function, for $t \in [0, z]$.

The following theorem follows from utilizing STEFFENSEN's result as depicted in Theorem 1 [12], see also [17] for details.

Theorem 5. *For $x > 1$ and $y > 1$ with $0 \leq z \leq 1$ we have the incomplete Beta function defined by (3.1) satisfying the following bounds*

$$(3.3) \quad \max\{L_1(z), L_2(z)\} \leq B(x, y; z) \leq \min\{U_1(z), U_2(z)\},$$

where

$$(3.4) \quad L_1(z) = \frac{z^{x-1}}{y} \left[\left(1 - z + \frac{z}{x}\right)^y - (1-z)^y \right], \quad U_1(z) = \frac{z^{x-1}}{y} \left[1 - \left(1 - \frac{z}{x}\right)^y \right]$$

and

$$(3.5) \quad L_2(z) = \frac{\lambda_2^x(z)}{x} + (1-z)^{y-1} \frac{z^x - \lambda_2^x(z)}{x},$$

$$U_2(z) = (1-z)^{y-1} \frac{(x - \lambda_2(z))^x}{x} + \frac{z^x - (z - \lambda_2(z))^x}{x}$$

with

$$(3.6) \quad \lambda_2(z) = \frac{1 - (1-z)[1 - z(1-y)]}{y[1 - (1-z)^{y-1}]}.$$

Proof. (Using STEFFENSEN's inequality) If we take $h(t) = (1-t)^{y-1}$ and $g(t) = t^{x-1}$, then for $y > 1$ and $x > 1$, $h(t)$ is a decreasing function of t and $0 \leq g(t) \leq z^{x-1}$. Thus, from (1.1)

$$(3.7) \quad z^{x-1} \int_{z-\lambda_1}^z (1-t)^{y-1} dt \leq \int_0^z t^{x-1} (1-t)^{y-1} dt \leq z^{x-1} \int_0^{\lambda_1} (1-t)^{y-1} dt,$$

where

$$\lambda_1 = \lambda_1(z) = \int_0^z \frac{t^{x-1}}{z^{x-1}} dt = \frac{z}{x}. \quad \square$$

Corollary 1. For $x > 1$ and $y > 1$ we have the Beta function

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt,$$

which is symmetric in x and y , satisfies the following bounds,

$$(3.8) \quad \max \left\{ \frac{1}{xy^x}, \frac{1}{yx^y} \right\} \leq B(x, y) \leq \min \left\{ \frac{1}{y} \left[1 - \left(1 - \frac{1}{x}\right)^y \right], \frac{1}{x} \left[1 - \left(1 - \frac{1}{y}\right)^x \right] \right\}.$$

Proof. Put $z = 1$ in (3.6) to give $\lambda_2(1) = \frac{1}{y}$ followed by the obvious correspondences from (3.3)–(3.5). \square

The following theorem relates to the Beta function [17] and is a correction of the result in [12].

Theorem 6. For $x > 1$ and $y > 1$ the following bounds hold for the Beta function, namely,

$$(3.9) \quad 0 \leq \frac{1}{xy} - B(x, y) \leq 2 \min \{A(x), A(y)\},$$

where

$$(3.10) \quad A(x) = \frac{x-1}{x\left(1+\frac{x}{x-1}\right)}.$$

Proof. (Sketch. Using the ČEBYŠEV functional and SONIN identity). We have from (1.16)–(1.17) with $p(\cdot) \equiv 1$,

$$0 \leq |T(f, g)| = |\mathcal{M}(fg) - \mathcal{M}(f)\mathcal{M}(g)| \leq \mathcal{M}(|f(\cdot) - \gamma||g(\cdot) - \mathcal{M}(g)|).$$

That is,

$$(3.11) \quad |T(f, g)| \leq \inf_{\gamma} \|f(\cdot) - \gamma\|_{\infty} \mathcal{M}|g(\cdot) - \mathcal{M}(g)|.$$

If we take $f(t) = t^{x-1}$, $g(t) = (1-t)^{y-1}$ then $\mathcal{M}(f) = \frac{1}{x}$ and $\mathcal{M}(g) = \frac{1}{y}$. \square

The following pleasing result is valid ([12], [17]).

Theorem 7. For $x > 1$ and $y > 1$ we have

$$(3.12) \quad 0 \leq \frac{1}{xy} - B(x, y) \leq \frac{x-1}{x\sqrt{2x-1}} \cdot \frac{y-1}{y\sqrt{2y-1}} \leq 0.090169437\dots,$$

where the upper bound is obtained at $x = y = \frac{3+\sqrt{5}}{2} = 2.618033988\dots$

Proof. (Using the 2-norm bound for the ČEBYŠEV functional) We have from (1.17)–(1.19)

$$(b-a)|T(f, g)| \leq \left(\int_a^b f^2(t) dt - \mathcal{M}^2(f) \right)^{1/2} \times \left(\int_a^b g^2(t) dt - \mathcal{M}^2(g) \right)^{1/2}.$$

That is, taking $f(t) = t^{x-1}$, $g(t) = (1-t)^{y-1}$.

Now, consider

$$(3.13) \quad C(x) = \frac{x-1}{x\sqrt{2x-1}}.$$

The maximum occurs when $x = x^* = \frac{3+\sqrt{5}}{2}$ to give $C(x^*) = 0.3002831\dots$. Hence, because of the symmetry we have the upper bound as stated in (3.12). \square

REMARK 3. In a recent paper ALZER [4] shows that

$$0 \leq \frac{1}{xy} - B(x, y) \leq b_A = \max_{x \geq 1} \left(\frac{1}{x^2} - \frac{\Gamma^2(x)}{\Gamma(2x)} \right) = 0.08731 \dots,$$

where 0 and b_A are shown to be the best constants. This uniform bound of ALZER is only smaller for a small area around $\left(\frac{3 + \sqrt{5}}{2}, \frac{3 + \sqrt{5}}{2} \right)$ while the first upper bound in (3.12) provides a better bound over a much larger region of the $x - y$ plane.

We may state the following corollary given the results above.

Corollary 2. For $x > 1$ and $y > 1$ we have

$$0 \leq \frac{1}{xy} - B(x, y) \leq \min \{C(x)C(y), b_A\},$$

where $C(x)$ is defined by (3.13) and b_A by (3.14).

REMARK 4. The upper bound in Theorem 6 by numerical investigation, seems not to be as good as that given in Theorem 7. Analytically, the transformation $\chi = \frac{x-1}{x}$ and $\eta = \frac{y-1}{y}$ in (3.9)–(3.12) results in requiring to show that

$$H(\chi, \eta) = 2(1 - \chi)^{\frac{1}{\chi}} - \eta \sqrt{\frac{1 - \chi}{1 + \chi} \cdot \frac{1 - \eta}{1 + \eta}} \geq 0$$

for $0 \leq \chi, \eta \leq 1$.

4. BOUNDS FOR THE EULER ZETA AND RELATED FUNCTIONS

4.1. BACKGROUND TO ZETA AND RELATED FUNCTIONS

The Zeta function ([10])

$$(4.1) \quad \zeta(x) := \sum_{n=1}^{\infty} \frac{1}{n^x}, \quad x > 1$$

was originally introduced in 1737 by the Swiss mathematician LEONHARD EULER (1707-1783) for real x who proved the identity

$$(4.2) \quad \zeta(x) := \prod_p \left(1 - \frac{1}{p^x} \right)^{-1}, \quad x > 1,$$

where p runs through all primes. It was RIEMANN who allowed x to be a complex variable z and showed that even though both sides of (4.1) and (4.2) diverge for $\operatorname{Re}(z) \leq 1$, the function has a continuation to the whole complex plane with a simple pole at $z = 1$ with residue 1. The function plays a very significant role

in the theory of the distribution of primes (see [5], [7], [27], [32], [37] and [46]). One of the most striking properties of the zeta function, discovered by RIEMANN himself, is the functional equation

$$(4.3) \quad \zeta(z) = 2^z \pi^{z-1} \sin\left(\frac{\pi z}{2}\right) \Gamma(1-z) \zeta(1-z)$$

that can be written in symmetric form to give

$$(4.4) \quad \pi^{-\frac{z}{2}} \Gamma\left(\frac{z}{2}\right) \zeta(z) = \pi^{-\left(\frac{1-z}{2}\right)} \Gamma\left(\frac{1-z}{2}\right) \zeta(1-z).$$

$\zeta(s)$ is commonly referred to as the RIEMANN Zeta function and if s is restricted to a real variable x , it is referred to as the EULER Zeta function.

In addition to the relation (4.3) between the zeta and the gamma function, these functions are also connected via the integrals [32]

$$(4.5) \quad \zeta(x) = \frac{1}{\Gamma(x)} \int_0^{\infty} \frac{t^{x-1}}{e^t - 1} dt, \quad x > 1,$$

and

$$(4.6) \quad \zeta(x) = \frac{1}{C(x)} \int_0^{\infty} \frac{t^{x-1}}{e^t + 1} dt, \quad x > 0,$$

where

$$(4.7) \quad C(x) := \Gamma(x) (1 - 2^{1-x}) \quad \text{and} \quad \Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt.$$

In the series expansion

$$(4.8) \quad \frac{te^{xt}}{e^t - 1} = \sum_{m=0}^{\infty} B_m(x) \frac{t^m}{m!},$$

where $B_m(x)$ are the BERNOULLI polynomials (after JACOB BERNOULLI), $B_m(0) = B_m$ are the BERNOULLI numbers. They occurred for the first time in the formula [1, p. 804]

$$(4.9) \quad \sum_{k=1}^m k^n = \frac{B_{n+1}(m+1) - B_{n+1}}{n+1}, \quad n, m = 1, 2, 3, \dots$$

One of EULER's most celebrated theorems discovered in 1736 (Institutiones Calculi Differentialis, Opera (1), Vol. 10) is

$$(4.10) \quad \zeta(2n) = (-1)^{n-1} \frac{2^{2n-1} \pi^{2n}}{(2n)!} B_{2n}; \quad n = 1, 2, 3, \dots$$

The Zeta function is also explicitly known at the non-positive integers by

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}, \quad \text{for } n = 1, 2, \dots$$

The result may also be obtained in a straight forward fashion from (4.6) and a change of variable on using the fact that

$$(4.11) \quad B_{2n} = (-1)^{n-1} \cdot 4n \int_0^{\infty} \frac{t^{2n-1}}{e^{2\pi t} - 1} dt$$

from WHITTAKER and WATSON [48, p. 126].

We note here that

$$\zeta(2n) = A_n \pi^{2n},$$

where

$$A_n = (-1)^{n-1} \cdot \frac{n}{(2n+1)!} + \sum_{j=1}^{n-1} \frac{(-1)^{j-1}}{(2j+1)!} A_{n-j}$$

and $A_1 = \frac{1}{3!}$.

Further, the Zeta function for even integers satisfy the relation (BORWEIN et al. [7], SRIVASTAVA [43])

$$\zeta(2n) = \left(n + \frac{1}{2}\right)^{-1} \sum_{j=1}^{n-1} \zeta(2j) \zeta(2n-2j), \quad n \in \mathbb{N} \setminus \{1\}.$$

Despite several efforts to find a formula for $\zeta(2n+1)$, there seems to be no elegant closed form representation for the zeta function at the odd integer values. Several series representations for the value $\zeta(2n+1)$ have been proved by SRIVASTAVA and co-workers in particular, see [43], [44].

There are also integral representations for $\zeta(n+1)$, see [1, p. 807] and [28].

Both series representations and the integral representations are however somewhat difficult in terms of computational aspects and time considerations.

We note that there are functions that are closely related to $\zeta(x)$. Namely, the DIRICHLET $\eta(\cdot)$ and $\lambda(\cdot)$ functions given by

$$(4.12) \quad \eta(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^x} = \frac{1}{\Gamma(x)} \int_0^{\infty} \frac{t^{x-1}}{e^t + 1} dt, \quad x > 0$$

and

$$(4.13) \quad \lambda(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^x} = \frac{1}{\Gamma(x)} \int_0^{\infty} \frac{t^{x-1}}{e^t - e^{-t}} dt, \quad x > 0.$$

These are related to $\zeta(x)$ by

$$(4.14) \quad \eta(x) = (1 - 2^{1-x}) \zeta(x) \quad \text{and} \quad \lambda(x) = (1 - 2^{-x}) \zeta(x)$$

satisfying the identity

$$(4.15) \quad \zeta(x) + \eta(x) = 2\lambda(x).$$

It should be further noted that explicit expressions for both of $\eta(2n)$ and $\lambda(2n)$ exist as a consequence of the relation to $\zeta(2n)$ via (4.14).

4.2. RESULTS FOR THE ZETA FUNCTION

Lemma 1. *The following identity involving the Zeta function holds. Namely,*

$$(4.16) \quad \int_0^{\infty} \frac{t^x}{(e^t + 1)^2} dt = C(x+1) \zeta(x+1) - xC(x) \zeta(x), \quad x > 0,$$

where $C(x)$ is as given by (4.7).

Based on the identity in Lemma 1, the following theorem was developed (see ALZER [2], CERONE et al. [18], and also [10] where the constants in the bounds of (4.17) were developed.

Theorem 8. *For real numbers $x > 0$ we have*

$$(4.17) \quad \left(\ln 2 - \frac{1}{2} \right) b(x) < \zeta(x+1) - (1 - b(x)) \zeta(x) < \frac{b(x)}{2},$$

where

$$(4.18) \quad b(x) = \frac{1}{2^x - 1},$$

and the constants $\ln 2 - \frac{1}{2}$ and $\frac{1}{2}$ are sharp.

The following is a correction of a result obtained by the author [13] by utilising the ČEBYŠEV functional bounds given by (1.17) and (4.5).

Theorem 9. *For $\alpha > 0$ the Zeta function satisfies the inequality*

$$(4.19) \quad \left| \zeta(\alpha+1) - \frac{2^{\alpha-1}}{\alpha} \cdot \frac{\pi^2}{6} \right| \leq \frac{\kappa \cdot 2^{\alpha-\frac{1}{2}}}{\Gamma(\alpha+1)} (\Gamma(2\alpha-1) - \Gamma^2(\alpha))^{\frac{1}{2}},$$

where

$$(4.20) \quad \kappa = \left[\pi^2 \left(1 - \frac{\pi^2}{72} \right) - 7\zeta(3) \right]^{\frac{1}{2}} = 0.319846901 \dots$$

with equality obtained at $\alpha = 1$.

The following theorem was obtained in [17] utilising bounds for the ČEBYŠEV functional.

Theorem 10. For $\alpha > 1$ and $m = \lfloor \alpha \rfloor$ the zeta function satisfies the inequality

$$(4.21) \quad \left| \zeta(\alpha + 1) - 2^{\alpha-m} \frac{\Gamma(m+1)}{\Gamma(\alpha+1)} \zeta(m+1) \Gamma(\alpha-m+1) \right| \\ \leq \frac{2^{(\alpha-m+\frac{1}{2})}}{\Gamma(\alpha+1)} \cdot E_m \cdot (\Gamma(2\alpha-2m+1) - \Gamma^2(\alpha-m+1))^{\frac{1}{2}},$$

where

$$(4.22) \quad E_m^2 = 2^{2m} \Gamma(2m+1) (\lambda(2m) - \lambda(2m+1)) - \frac{1}{2} \Gamma^2(m+1) \zeta^2(m+1),$$

with $\lambda(\cdot)$ given by (4.13). Equality in (4.21) results when $\alpha = m$.

Proof. (Sketch using the ČEBYŠEV Functional Approach). Let

$$(4.23) \quad \tau(\alpha) = \Gamma(\alpha+1) \zeta(\alpha+1) = \int_0^{\infty} \frac{x^\alpha}{e^x - 1} dx \\ = \int_0^{\infty} e^{-x/2} \frac{x^m}{e^{x/2} - e^{-x/2}} \cdot x^{\alpha-m} dx, \quad \alpha > 1$$

where $m = \lfloor \alpha \rfloor$.

Make the associations

$$(4.24) \quad p(x) = e^{-x/2}, \quad f(x) = \frac{x^m}{e^{x/2} - e^{-x/2}}, \quad g(x) = x^{\alpha-m}$$

then we have from (1.17)

$$(4.25) \quad \left\{ \begin{array}{l} P = \int_0^{\infty} e^{-x/2} dx = 2, \\ \mathcal{M}(f; p) = \frac{1}{2} \int_0^{\infty} \frac{e^{-x/2} x^m}{e^{x/2} - e^{-x/2}} dx = \frac{1}{2} \Gamma(m+1) \zeta(m+1), \\ \mathcal{M}(g; p) = \frac{1}{2} \int_0^{\infty} e^{-x/2} x^{\alpha-m} dx = 2^{\alpha-m} \Gamma(\alpha-m+1). \quad \square \end{array} \right.$$

The following corollary provides upper bounds for the zeta function at odd integers.

Corollary 3. *The inequality*

$$(4.26) \quad \Gamma(2m+1) \left(2 \cdot (2^{2m} - 1) \zeta(2m) - (2^{2m+1} - 1) \zeta(2m+1) \right) \\ - \Gamma^2(m+1) \zeta^2(m+1) > 0$$

holds for $m = 1, 2, \dots$

Proof. From equation (4.22) of Theorem 10, we have $E_m^2 > 0$. Utilising the relationship between $\lambda(\cdot)$ and $\zeta(\cdot)$ given by (4.14) readily gives the inequality (4.26). \square

REMARK 5. In (4.26), if m is odd, then $2m$ and $m+1$ are even so that an expression in the form

$$(4.27) \quad \alpha(m) \zeta(2m) - \beta(m) \zeta(2m+1) - \gamma(m) \zeta^2(m+1) > 0,$$

results, where

$$(4.28) \quad \begin{aligned} \alpha(m) &= 2(2^{2m} - 1) \Gamma(2m+1), \\ \beta(m) &= (2^{2m+1} - 1) \Gamma(2m+1) \quad \text{and} \\ \gamma(m) &= \Gamma^2(m+1). \end{aligned}$$

Thus for m **odd** we have

$$(4.29) \quad \zeta(2m+1) < \frac{\alpha(m) \zeta(2m) - \gamma(m) \zeta^2(m+1)}{\beta(m)}.$$

That is, for $m = 2k - 1$, we have from (4.29)

$$(4.30) \quad \zeta(4k-1) < \frac{\alpha(2k-1) \zeta(4k-2) - \gamma(2k-1) \zeta^2(2k)}{\beta(2k-1)}$$

giving for $k = 1, 2, 3$, for example,

$$\begin{aligned} \zeta(3) &< \frac{\pi^2}{7} \left(1 - \frac{\pi^2}{72} \right) = 1.21667148, \\ \zeta(7) &< \frac{2\pi^6}{1905} \left(1 - \frac{\pi^2}{2160} \right) = 1.00887130, \\ \zeta(11) &< \frac{62\pi^{10}}{5803245} \left(1 - \frac{\pi^2}{492150} \right) = 1.00050356. \end{aligned}$$

The above bound for $\zeta(3)$ was obtained previously by the author in [13] from (4.20).

If m is **even** then for $m = 2k$ we have from (4.29)

$$(4.31) \quad \zeta(4k+1) < \frac{\alpha(2k) \zeta(4k) - \gamma(2k) \zeta^2(2k+1)}{\beta(2k)}, \quad k = 1, 2, \dots$$

We notice that in (4.31), or equivalently (4.27) with $m = 2k$ there are two zeta functions with odd arguments. There are a number of possibilities for resolving this, but firstly it should be noticed that $\zeta(x)$ is monotonically decreasing for $x > 1$ so that $\zeta(x_1) > \zeta(x_2)$ for $1 < x_1 < x_2$.

Firstly, we may use lower bounds obtained in [10], namely

$$L(x) = (1 - b(x))\zeta(x) + \left(\ln 2 - \frac{1}{2}\right)b(x) \quad \text{or} \quad L_2(x) = \frac{\zeta(x+2) - \frac{b(x+1)}{2}}{1 - b(x+1)},$$

where $b(x)$ is given by (4.18).

However, from numerical investigation in [10], it seems that $L_2(x) > L(x)$ for positive integer x and so we have from (4.31)

$$(4.32) \quad \zeta_L(4k+1) < \frac{\alpha(2k)\zeta(2k) - \gamma(2k)L_2^2(2k)}{\beta(2k)},$$

where we have used the fact that $L_2(x) < \zeta(x+1)$.

Secondly, since the even argument $\zeta(2k+2) < \zeta(2k+1)$, then from (4.31) we have

$$(4.33) \quad \zeta_E(4k+1) < \frac{\alpha(2k)\zeta(4k) - \gamma(2k)\zeta^2(2k+2)}{\beta(2k)}.$$

Finally, we have that $\zeta(m+1) > \zeta(2m+1)$ so that from (4.27) we have, with $m = 2k$ on solving the resulting quadratic equation that

$$(4.34) \quad \zeta_Q(4k+1) < \frac{-\beta(2k) + \sqrt{\beta^2(2k) + 4\gamma(2k)\alpha(2k)\zeta(4k)}}{2\gamma(2k)}.$$

For $k = 1$ we have from (4.32)–(4.34) that

$$\begin{aligned} \zeta_L(5) &< \frac{\pi^4}{93} - \frac{1}{186} \left(\frac{7\pi^4}{540} - \frac{1}{12} \right)^2 = 1.039931461, \\ \zeta_E(5) &< \frac{\pi^4}{93} \left(1 - \frac{\pi^4}{16200} \right) = 1.041111605, \\ \zeta_Q(5) &< -93 + \sqrt{8649 + 2\pi^4} = 1.04157688; \end{aligned}$$

and for $k = 2$

$$\begin{aligned} \zeta_L(9) &< \frac{17}{160965}\pi^8 - \frac{1}{35770} \left(\frac{31}{28350}\pi^6 - \frac{1}{60} \right)^2 = 1.002082506, \\ \zeta_E(9) &< \frac{17}{160965}\pi^8 \left(1 - \frac{\pi^4}{337650} \right) = 1.0020834954, \\ \zeta_Q(9) &< -17885 + \frac{1}{3}\sqrt{2878859025 + 34\pi^8} = 1.00208436. \end{aligned}$$

It should be noted that the above results give tighter upper bounds for the odd zeta function evaluations than were possible using the methodology utilising techniques based around Theorem 8 as demonstrated by the numerics which are presented in Table 1 of [10].

Numerical experimentation using Maple seems to indicate that the upper bounds for

$$\zeta_L(4k+1), \zeta_E(4k+1) \quad \text{and} \quad \zeta_Q(4k+1)$$

are in increasing order. Analytic demonstration that $\zeta_L(4k+1)$ is better remains an open problem.

5. CONCLUDING REMARKS

In the paper the usefulness of some recent results in the analysis of inequalities, has been demonstrated through application to some special functions. Although these techniques have been applied in a variety of areas of applied mathematics, their application to special functions does not seem to have received much attention to date. There are many special functions which may be represented as the integral of products of functions. The investigation in the current article has restricted itself to the investigation of the BESSEL function of the first kind, the Beta function and the Zeta function.

It may be surmised from the above investigations that the accuracy of the bounds over particular regions of parameters cannot be ascertained *a priori*. It has been demonstrated, however, that some useful bounds may be obtained which have hitherto do not seem to have been discovered. The approach of utilising developments in the field of inequalities to special functions has been shown to have the potential for further development.

A general investigation of DIRICHLET series has also been undertaken in [20], [21] utilising convexity arguments and it is shown that in particular

$$(5.1) \quad \zeta(s+1) \leq \mathcal{A}\left(\frac{1}{\zeta(s)}, \frac{1}{\zeta(s+2)}\right) \leq \mathcal{G}(\zeta(s), \zeta(s+2))$$

where $\mathcal{A}(\cdot, \cdot)$ is the arithmetic mean and $\mathcal{G}(\cdot, \cdot)$ the geometric mean.

Specifically, for $s = 2n$, then

$$(5.2) \quad \zeta(2n+1) \leq \mathcal{H}(\zeta(2n), \zeta(2n+2)) \leq \mathcal{G}(\zeta(2n), \zeta(2n+2)),$$

where the harmonic mean

$$\mathcal{H}(\alpha, \beta) = \frac{\mathcal{G}^2(\alpha, \beta)}{\mathcal{A}(\alpha, \beta)} = \mathcal{A}\left(\frac{1}{\alpha}, \frac{1}{\beta}\right).$$

The reader may also wish to refer to the papers [3] and [6] which provide some results using monotonicity and convexity arguments.

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