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# TRAVELING WAVE SOLUTIONS OF AN ORDINARY-PARABOLIC SYSTEM IN $\mathbf{R}^2$ AND A 2D-STRIP

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We investigate a prey-predator model, which we describe by an ordinary-parabolic system. We obtain four types of wave solutions of this system, which are connecting different equilibria. To establish the existence of four types of traveling wave solutions with double wave speeds, we introduce a new approach to constructing monotonous iteration schemes. Moreover, by using spreading speeds, we establish the non-existence of traveling wave solutions. Our results provide insight into the dynamics of this model system.

## 1. INTRODUCTION

One of the dominant themes in both ecology and mathematical ecology is the dynamic relationship between predators and their prey due to its universal existence and importance in population dynamics. (see [**3, 7, 8, 9, 11, 13, 15**]). Taking into consideration the fact that the natural enemies are not completely dependent on the prey, we study a diffusive prey-predator model with modified Leslie-Gower and Holling II schemes which was proposed in [**13**], and the corresponding ODE, proposed by Aziz in [**3**] and considered by Nindjin in [**8, 9**]. If the prey moves slower than the predator, we can regard the diffusive coefficient of the prey as 0. Thus we consider an ordinary-parabolic system as follows.

$$
\begin{aligned}
&\frac{\partial u}{\partial t} = u\left(1 - u - \frac{\beta_1 w}{u + k_1}\right),\ t > 0,\ \mathbf{x} \in \mathbf{R}^2,\\
&\frac{\partial w}{\partial t} = D\Delta w + \alpha w\left(1 - \frac{\beta_2 w}{u + k_2}\right),\ t > 0,\ \mathbf{x} \in \mathbf{R}^2,
\end{aligned} \tag{1}
$$

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where $D, \alpha, \beta_1, \beta_2, k_1, k_2$ are positive constants, $u = u(t, \mathbf{x})$, $w = w(t, \mathbf{x})$, $\mathbf{x} = (x_1, x_2)$, refer to [**13**].

Obviously, $(0,0), (1,0), (0, k_2/\beta_2)$ are the three steady states of (1). The following proposition (see [**8, 13**]) guarantees the existence, non-existence and uniqueness of the positive constant steady state.

**Proposition 1.** *System* (1) *has a unique interior equilibrium* $E_1 = (u^*, w^*)$ (*i.e* $u^* > 0, w^* > 0$) *if the following condition holds*

$$\frac{k_2}{\beta_2} < \frac{k_1}{\beta_1}, \tag{2}$$

*where* $u^*, w^*$ *are given by* $u^* = \frac{1}{2\beta_2}\big[-(\beta_1 - \beta_2 + \beta_2 k_1) + \Delta^{1/2}\big]$, $w^* = \frac{u^* + k_2}{\beta_2}$, *and* $\Delta = (\beta_1 - \beta_2 + \beta_2 k_1)^2 - 4\beta_2(\beta_1 k_2 - \beta_2 k_1)$.

*System* (1) *has no interior equilibrium if one of the following conditions is satisfied.*

$$\text{(i) } k_1 \geq 1 \textit{ and } \frac{k_1}{\beta_1} < \frac{k_2}{\beta_2};\ \text{(ii) } k_1 < 1 \textit{ and } \left(\frac{1+k_1}{2}\right)^2 < \frac{\beta_1 k_2}{\beta_2}. \tag{3}$$

Moreover, if the prey and the predator live in a 2D-strip, we also study the following system

$$\begin{aligned}
&\frac{\partial u}{\partial t} = u\left(1 - u - \frac{\beta_1 w}{u + k_1}\right),\ t > 0,\ x \in \mathbf{R},\ y \in (-L, L),\\
&\frac{\partial w}{\partial t} = D\Delta w + \alpha w\left(1 - \frac{\beta_2 w}{u + k_2}\right),\ t > 0,\ x \in \mathbf{R},\ y \in (-L, L),\\
&\frac{\partial w}{\partial y} = 0, t > 0,\ x \in \mathbf{R},\ y \in \{-L, L\}.
\end{aligned} \tag{4}$$

Proposition 1 is valid for system (4).

Since there are four equilibria for the models, there are multiple spreading patterns for those models. We study these patterns by investigating the traveling wave solutions connecting different equilibria. We also refer the reader to many important results on the topic, see [**1, 2, 4, 5, 6, 7, 12, 14, 16, 17, 18**]. Many of them obtain the existence and the minimal wave speed of the traveling wave solution when the considered dynamics are monotonous. Clearly, the prey-predator model is not a monotonous dynamic, so the results mentioned can not be applied to our system. In [**4**], the system was discussed without quasi-monotonicity by using phase plane analysis method, which is more complicated. Motivated by [**10**], we analyze the model (1) in $\mathbf{R}^2$ and a 2D strip by using the weak coupled upper-lower solution method and the crossing iteration method, which are different from [**4, 17, 18**]. First, we establish the existence and non-existence of four types of wave solutions with double speeds for system (1) in $\mathbf{R}^2$, connecting different equilibria, to describe multiple invasion processes of the model. The term "double speeds" here means that the wave solutions of the prey and the predator have different speeds. Second, we obtain two categories of the wave solutions for (4). One is independent of $y$, which can be obtained in a similar way for (1) in $\mathbf{R}^2$ whereas the second category is dependent on $y$. Regarding the wave system of (4) as an elliptic system, we develop the weak upper and weak lower solutions method proposed by WANG in [**15**] to obtain the existence of the second category wave solution by using iterative schemes. Moreover, the minimal wave speeds are also investigated in our paper. Thus, the results and the methods in our paper are new for the prey-predator system. In addition, the methods in our paper can be applied to more generalized prey-predator systems.

Our paper is organized as follows. The existence and the non-existence of four types of traveling wave solutions in $\mathbf{R}^2$ and a 2D-strip are established in sections 2 and 3, respectively. We point out that the traveling wave solutions in a 2D strip are different from those in $\mathbf{R}^2$, and we provide a more detailed explanation in section 3. Our results reveal multiple invasion patterns of the system.

A well known result, first introduced in [**2**], is as follows.

**Lemma 1.** *Let* $\hat{d} \geq 0$, *consider the system*

$$
\begin{aligned}
&\frac{\partial w}{\partial t} = D\Delta w + \alpha w\left(1 - \frac{\beta_2 w}{\hat{d} + k_2}\right),\ t > 0,\ x \in \mathbf{R}, \\
&w(0, x) = w_0(x),\ x \in \mathbf{R},
\end{aligned} \tag{5}
$$

*then the following statements are valid*:

$$
\liminf_{t\to+\infty, |x|<ct} w(t, x) = \frac{\hat{d} + k_2}{\beta_2} \tag{i}
$$

*for each* $0 < c < 2\sqrt{D\alpha}$ *if* $w_0(x) \geq 0,\ w_0(x) \not\equiv 0$;

$$
\liminf_{t\to+\infty, |x|>ct} w(t, x) = 0 \tag{ii}
$$

*for each* $c > 2\sqrt{D\alpha}$ *if* $w_0(x)$ *has a compact support set.*

## 2. TRAVELING WAVE SOLUTIONS AND MINIMAL WAVE SPEEDS IN $\mathbf{R}^2$

We observe traveling wave solutions of (1) in the form

$$
(u, w) = (\phi_1^{c_1}, \phi_2^{c_2}), \qquad \phi_i^{c_i}(t, x) = p_i(e_1 x_1 + e_2 x_2 + c_i t),\ i = 1, 2, \tag{6}
$$

where $(e_1, e_2)$ is the unit vector in $\mathbf{R}^2$. We discuss four types of such solutions. Solutions of the form (6) that connect $(0, 0)$ with $(u^*, w^*)$ will be referred to as being type I solutions, those that connect $(1, 0)$ with $(u^*, w^*)$ - type II, solutions that connect $(0, k_2/\beta_2)$ with $(u^*, w^*)$ - type III and finally, those that connect $(1, 0)$ with $(0, k_2/\beta_2)$ will be type IV.

For convenience, in the following we write $\phi_i(t, x)$ instead of $\phi_i^{c_i}(t, x)$. With $s_i = (e_1, e_2)\binom{x_1}{x_2} + c_i t,\ \phi_i(t, x) = p_i(s_i) = p_i(e_1 x_1 + e_2 x_2 + c_i t) = \psi_i(t)$, we have that

$$\frac{\partial \phi_i}{\partial t} = c_i \frac{dp_i}{ds_i} = \frac{d\psi_i}{dt},\ \frac{\partial^2 \phi_i}{\partial x_1^2} = \frac{e_1^2}{c_i^2}\frac{d^2\psi_i}{dt^2},\ \frac{\partial^2 \phi_i}{\partial x_2^2} = \frac{e_2^2}{c_i^2}\frac{d^2\psi_i}{dt^2}.$$

By substitution into (1), we obtain the following wave system of (1)

$$\begin{aligned} &-\psi_1'(t) + \psi_1(t)\left(1 - \psi_1(t) - \frac{\beta_1\psi_2(t)}{\psi_1(t) + k_1}\right) = 0, \\ &\frac{D}{c_2^2}\psi_2''(t) - \psi_2'(t) + \alpha\psi_2(t)\left(1 - \frac{\beta_2\psi_2(t)}{\psi_1(t) + k_2}\right) = 0, \end{aligned} \tag{7}$$

Thus the four types of wave solutions corresponding to (7) can be defined in a similar way as for (1), as positive solutions of (7) that connect $(0, 0)$ with $(u^*, w^*)$ (type I), $(1, 0)$ with $(u^*, w^*)$ (type II), $(0, k_2/\beta_2)$ with $(u^*, w^*)$ (type III) or $(1, 0)$ with $(0, k_2/\beta_2)$ (type IV).

In what follows, we seek the solutions of (7) with four types of boundary conditions using iteration method and weak coupled upper and lower solutions.

### 2.1 Weak coupled upper and lower solutions of (7)

The definition of the weak coupled upper and lower solutions of (7) is given as follows.

**Definition 1.** *Let*

$$\Lambda := \{\psi : \mathbf{R} \to \mathbf{R}\},$$

*where $\psi'$ and $\psi''$ exist and are essentially bounded for all $t \in \mathbf{R}\backslash\{T_i : i = 1, \ldots, m\}$. Suppose that $\bar{\psi}_i, \underline{\psi}_i \in \Lambda, i = 1, 2$. Two pairs of continuous functions, $(\bar{\psi}_1, \bar{\psi}_2)$ and $(\underline{\psi}_1, \underline{\psi}_2)$, are called weak coupled upper and lower solutions of (7) if they satisfy*

$$\begin{aligned} &-\bar{\psi}_1' + \bar{\psi}_1\left(1 - \bar{\psi}_1 - \frac{\beta_1\underline{\psi}_2}{\bar{\psi}_1 + k_1}\right) \le 0,\ t \in \mathbf{R}\setminus\{T_i : i = 1, \ldots, m\}, \\ &-\underline{\psi}_1' + \underline{\psi}_1\left(1 - \underline{\psi}_1 - \frac{\beta_1\bar{\psi}_2}{\underline{\psi}_1 + k_1}\right) \ge 0,\ t \in \mathbf{R}\setminus\{T_i : i = 1, \ldots, m\}, \\ &\frac{D}{c_2^2}\bar{\psi}_2'' - \bar{\psi}_2' + \alpha\bar{\psi}_2\left(1 - \frac{\beta_2\bar{\psi}_2}{\bar{\psi}_1 + k_2}\right) \le 0,\ t \in \mathbf{R}\setminus\{T_i : i = 1, \ldots, m\},\ \bar{\psi}_2'(t^+) \le \bar{\psi}_2'(t^-), \\ &\frac{D}{c_2^2}\underline{\psi}_2'' - \underline{\psi}_2' + \alpha\underline{\psi}_2\left(1 - \frac{\beta_2\underline{\psi}_2}{\underline{\psi}_1 + k_2}\right) \ge 0,\ t \in \mathbf{R}\setminus\{T_i : i = 1, \ldots, m\},\ \underline{\psi}_2'(t^+) \ge \underline{\psi}_2'(t^-). \end{aligned} \tag{8}$$

Next, we construct the upper and lower solutions of (7). Define

$$\bar{\phi}_1 = \begin{cases} u^*(1+\hat{\ell}_1 e^{-st}), & t>0,\\ u^*(1+\hat{\ell}_1)e^{t}, & t\le 0,\end{cases} \qquad \bar{\phi}_2 = \begin{cases} w^*(1+\hat{\ell}_2 e^{-st}), & t>0,\\ w^*(1+\hat{\ell}_2)e^{\bar{\zeta}_1 t}, & t\le 0,\end{cases}$$

$$\underline{\phi}_1 = \begin{cases} u^*(1-\ell_1 e^{-st}), & t>0,\\ u^*(1-\ell_1)e^{(1-\varepsilon_1)t}, & t\le 0,\end{cases} \qquad \underline{\phi}_2 = \begin{cases} w^*(1-\ell_2 e^{-st}), & t>0,\\ w^*(1-\ell_2 e^{\varepsilon_2 t})e^{\kappa_1 t}, & t\le 0.\end{cases}$$

The assumption (H1) is proposed as

$$\text{(H1)}\ \frac{1+k_2}{\beta_2} < \frac{k_1}{\beta_1},\ u^*+k_1-1-\frac{\beta_1 w^*}{u^*} > 0. \tag{9}$$

Under the assumption (H1), the parameters $\hat{\ell}_1,\ \hat{\ell}_2,\ \ell_1,\ \ell_2,\ s,\ \bar{\zeta}_1, \kappa_1,\ \varepsilon_1,\ \varepsilon_2$ are chosen as follows.

(I1) $\beta_2 w^*\ell_2 > \max\{u^*\ell_1,\ \beta_2 w^* - k_2\}$,

(I2) $2u^*+k_1-1-\dfrac{\beta_1 w^*\ell_2}{u^*\hat{\ell}_1} > 0$, $2u^*+k_1-1-\dfrac{u^*}{\ell_1}-\dfrac{\beta_1 w^*\hat{\ell}_2}{u^*\ell_1} > 0$,

(I3) Since $(1-u^*)(u^*+k_1)-\beta_1 w^* = 0$, $u^* > \dfrac{1-k_1}{2}$, then $\big[1-u^*(1-\ell_1)\big]\big[u^*(1-\ell_1)+k_1\big]-\beta_1 w^* > 0$. Choose $\varepsilon_1 \in (u^*(1-\ell_1),1)$ such that $\big[\varepsilon_1 - u^*(1-\ell_1)\big]\big[u^*(1-\ell_1)+k_1\big]-\beta_1 w^*(1+\hat{\ell}_2) > 0$.

(I4) Define $\Delta(c_2,\zeta) := \dfrac{D}{c_2^2}\zeta^2-\zeta+\alpha$, then $\Delta(c_2,\zeta)=0$ has two positive roots $0<\bar{\zeta}_1<\bar{\zeta}_2$ if $c_2 > 2\sqrt{D\alpha}$,

(I5) Define $B := 1-\dfrac{\beta_2 w^*(1-\ell_2)}{k_2} > 0$, define $\bar{\Delta}(c_2,\kappa) = \dfrac{D}{c_2^2}\kappa^2-\kappa+\alpha B$, then $\bar{\Delta}(c_2,\kappa)=0$ has two positive roots $0<\kappa_1<\kappa_2$ if $c_2>2\sqrt{D\alpha}$. Choose $\varepsilon_2>0$ such that $\kappa_1+\varepsilon_2<\kappa_2$.

(I6) Define

$$\gamma_1 = \frac{u^*}{u^*+k_1}\left(2u^*+k_1-1-\frac{\beta_1 w^*\ell_2}{u^*\hat{\ell}_1}\right),$$
$$\gamma_2 = \frac{u^*(1-\ell_1)}{u^*+k_1}\left(2u^*+k_1-1-u^*\ell_1-\frac{\beta_1 w^*\hat{\ell}_2}{u^*\ell_1}\right)$$
$$\gamma_3 = \frac{\alpha(\beta_2 w^*\hat{\ell}_2-u^*\hat{\ell}_1)}{u^*[(1+\hat{\ell}_1)+k_2]\hat{\ell}_2},\ \gamma_4 = \frac{\alpha(1-\ell_2)}{(u^*+k_2)\ell_2}(\beta_2 w^*\ell_2-u^*\ell_1),$$

then $\gamma_i>0$ $(i=1,2,3,4)$ from (I1) and (I2). Define the function $\tilde{\Delta}(s,\gamma_{3,4}) := \dfrac{D}{c_2^2}s^2+s-\min\{\gamma_3,\gamma_4\}$, $s_1$ is the root of $\tilde{\Delta}(s,\gamma_{3,4})=0$, and choose $s$ satisfying $0<s<\min\{s_1,\ \gamma_1,\ \gamma_2\}$.

**Lemma 2.** *Suppose that* (H1) *holds. Then, continuous functions* $(\bar{\phi}_1, \bar{\phi}_2)$ *and* $(\underline{\phi}_1, \underline{\phi}_2)$ *are a pair of weak coupled upper and lower solutions of* (7) *with*

$$\begin{aligned}
\big(\bar{\phi}_1(+\infty), \bar{\phi}_2(+\infty)\big) &= \big(\underline{\phi}_1(+\infty), \underline{\phi}_2(+\infty)\big) = (u^*, w^*),\\
\big(\bar{\phi}_1(-\infty), \bar{\phi}_2(-\infty)\big) &= \big(\underline{\phi}_1(-\infty), \underline{\phi}_2(-\infty)\big) = (0, 0)
\end{aligned}$$

*if* $c_2 \geq 2\sqrt{\alpha}$.

**Proof.** It is obvious that $\bar{\phi}_2'(0^+) < \bar{\phi}_2'(0^-)$ and $\underline{\phi}_2'(0^+) > \underline{\phi}_2'(0^-)$ hold. Next, we prove that $(\bar{\phi}_1, \bar{\phi}_2), (\underline{\phi}_1, \underline{\phi}_2)$ satisfy (8). For $t > 0$, from (I2) and (I6), we have

$$\begin{aligned}
&-\bar{\phi}_1' + \bar{\phi}_1 \left(1 - \bar{\phi}_1 - \frac{\beta_1 \underline{\phi}_2}{\bar{\phi}_1 + k_1}\right)\\
&= u^* \hat{\ell}_1 e^{-st} \left[s - \frac{u^*}{u^* + k_1}\Big(2u^* + k_1 - 1 - \frac{\beta_1 w^* \ell_2}{u^* \hat{\ell}_1}\Big)\right] \leq 0.
\end{aligned}$$

For $t \leq 0$,

$$-\bar{\phi}_1' + \bar{\phi}_1 \left(1 - \bar{\phi}_1 - \frac{\beta_1 \underline{\phi}_2}{\bar{\phi}_1 + k_1}\right) \leq u^*(1 + \hat{\ell}_1)(-1 + 1)e^t = 0.$$

Thus the first inequality of (8) holds. For $t > 0$, from (I2) and (I6),

$$\begin{aligned}
&-\underline{\phi}_1' + \underline{\phi}_1 \left(1 - \underline{\phi}_1 - \frac{\beta_1 \bar{\phi}_2}{\underline{\phi}_1 + k_1}\right)\\
&\geq u^* \ell_1 e^{-st} \left[- s + \frac{u^*\,(1 - \ell_1)}{u^* + k_1}\Big(2u^* + k_1 - 1 - u^* \ell_1 - \frac{\beta_1 \hat{\ell}_2 w^*}{u^* \ell_1}\Big)\right] \geq 0.
\end{aligned}$$

For $t \leq 0$, in view of (I3), we get

$$\begin{aligned}
&-\underline{\phi}_1' + \underline{\phi}_1 \left(1 - \underline{\phi}_1 - \frac{\beta_1 \bar{\phi}_2}{\underline{\phi}_1 + k_1}\right)\\
&= -A(1 - \varepsilon_1)e^{(1-\varepsilon_1)t} + Ae^{(1-\varepsilon_1)t}\left(1 - \underline{\phi}_1 - \frac{\beta_1 \bar{\phi}_2}{\underline{\phi}_1 + k_1}\right)\\
&\geq \frac{Ae^{(1-\varepsilon_1)t}}{\underline{\phi}_1 + k_1}\big[[\varepsilon_1 - u^*(1 - \ell_1)][u^*(1 - \ell_1) + k_1] - \beta_1 w^*(1 + \hat{\ell}_2)\big] > 0.
\end{aligned}$$

Similarly, for $t > 0$, from (I1) and (I6), we obtain

$$\begin{aligned}
&\frac{D}{c_2^2}\bar{\phi}_2'' - \bar{\phi}_2' + \alpha \bar{\phi}_2 \left(1 - \frac{\beta_2 \bar{\phi}_2}{\bar{\phi}_1 + k_2}\right)\\
&\leq w^* \hat{\ell}_2 e^{-st}\left[\frac{D}{c_2^2}s^2 + s - \frac{\alpha(\beta_2 w^* \hat{\ell}_2 - u^* \hat{\ell}_1)}{[u^*(1 + \hat{\ell}_1) + k_2]\hat{\ell}_2}\right] \leq w^* \hat{\ell}_2 e^{-st}\left[\frac{D}{c_2^2}s^2 + s - \gamma\right] \leq 0.
\end{aligned}$$

For $t \le 0$, from (I4), we have

$$\begin{aligned}
&\frac{1}{c_2^2}\bar{\phi}_2'' - \bar{\phi}_2' + \alpha\bar{\phi}_2\left(1 - \frac{\beta_2\bar{\phi}_2}{\bar{\phi}_1 + k_2}\right) \\
&\le \frac{1}{c_2^2}\bar{\phi}_2'' - \bar{\phi}_2' + \alpha\bar{\phi}_2 = w^*(1+\hat{\ell}_2)e^{\bar{\zeta}_1 t}\left[\frac{1}{c_2^2}\bar{\zeta}_1^2 - \bar{\zeta}_1 + \alpha\right] = 0.
\end{aligned}$$

For $t > 0$, from (I1) and (I6), it follows that

$$\begin{aligned}
&\frac{1}{c_2^2}\underline{\phi}_2'' - \underline{\phi}_2' + \alpha\underline{\phi}_2\left(1 - \frac{\beta_2\underline{\phi}_2}{\underline{\phi}_1 + k_2}\right) \\
&\ge w^*\ell_2 e^{-st}\left[-\frac{D}{c_2^2}s^2 - s + \frac{\alpha(1-\ell_2)(\beta_2 w^*\ell_2 - u^*\ell_1)}{(u^*+k_2)\ell_2}\right] \ge 0.
\end{aligned}$$

For $t \le 0$, from (I5),

$$\begin{aligned}
&\frac{D}{c_2^2}\underline{\phi}_2'' - \underline{\phi}_2' + \alpha\underline{\phi}_2\left(1 - \frac{\beta_2\underline{\phi}_2}{\underline{\phi}_1 + k_2}\right) \\
&\ge w^* e^{\kappa_1 t}\left(\frac{D}{c_2^2}\kappa_1^2 - \kappa_1\right) - w^*\ell_2 e^{(\kappa_1+\varepsilon_2 t)}\left(\frac{D}{c_2^2}(\kappa_1+\varepsilon_2)^2 - (\kappa_1+\varepsilon_2)\right) + \underline{\phi}_2\alpha B \\
&\ge w^* e^{\kappa_1 t}\left(\frac{D}{c_2^2}\kappa_1^2 - \kappa_1 + \alpha B\right) - w^*\ell_2 e^{(\kappa_1+\varepsilon_1 t)}\left(\frac{D}{c_2^2}(\kappa_1+\varepsilon_2)^2 - (\kappa_1+\varepsilon_2) + \alpha B\right) \\
&\ge 0.
\end{aligned}$$

Therefore, we proved (8) and the lemma holds. □

Next, we construct other weak coupled upper and lower solutions of (7), $(\bar{\chi}_1,\ \bar{\chi}_2) - (\underline{\chi}_1,\ \underline{\chi}_2)$, $(\bar{\varphi}_1,\ \bar{\varphi}_2) - (\underline{\varphi}_1,\ \underline{\varphi}_2)$, corresponding to type II and type III, respectively.

$$\bar{\chi}_1 = \begin{cases} u^*(1+h_1e^{-st}), & t > 0, \\ 1, & t \le 0, \end{cases} \qquad \bar{\chi}_2 = \begin{cases} w^*(1+\hat{\ell}_2 e^{-st}), & t > 0, \\ w^*(1+\hat{\ell}_2)e^{\bar{\zeta}_1 t}, & t \le 0, \end{cases}$$

$$\underline{\chi}_1 = \begin{cases} u^*(1-\ell_1 e^{-st}), & t > 0, \\ 1-\ell_3 e^{\bar{\gamma}t}, & t \le 0, \end{cases} \qquad \underline{\chi}_2 = \begin{cases} w^*(1-\ell_2 e^{-st}), & t > 0, \\ w^*(1-\ell_2 e^{\varepsilon_2 t})e^{\kappa_1 t}, & t \le 0, \end{cases}$$

where

$$u^*(1+h_1) = 1,\ u^*(1-\ell_1) = 1-\ell_3,\ \bar{\gamma} = \frac{c}{D}.$$

$$\bar{\varphi}_1 = \begin{cases} u^*(1+\hat{\ell}_1 e^{-st}), & t > 0, \\ u^*(1+\hat{\ell}_1)e^{t}, & t \le 0, \end{cases} \qquad \bar{\varphi}_2 = \begin{cases} w^*(1+\hat{\ell}_2 e^{-st}), & t > 0, \\ k_2/\beta_2(1+\hat{\ell}_4 e^{\bar{\eta}t}), & t \le 0, \end{cases}$$

$$\underline{\varphi}_1 = \begin{cases} u^*(1-\ell_1 e^{-st}), & t > 0, \\ u^*(1-\ell_1)e^{(1-\varepsilon_1)t}, & t \le 0, \end{cases} \qquad \underline{\varphi}_2 = \begin{cases} w^*(1-h_2 e^{-st}), & t > 0, \\ k_2/\beta_2, & t \le 0, \end{cases}$$

and

$$w^*(1+\hat{\ell}_2) = \frac{k_2}{\beta_2}(1+\hat{\ell}_4),\ \bar{\eta} = c,\ w^*(1-h_2) = \frac{k_2}{\beta_2}.$$

To obtain the upper-lower solutions connecting $(1,0)$ and $(0,k_2/\beta_2)$, we give an assumption (H2) as follows. If (H2) holds, there is no interior equilibrium for system (1).

$$\text{(H2)} \quad k_1 + \frac{(1-k_1)^2}{4} - \frac{\beta_1 k_2}{\beta_2} < 0. \tag{10}$$

The solutions $\bar{v}_i,\ \underline{v}_i (i=1,2)$, connecting $(1,0)$ and $(0,k_2/\beta_2)$, are defined as follows.

$$\bar{v}_1 = \begin{cases} e^{-\bar{s}t}, & t>0, \\ 1, & t \le 0, \end{cases} \qquad \bar{v}_2 = \begin{cases} k_2/\beta_2\left(1+d_1 e^{-\bar{s}t}\right), & t>0, \\ d_2 e^{\bar{\zeta}_1 t}, & t\le 0, \end{cases}$$

$$\underline{v}_1 = \begin{cases} d_3 e^{-\bar{s}t}, & t>0, \\ 1-d_4 e^{\bar{\rho}t}, & t \le 0, \end{cases} \qquad \underline{v}_2 = \begin{cases} k_2/\beta_2(1-d_5 e^{-\bar{s}t}), & t\ge 0, \\ B(1-d_6 e^{\varepsilon_3 t})e^{\tau_1 t}, & t< 0. \end{cases}$$

The parameters $\bar{s},\ d_i\ (i=1,2,3,4,5,6),\ \bar{\rho},\ \varepsilon_3$ are chosen as

(T1) $\bar{s} = \dfrac{\beta_1 k_2}{\beta_2}(1-d_5) - k_1 - \dfrac{(1-k_1)^2}{4} > 0;$

(T2) $0<\bar{\rho}<\bar{\zeta}_1;$

(T3) $d_2 = \dfrac{k_2}{\beta_2}(1+d_1),\ d_3 = 1-d_4;$

(T4) $\tilde{\Delta}(c_2,\tau) = \dfrac{D}{c_2^2}\tau^2 - \tau + \dfrac{\alpha}{1+k_2} = 0$ has two positive roots $0<\tau_1<\tau_2,$ , choose $\varepsilon \in (0, \tau_2-\tau_1)$ and $B(1-d_6) = \dfrac{k_2}{\beta_2}(1-d_5).$

As in Lemma 2, we prove that $(\bar{\chi}_1,\bar{\chi}_2)-(\underline{\chi}_1,\underline{\chi}_2)$, $(\bar{\phi}_1,\bar{\phi}_2)-(\underline{\phi}_1,\underline{\phi}_2)$ and $(\bar{v}_1,\bar{v}_2)-(\underline{v}_1,\underline{v}_2)$ satisfy (8). Based on the above, we obtain the following theorem.

**Theorem 1.** (a) *Continuous functions* $(\bar{\phi}_1,\bar{\phi}_2)$ - $(\underline{\phi}_1,\underline{\phi}_2)$ *are a pair of weak coupled upper and lower solutions of* (7) *with*

$$\big(\bar{\phi}_1(+\infty),\bar{\phi}_2(+\infty)\big) = \big(\underline{\phi}_1(+\infty),\underline{\phi}_2(+\infty)\big) = (u^*,w^*),$$
$$\big(\bar{\phi}_1(-\infty),\bar{\phi}_2(-\infty)\big) = \big(\underline{\phi}_1(-\infty),\underline{\phi}_2(-\infty)\big) = (0,0)$$

*if* $c_2 \ge 2\sqrt{\alpha}$ *and* (H1) *holds.*

(b) *Continuous functions* $(\bar{\chi}_1,\bar{\chi}_2)$ - $(\underline{\chi}_1,\underline{\chi}_2)$ *are a pair of weak coupled upper and lower solutions of* (7) *with*

$$\big(\bar{\chi}_1(+\infty),\bar{\chi}_2(+\infty)\big) = \big(\underline{\chi}_1(+\infty),\underline{\chi}_2(+\infty)\big) = (u^*,w^*),$$
$$\big(\bar{\chi}_1(-\infty),\bar{\chi}_2(-\infty)\big) = \big(\underline{\chi}_1(-\infty),\underline{\chi}_2(-\infty)\big) = (1,0)$$

*if* $c_2 \ge 2\sqrt{\alpha}$ *and* (H1) *holds.*

(c) *Continuous functions* $(\bar{\varphi}_1, \bar{\varphi}_2)$ - $(\underline{\varphi}_1, \underline{\varphi}_2)$ *are a pair of weak coupled upper and lower solutions of* (7) *with*

$$\big(\bar{\varphi}_1(+\infty), \bar{\varphi}_2(+\infty)\big) = \big(\underline{\varphi}_1(+\infty), \underline{\varphi}_2(+\infty)\big) = (u^*, w^*),$$
$$\big(\bar{\varphi}_1(-\infty), \bar{\varphi}_2(-\infty)\big) = \big(\underline{\varphi}_1(-\infty), \underline{\varphi}_2(-\infty)\big) = (0, k_2/\beta_2)$$

*if* $c_1 > 0$ *and* (H1) *holds.*

(d) *Continuous functions* $(\bar{\upsilon}_1, \bar{\upsilon}_2)$ - $(\underline{\upsilon}_1, \underline{\upsilon}_2)$ *are a pair of weak coupled upper and lower solutions of* (7) *with*

$$\big(\bar{\upsilon}_1(+\infty), \bar{\upsilon}_2(+\infty)\big) = \big(\underline{\upsilon}_1(+\infty), \underline{\upsilon}_2(+\infty)\big) = (0, k_2/\beta_2),$$
$$\big(\bar{\upsilon}_1(-\infty), \bar{\upsilon}_2(-\infty)\big) = \big(\underline{\upsilon}_1(-\infty), \underline{\upsilon}_2(-\infty)\big) = (1, 0)$$

*if* $c_2 \geq 2\sqrt{\alpha}$ *and* (H2) *holds.*

### 2.2 Existence and Non-existence of the four types of traveling wave solutions

In this subsection, we establish the existence of four types of traveling wave solutions by using iteration sequences. Suppose that $(\bar{\psi}_1, \bar{\psi}_2)$ and $(\underline{\psi}_1, \underline{\psi}_2)$ are a pair of coupled upper and lower solutions of (7). Similarly as in [**17**], a lemma is given to construct the monotonous iteration schemes between $(\bar{\psi}_1, \bar{\psi}_2)$ and $(\underline{\psi}_1, \underline{\psi}_2)$.

**Lemma 3.** *Let* $y : \mathbf{R} \to \mathbf{R}$ *be a bounded classical solution of the following impulsive equation*

$$dy'' + ay' + by + f(t) = 0,\ (t \neq t_j),\ y'(t_j^+) - y'(t_j^-) = \beta_j,\ j = 1, 2, \ldots, m.$$

*where* $\{t_j\}$ *is a finite increasing sequence,* $f : \mathbf{R} \to \mathbf{R}$ *is bounded and continuous for every* $t \neq t_j$*. Assume that* $dz^2 + az + b = 0$ *has one negative and one positive root* $\lambda < 0 < \mu$*. Then*

$$y(t) = \frac{1}{d(\mu - \lambda)} \left[\int_{-\infty}^{t} \varrho(t,s) f(s) \mathrm{d}s + \int_{t}^{+\infty} \varrho(t,s) f(s) \mathrm{d}s\right] - \frac{1}{(\mu - \lambda)} \sum_{j=1}^{m} \min\left\{e^{\lambda(t-t_j)}, e^{\mu(t-t_j)}\right\} \beta_j,$$

*where*

$$\varrho(t,s) = \begin{cases} e^{\lambda(t-s)}, & s \leq t, \\ e^{\mu(t-s)}, & s \geq t. \end{cases}$$

**Lemma 4.** *Let* $y : \mathbf{R} \to \mathbf{R}$ *be a bounded classical solution of the following equation*

$$y' + \delta y = f(t),$$

*where* $f : \mathbf{R} \to \mathbf{R}$ *is bounded and continuous on* $\mathbf{R}$*. Then*

$$y(t) = e^{-\delta t} \int_{-\infty}^{t} e^{\delta h} f(h) \mathrm{d}h.$$

Lemmas 3 and 4 can be easily proved by direct calculation, so their proofs are omitted.

Suppose that $z, \hat{\varphi}, \hat{\psi}, \ \check{\varphi}, \check{\psi}$ are bounded continuous function. We define the following operators.

$$
\begin{aligned}
&\bar{\mathfrak{L}}_1 z := -z' - Az,\ \bar{\mathfrak{L}}_2 z := \frac{1}{c_2^2} z'' - z' - Bz,\\
&T_1^{\hat{\psi}}(\hat{\varphi}) := \int_{-\infty}^{s} e^{-A(t-s)} H_1(\hat{\varphi}, \hat{\psi})(s) \mathrm{d}s,\\
&T_2^{\check{\varphi}}(\check{\psi}) := \frac{c_2^2}{(\lambda_{22} - \lambda_{21})} \int_{-\infty}^{+\infty} k(t,s) H_2(\check{\varphi}, \check{\psi})(s) \mathrm{d}s,\\
&H_1(\hat{\varphi}, \hat{\psi}) = A\hat{\varphi} + \hat{\varphi}\left(1 - \hat{\varphi} - \frac{\beta_1 \hat{\psi}}{\hat{\varphi} + k_1}\right),\\
&H_2(\check{\varphi}, \check{\psi}) = B\check{\psi} + \alpha\check{\psi}\left(1 - \frac{\beta_2 \check{\psi}}{\check{\varphi} + k_2}\right),
\end{aligned} \tag{11}
$$

where $A = 1 + \dfrac{\beta_1(1+k_2)}{k_1\beta_2},\ B = \alpha(1 + 2/k_2)$, and

$$
\lambda_{21} = \frac{c_2^2 - c_2\sqrt{c_2^2 + 4B}}{2}, \lambda_{22} = \frac{c_2^2 + c_2\sqrt{c_2^2 + 4B}}{2},\ k(t,s) = \begin{cases} e^{\lambda_{21}(t-s)}, & s \le t,\\ e^{\lambda_{22}(t-s)}, & s \ge t. \end{cases} \tag{12}
$$

Suppose that $(\bar{\psi}_1, \bar{\psi}_2) - (\underline{\psi}_1, \underline{\psi}_2)$ are a pair of weak upper-lower solutions of (7), we define the sets

$$
\begin{aligned}
&\Lambda_1 := \{\varphi \in \mathbf{C}(\mathbf{R}) | \underline{\psi}_1(t) \le \varphi(t) \le \bar{\psi}_1(t),\ t \in \mathbf{R}\},\\
&\Lambda_2 := \{\varphi \in \mathbf{C}(\mathbf{R}) | \underline{\psi}_2(t) \le \varphi(t) \le \bar{\psi}_2(t),\ t \in \mathbf{R}\}.
\end{aligned} \tag{13}
$$

**Lemma 5.** *Suppose that* $z_1 = T_1^{\hat{\psi}}(\hat{\varphi}),\ z_2 = T_2^{\check{\varphi}}(\check{\psi})$, *then* $(z_1, z_2)$ *are the bounded solutions of the system*

$$
\bar{\mathfrak{L}}_1 z_1 + H_1(\hat{\varphi}, \hat{\psi}) = 0,\ \bar{\mathfrak{L}}_2 z_2 + H_2(\check{\varphi}, \check{\psi}) = 0. \tag{14}
$$

*Moreover, if* $\hat{\varphi}, \check{\varphi} \in \Lambda_1,\ \hat{\psi}, \check{\psi} \in \Lambda_2$, *then* $z_1 \in \Lambda_1,\ z_2 \in \Lambda_2$.

**Proof.** By direct calculation, we can easily prove that (14) holds, so we omit it here. We prove $z_2 \le \bar{\psi}_2$. Let $\beta(t) = z_2 - \bar{\psi}_2$, then there exists $h(t) \le 0$ such that

$$
\frac{D}{c_2^2}\beta''(t) - \beta'(t) - B\beta(t) + h(t) = 0,\ t \ne t_j,\ \beta(t_j^+) - \beta(t_j^-) = \bar{\psi}_2(t_j^-) - \bar{\psi}_2(t_j^+).
$$

Since $\beta(t)$ is bounded, then from Lemma 3, it follows that

$$
\beta(t) = \frac{c_2^2}{D(\lambda_{22} - \lambda_{21})} \left[\int_{-\infty}^{t} e^{\lambda_{21}(t-s)} h(s) \mathrm{d}s + \int_{t}^{+\infty} e^{\lambda_{22}(t-s)} h(s) \mathrm{d}s\right]
$$

$$-\frac{1}{(\lambda_{22}-\lambda_{21}))}\sum_{k=1}^{m}\min\left\{e^{\lambda_{21}(t-t_j)}, e^{\lambda_{22}(t-t_j)}\right\}\left(\bar{\psi}_2(t_j^-)-\bar{\psi}_2(t_j^+)\right).$$

Thus $\beta(t)\le 0$, i.e $z_2\le\bar{\psi}_2$. Similarly, we prove that $z_1\le\bar{\psi}_1$ and $\underline{\psi}_i\le z_i$ $(i=1,2)$ which completes the proof. $\square$

Now, we can construct the monotonous schemes.

Step 1. We define $s_1^{(0)}=\bar{\psi}_1$, $s_1^{(n)}:=T_1^{\underline{\psi}_2}(s_1^{(n-1)})$. By using comparison principle, the scheme $\{s_1^{(n)}\}$ satisfies

$$\underline{\psi}_1\le s_1^{(n+1)}\le s_1^{(n)}\le\cdots\le s_1^{(2)}\le s_1^{(1)}\le s_1^{(0)}=\bar{\psi}_1.$$

Thus there exists $\bar{s}_1$ such that

$$\underline{\psi}_1\le\bar{s}_1\le\bar{\psi}_1,\ \bar{s}_1=T_1^{\underline{\psi}_2}(\bar{s}_1).$$

Define $v_1^{(0)}=\bar{\psi}_2$, $v_1^{(n)}=T_2^{\bar{s}_1}(v_1^{(n-1)})$. Similarly, we have

$$\underline{\psi}_2\le v_1^{(n+1)}\le v_1^{(n)}\le\cdots\le v_1^{(2)}\le v_1^{(1)}\le v_1^{(0)}=\bar{\psi}_2.$$

Consequently, there exists $\bar{v}_1$ such that

$$\underline{\psi}_2\le\bar{v}_1\le\bar{\psi}_2,\ \bar{v}_1=T_2^{\bar{s}_1}(\bar{v}_1).$$

Step 2. We define $\bar{s}_k$ $(k\ge 3)$, such that $\bar{s}_k=T_1^{\bar{v}_{k-1}}(\bar{s}_k),\bar{v}_k=T_2^{\bar{s}_k}(\bar{v}_k)$. The comparison principle leads to

$$\begin{aligned}&\underline{\psi}_1\le\bar{s}_{k+1}\le\bar{s}_k\le\cdots\le\bar{s}_1\le\bar{\psi}_1,\ \text{for } k\ge 1,\\&\underline{\psi}_2\le\bar{v}_{k+1}\le\bar{v}_k\le\cdots\le\bar{v}_1\le\bar{\psi}_2,\ \text{for } k\ge 1.\end{aligned}$$

Step 3. We obtain the limits of $\{\bar{s}_k\}$ and $\{\bar{v}_k\}$ for $k\to\infty$, then there exist $\bar{s}\in\Lambda_1$ and $\bar{v}\in\Lambda_2$ such that $\bar{s}=\lim\limits_{k\to\infty}\bar{s}_k$, $\bar{v}=\lim\limits_{k\to\infty}\bar{v}_k$ and $\bar{s}=T_1^{\bar{v}}(\bar{s})$, $\bar{v}=T_2^{\bar{s}}(\bar{v})$, which are the solutions of

$$\bar{\mathfrak{L}}_1\bar{s}+H_1(\bar{s},\bar{v})=0,\ \bar{\mathfrak{L}}_2\bar{v}+H_2(\bar{s},\bar{v})=0.$$

Thus $(\bar{s},\bar{v})$ is the solution of (7) between $(\bar{\psi}_1,\bar{\psi}_2)$ and $(\underline{\psi}_1,\underline{\psi}_2)$. Based on the above, we can now give the following theorem.

**Theorem 2.** *Suppose that* (7) *has a pair of weak upper solution and weak lower solution* $(\bar{\psi}_1,\bar{\psi}_2)-(\underline{\psi}_1,\underline{\psi}_2)$ *with* $0\le\underline{\psi}_i\le\bar{\psi}_i$ *and* $\underline{\psi}_i\not\equiv 0$ *for* $i=1,2$. *Then* (7) *has a solution* $(\psi_1,\psi_2)$ *with* $\underline{\psi}_i\le\psi_i\le\bar{\psi}_i$ *for* $i=1,2$.

Theorem 1 and Theorem 2 lead to

**Theorem 3.** *Suppose that* $c_1,c_2>0$, *then the following statements hold.*

(i) *Suppose that* (H1) *holds, then type I-II wave solutions exist if* $c_2 \geq 2\sqrt{\alpha}$, *neither of them exists if* $c_2 < 2\sqrt{\alpha}$;

(ii) *Suppose that* (H1) *and* $u^* < \beta_2$ *hold, then type III wave solution exists if* $c_1 > 0$.

(iii) *Suppose that* (H2) *holds, then type IV wave solution exists if* $c_2 \geq 2\sqrt{\alpha}$ *and does not exist if* $c_2 < 2\sqrt{\alpha}$.

**Proof.** The existence of the four types of wave solutions follows from Theorem 1 and Theorem 2. Next, we prove the non-existence by using reduction. From Lemma 1, we have $\lim\limits_{t\to+\infty} \inf_{|x|<ct} w(t,x) \geq \dfrac{k_2}{\beta_2}$ for each $c < 2\sqrt{\alpha}$ if $w_0(x) \geq 0$ and $w_0(x) \not\equiv 0$. Suppose that (1) has a traveling wave solution $(U(x+c_1t), W(x+c_1t))$ with the speed $c_1 < 2\sqrt{\alpha}$ and the boundary condition $W(-\infty) = 0$. Choose $c_1 < c_2 < 2\sqrt{\alpha}$, then $w(t,x) = W(x+c_1t)$ has the following property:

$$\frac{k_2}{\beta_2} \leq \lim_{t\to+\infty} \inf_{x=-c_2t} w(t,x) = \liminf_{t\to+\infty} W((c_1-c_2)t) = W(-\infty) = 0,$$

which is a contradiction. Thus (1) has no traveling wave solutions with the boundary condition $W(-\infty) = 0$ if $c < 2\sqrt{\alpha}$. Hence the non-existence of types (I),(II) and (IV) is established which completes the proof.

## 3. TRAVELING WAVE SOLUTIONS AND MINIMAL WAVE SPEEDS IN A 2D STRIP

We consider separately traveling wave solutions of (4) that are independent of $y$ and those that depend on $y$.

### 3.1. Wave solutions independent of $y$

Set $s_i = x + c_it,\ \phi_i(t,x) = q_i(x+c_it) = \psi_i^x(t), (i = 1,2)$, then

$$\frac{\partial \phi_i}{\partial t} = c_i \frac{\mathrm{d}q_i}{\mathrm{d}s_i} = \frac{\mathrm{d}\psi_i}{\mathrm{d}t},\ \frac{\partial^2 \phi_i}{\partial x^2} = \frac{1}{c_i^2}\frac{\mathrm{d}^2\psi_i}{\mathrm{d}t^2},$$

if we substitute $\phi_i(x+c_it)$ into the system (4), we obtain the following wave system of (4)

$$\begin{aligned} &-\psi_1'(t) + \psi_1(t)\left(1 - \psi_1(t) - \frac{\beta_1\psi_2(t)}{\psi_1(t)+k_1}\right) = 0,\\ &\frac{D}{c_2^2}\psi_2''(t) - \psi_2'(t) + \alpha\psi_2(t)\left(1 - \frac{\beta_2\psi_2(t)}{\psi_1(t)+k_2}\right) = 0. \end{aligned} \tag{15}$$

here we write $\psi_i^x(t)$ as $\psi_i(t)$ for convenience. According to type I-type IV wave solutions in section 2, there are four types of wave solutions for this category. We denote them by Type IA - Type IVA. Their existence can be obtained in a similar way as described in Section 2, therefore we only give the theorem and omit the proof.

**Theorem 4.** *Suppose that $c_1, c_2 > 0$, then the following statements hold.*

(i) *Suppose that* (H1) *holds, then type IA-IIA wave solutions exist if $c_2 \geq 2\sqrt{\alpha}$, neither of them exists if $c_2 < 2\sqrt{\alpha}$;*

(ii) *Suppose that* (H1) *and $u^* < \beta_2$ hold, then type IIIA wave solution exists if $c_1 > 0$.*

(iii) *Suppose that* (H2) *holds, then type IVA wave solution exists if $c_2 \geq 2\sqrt{\alpha}$ and does not exist if $c_2 < 2\sqrt{\alpha}$.*

### 3.2. Wave solutions dependent on $y$

In this subsection we find the solution which is dependent on $y$. Set $s_i = x + c_i t$, $\phi_i(t,x,y) = R_i(x + c_i t, y) = \psi_i^x(t,y), (i = 1,2)$, then

$$\frac{\partial \phi_i}{\partial t} = c_i \frac{\partial R_i}{\partial s_i} = \frac{\partial \psi_i}{\partial t}, \ \frac{\partial^2 \phi_i}{\partial x^2} = \frac{1}{c_i^2} \frac{\partial^2 \psi_i}{\partial t^2},$$

if we substitute $\phi_i(t,x,y)$ into the system (4), we obtain the wave system of (4) as follows:

$$(16)\quad \begin{aligned} & -\frac{\partial \psi_1}{\partial t} + \psi_1(t,y)\left(1 - \psi_1(t,y) - \frac{\beta_1 \psi_2(t,y)}{\psi_1(t,y) + k_1}\right) = 0, \ t \in \mathbf{R}, \ y \in (-L, L), \\ & \frac{D}{c_2^2}\left(\frac{\partial^2 \psi_2}{\partial t^2} + \frac{\partial^2 \psi_2}{\partial y^2}\right) - \frac{\partial \psi_2}{\partial t} + \alpha \psi_2(t,y)\left(1 - \frac{\beta_2 \psi_2(t,y)}{\psi_1(t,y) + k_2}\right) = 0, \\ & t \in \mathbf{R}, \ y \in (-L, L), \\ & \frac{\partial \psi_2}{\partial y} = 0, \ t \in \mathbf{R}, \ y \in \{-L, L\}, \end{aligned}$$

here we write $\psi_i^x(t,y)$ as $\psi_i(t,y)$ for convenience. Similar to section 3.1, we denote the corresponding four types of solutions by type IB - type IVB.

If we consider (16) as an elliptic system, some previous results on weak upper and lower solution should be provided first. Suppose that $\Omega$ is a smooth region in $\mathbf{R}^2$ with $\partial\Omega$ defined as follows.

$$\partial\Omega := L_1 \cup L_2 \cup \Gamma_1 \cup \Gamma_2,$$

where $L_1 : \ y = L, \ -T \leq t \leq T$; $L_2 : \ y = -L, \ -T \leq t \leq T$; $\Gamma_1$ and $\Gamma_2$ are the corresponding curves connecting $L_1$ and $L_2$ so that $\partial\Omega$ is smooth.

Suppose that $f(t,y,u) \in \mathbf{C}^\alpha(\bar{\Omega} \times \mathbf{R})$, consider the equation

$$(17)\quad \begin{aligned} & -\Delta u + \frac{\partial u}{\partial t} = f(t,y,u), (t,y) \in \Omega; \\ & \frac{\partial u}{\partial \mathbf{n}} = 0, \ (t,y) \in \partial\Omega, \ \mathbf{n} \text{ is the outer normal vector of } \partial\Omega, \end{aligned}$$

based on Theorem 3.10.2 in [**15**], we have the following lemma.

**Lemma 6.** *Suppose that* $\mathcal{D}_i \subset\subset \Omega$, $(i = 1, 2, \ldots, m)$, $\Omega = \sum_{i=1}^{m} \bar{\mathcal{D}}_i$, $\bar{\mathcal{D}}_i \bigcap \bar{\mathcal{D}}_{i+1} = \Sigma_i, (i = 1, 2, \ldots, m-1)$, $\bar{\mathcal{D}}_i \bigcap \bar{\mathcal{D}}_j = \emptyset$ *if* $|j - i| \neq 1$. *Then if*

$$
\begin{aligned}
& -\Delta w + \frac{\partial w}{\partial t} \geq f(t, y, w), (t, y) \in \mathcal{D}_i, i = 1, 2, \ldots, m; \\
& \frac{\partial w}{\partial \ell_i^+} + \frac{\partial w}{\partial \ell_i^-} \geq 0,\ (t, y) \in \Sigma_i,\ i = 1, 2, \ldots, m-1; \\
& \frac{\partial w}{\partial \mathbf{n}} \geq 0,\ (t, y) \in \partial\Omega,
\end{aligned}
\tag{18}
$$

*where* $\ell_i^+$ *is the normal vector of the curve* $\Sigma_i$ *along the direction in which* $t$ *is increasing and* $\ell_i^-$ *is the normal vector of the curve* $\Sigma_i$ *along the direction in which* $t$ *is decreasing. Then* $w$ *is a weak upper solution of* (17) *and we can define the weak lower solution of* (17) *if we converse the signs of inequalities in* (18).

**Lemma 7** ([**15**]). *Suppose that* $\bar{w}$ *and* $\underline{w}$ *are the weak upper-lower solution of* (17) *where* $f(t, y, \psi) \in \mathbf{C}(\Omega \times [\underline{L}, \bar{L}])$. *If* $\bar{w}, \underline{w} \in \mathbf{L}^\infty(\boldsymbol{\Omega}), \bar{w} \geq \underline{w}$, *then there exist* $\hat{w}, \check{w}$ *which are minimal and maximal solutions, respectively, of equation* (17) *between* $\underline{w}$ *and* $\bar{w}$.

To obtain the solution of (16), we consider the elliptic system in the domain $\Omega_n$, $n \in \mathbf{N}$. Define an increasing series $\{T_n\}$ with $\lim_{n\to+\infty} T_n = +\infty$, set the bounded $\Omega_n \in \mathbf{R} \times [-L, L]$ with the boundary as

$$\partial\Omega_n := L_1^{(n)} \cup L_2^{(n)} \cup \Gamma_1^{(n)} \cup \Gamma_2^{(n)},$$

where $L_1^{(n)}:\ y = L,\ -T_n \leq t \leq T_n;\ L_2^{(n)}:\ y = -L,\ -T_n \leq t \leq T_n;\ \Gamma_1^{(n)}$ and $\Gamma_2^{(n)}$ are the corresponding curves connecting $L_1^{(n)}$ and $L_2^{(n)}$ so that $\partial\Omega_n$ is smooth.

Define

$$< u_1, u_2 >_{\Omega_n} := \{\varphi \in \mathbf{C}(\Omega_n, \mathbf{R}) | u_1 \leq \varphi \leq u_2 \text{ for all } (t, y) \in \Omega_n\}.$$

We first consider the following system.

$$
\begin{aligned}
& -\frac{\partial \psi_1}{\partial t} + \psi_1(t, y)\left(1 - \psi_1(t, y) - \frac{\beta_1 \psi_2(t, y)}{\psi_1(t, y) + k_1}\right) = 0,\ (t, y) \in \Omega_n, \\
& \frac{D}{c_2^2}\left(\frac{\partial^2 \psi_2}{\partial t^2} + \frac{\partial^2 \psi_2}{\partial y^2}\right) - \frac{\partial \psi_2}{\partial t} + \alpha\psi_2(t, y)\left(1 - \frac{\beta_2 \psi_2(t, y)}{\psi_1(t, y) + k_2}\right) = 0,\ (t, y) \in \Omega_n, \\
& \frac{\partial \psi_2}{\partial \mathbf{n}} = 0,\ (t, y) \in \partial\Omega_n.
\end{aligned}
\tag{19}
$$

The definition of the weak coupled upper and lower solutions of (19) is given as follows.

**Definition 2.** *Define*

$$\Sigma_i := \big\{(t,y) : t = \tilde{T}_i,\ y \in (-L,L)\big\},\ i = 1,2,\ldots,m,\ \Sigma = \cup_{i=1}^m \Sigma_i;$$

$$\Lambda_1 := \left\{\psi : \Omega_n \to \mathbf{R}, \frac{\partial \psi}{\partial t}\ \begin{matrix}\textit{exist and are essentially}\\ \textit{bounded for all}\end{matrix}\ (t,y) \in \Omega_n \setminus \Sigma.\right\}$$

$$\Lambda_2 := \Big\{\psi : \Omega_n \to \mathbf{R},$$

$$\left.\frac{\partial \psi}{\partial t}, \frac{\partial \psi}{\partial y}\ \text{and}\ \frac{\partial^2 \psi}{\partial t^2}, \frac{\partial^2 \psi}{\partial y^2}\ \begin{matrix}\textit{exist and are essentially}\\ \textit{bounded for all}\end{matrix}\ (t,y) \in \Omega_n \setminus \Sigma.\right\}$$

*Suppose that* $\bar{\psi}_i, \underline{\psi}_i \in \Lambda_i, i = 1,2$. *A pair of continuous functions* $(\bar{\psi}_1, \bar{\psi}_2)$ *and* $(\underline{\psi}_1, \underline{\psi}_2)$ *are called weak coupled upper and lower solutions of* (19) *if they satisfy*

$$(20)\qquad \begin{aligned}
&-\frac{\partial \bar{\psi}_1}{\partial t} + \bar{\psi}_1\left(1 - \bar{\psi}_1 - \frac{\beta_1 \underline{\psi}_2}{\bar{\psi}_1 + k_1}\right) \le 0,\ (t,y) \in \Omega_n \setminus \Sigma,\\
&-\frac{\partial \underline{\psi}_1}{\partial t} + \underline{\psi}_1\left(1 - \underline{\psi}_1 - \frac{\beta_1 \bar{\psi}_2}{\underline{\psi}_1 + k_1}\right) \ge 0,\ (t,y) \in \Omega_n \setminus \Sigma,\\
&\frac{D}{c_2^2}\left(\frac{\partial^2 \bar{\psi}_2}{\partial t^2} + \frac{\partial^2 \bar{\psi}_2}{\partial y^2}\right) - \frac{\partial \bar{\psi}_2}{\partial t} + \alpha \bar{\psi}_2\left(1 - \frac{\beta_2 \bar{\psi}_2}{\bar{\psi}_1 + k_2}\right) \le 0,\ (t,y) \in \Omega_n \setminus \Sigma,\\
&\frac{\partial \bar{\psi}}{\partial \ell_i^+} - \frac{\partial \bar{\psi}}{\partial \ell_i^-} \ge 0,\ (t,y) \in \Sigma_i,\ i = 1,2\ldots,m,\\
&\frac{D}{c_2^2}\left(\frac{\partial^2 \underline{\psi}_2}{\partial t^2} + \frac{\partial^2 \underline{\psi}_2}{\partial y^2}\right) + \alpha \underline{\psi}_2\left(1 - \frac{\beta_2 \underline{\psi}_2}{\underline{\psi}_1 + k_2}\right) \ge 0,\ (t,y) \in \Omega_n \setminus \Sigma,\\
&\frac{\partial \underline{\psi}}{\partial \ell_i^+} - \frac{\partial \underline{\psi}}{\partial \ell_i^-} \le 0,\ (t,y) \in \Sigma_i,\ i = 1,2,\ldots,m,\\
&\frac{\partial \underline{\psi}_2}{\partial \mathbf{n}} \le 0 \le \frac{\partial \bar{\psi}_2}{\partial \mathbf{n}},\ (t,y) \in \partial\Omega_n.
\end{aligned}$$

*where* $\ell_i^+$ *is the normal vector of the curve* $\Sigma_i$ *along the direction in which* $t$ *is increasing and* $\ell_i^-$ *is the normal vector of the curve* $\Sigma_i$ *along the direction in which* $t$ *is decreasing.*

**Lemma 8.** *Suppose that functions* $(\bar{\psi}_1^{(n)},\ \bar{\psi}_2^{(n)}) - (\underline{\psi}_1^{(n)}, \underline{\psi}_2^{(n)})$ *are a pair of weak coupled upper-lower solution of* (19), $n \in \mathbf{N}$. *Then* (16) *has a solution* $(\psi_1, \psi_2)$ *with* $\underline{\psi}_i \le \psi_i \le \bar{\psi}_i$ *for* $i = 1,2$, *where* $\underline{\psi}_i = \inf_{n\in\mathbf{N}}\{\underline{\psi}_i^{(n)}\}, \bar{\psi}_i = \sup_{n\in\mathbf{N}}\{\bar{\psi}_i^{(n)}\}$.

**Proof.** Consider the system

$$(21)\qquad -\frac{\partial \psi}{\partial t} + \psi(t,y)\left(1 - \psi(t,y) - \frac{\beta_1 \underline{\psi}_2^{(n)}(t,y)}{\psi(t,y) + k_1}\right) = 0,\ (t,y) \in \Omega_n.$$

Since $\underline{\psi}_1^{(n)}, \bar{\psi}_1^{(n)}$ is a pair of weak lower-upper solutions of (21), there exists $\psi_{1n}^{(1)}(t,y)$ which is the solution of (21) and it holds that $\psi_{1n}^{(1)} \in < \underline{\psi}_1^{(n)}, \bar{\psi}_1^{(n)} >_{\Omega_n}$ which can be verified by simple calculation. Consider a convergent subsequence from

$\{\psi_{1n}^{(1)}(t,y)\}$, denoted for convenience by $\{\psi_{1n}^{(1)}(t,y)\}$, then it holds that $\psi_1^{(1)}(t,y) = \lim_{n\to+\infty} \psi_{1n}^{(1)}(t,y)$.

Consider the equation

$$\begin{aligned}
&\frac{D}{c_2^2}\left(\frac{\partial^2\psi}{\partial t^2}+\frac{\partial^2\psi}{\partial y^2}\right)-\frac{\partial\psi}{\partial t}+\alpha\psi(t,y)\left(1-\frac{\beta_2\psi(t,y)}{\psi_{1n}^{(1)}(t,y)+k_2}\right)=0,\ (t,y)\in\Omega_n,\\
&\frac{\partial\psi}{\partial\mathbf{n}}=0,\ (t,y)\in\partial\Omega_n.
\end{aligned}\tag{22}$$

Since $(\underline{\psi}_2^{(n)},\bar{\psi}_2^{(n)})$ is a pair of weak lower-upper solutions of (22), then, similar to above, there exists $\psi_2^{(1)}(t,y)=\lim_{n\to+\infty}\psi_{2n}^{(1)}(t,y)$. Moreover, $\psi_1^{(1)}(t,y)$ and $\psi_2^{(1)}(t,y)$ satisfies

$$\begin{aligned}
&\frac{D}{c_2^2}\Big(\frac{\partial^2\psi_2^{(1)}}{\partial t^2}+\frac{\partial^2\psi_2^{(1)}}{\partial y^2}\Big)-\frac{\partial\psi_2^{(1)}}{\partial t}+\alpha\psi_2^{(1)}\Big(1-\frac{\beta_2\psi_2^{(1)}}{\psi_1^{(1)}+k_2}\Big)=0,\ (t,y)\in\mathbf{R}\times(-L,L),\\
&\frac{\partial\psi_2^{(1)}}{\partial\mathbf{n}}=0,\ t\in\mathbf{R},\ y\in\{-L,L\}.
\end{aligned}\tag{23}$$

Consider the equation

$$-\frac{\partial\psi}{\partial t}+\psi(t,y)\left(1-\psi(t,y)-\frac{\beta_1\psi_{2n}^{(1)}(t,y)}{\psi(t,y)+k_1}\right)=0,\ (t,y)\in\Omega_n.\tag{24}$$

Since $(\underline{\psi}_1^{(n)},\psi_{1n}^{(1)})$ is a pair of weak lower-upper solutions of (24), then $\psi_{1n}^{(2)}(t,y)$ is the solution of (24) and $\psi_{1n}^{(2)}\in<\underline{\psi}_1^{(n)},\psi_{1n}^{(1)}>_{\Omega_n}$. Also, $\psi_1^{(2)}(t,y)=\lim_{n\to+\infty}\psi_{1n}^{(2)}(t,y)\le\psi_1^{(1)}(t,y)$ where $\psi_1^{(2)}(t,y)$ and $\psi_2^{(1)}(t,y)$ satisfy

$$-\frac{\partial\psi_1^{(2)}}{\partial t}+\psi_1^{(2)}(t,y)\left(1-\psi_1^{(2)}(t,y)-\frac{\beta_1\psi_2^{(1)}(t,y)}{\psi_1^{(2)}(t,y)+k_1}\right)=0,\ (t,y)\in\mathbf{R}\times(-L,L).\tag{25}$$

Consider the system

$$\begin{aligned}
&\frac{D}{c_2^2}\left(\frac{\partial^2\psi}{\partial t^2}+\frac{\partial^2\psi}{\partial y^2}\right)-\frac{\partial\psi}{\partial t}+\alpha\psi(t,y)\left(1-\frac{\beta_2\psi(t,y)}{\psi_{1n}^{(2)}(t,y)+k_2}\right)=0,\ (t,y)\in\Omega_n,\\
&\frac{\partial\psi}{\partial\mathbf{n}}=0,\ (t,y)\in\partial\Omega_n.
\end{aligned}\tag{26}$$

Since $(\underline{\psi}_2^{(n)},\psi_{2n}^{(1)})$ is a pair of weak lower-upper solutions of (26), then, similarly $\psi_2^{(2)}(t,y)=\lim_{n\to+\infty}\psi_{2n}^{(2)}(t,y)\le\psi_2^{(1)}(t,y)$. Moreover, $\psi_1^{(2)}(t,y)$ and $\psi_2^{(2)}(t,y)$ satisfy

$$\begin{aligned}
&\frac{D}{c_2^2}\Big(\frac{\partial^2\psi_2^{(2)}}{\partial t^2}+\frac{\partial^2\psi_2^{(2)}}{\partial y^2}\Big)-\frac{\partial\psi_2^{(2)}}{\partial t}+\alpha\psi_2^{(2)}\Big(1-\frac{\beta_2\psi_2^{(2)}}{\psi_1^{(2)}+k_2}\Big)=0,\ (t,y)\in\mathbf{R}\times(-L,L),\\
&\frac{\partial\psi_2^{(2)}}{\partial\mathbf{n}}=0,\ t\in\mathbf{R},\ y\in\{-L,L\}.
\end{aligned}\tag{27}$$

By repeating the above iterative steps, we obtain the following two monotonous schemes.

$$\underline{\psi}_1 \leq \cdots \leq \psi_1^{(n+1)} \leq \psi_1^{(n)} \leq \cdots \leq \psi_1^{(1)} \leq \bar{\psi}_1,$$
$$\underline{\psi}_2 \leq \cdots \leq \psi_2^{(n+1)} \leq \psi_2^{(n)} \leq \cdots \leq \psi_1^{(1)} \leq \bar{\psi}_2,$$

hence $\psi_1, \psi_2$ are the limits of $\psi_1^{(n)}, \psi_2^{(n)}$ such that (19) holds. Thus the lemma is proved. □

From Lemma 8, to establish the existence of type IB wave solution, we construct the weak upper and lower solutions of (19) by doing some modifications on $(\bar{\phi}_1, \bar{\phi}_2) - (\underline{\phi}_1, \underline{\phi}_2)$ proposed in section 2. Define

$$\begin{aligned}
\varphi_1^{(n)}(t) &= w^*\big(1 + \hat{\ell}_2 e^{-s(T_n-\delta)}\big) e^{\bar{\zeta}_1(t-T_n+\delta)},\\
\varphi_2(t) &= \frac{1+k_2}{\beta_2} - ae^{mt},\\
\varphi_3^{(n)}(t) &= w^*\big(1 - \ell_2 e^{-s(T_n-\delta)} e^{\varepsilon_2(t-(T_n-\delta))}\big) e^{\kappa_1(t-(T_n-\delta))},\\
\varphi^*(t) &= \frac{1+k_2}{\beta_2} - ae^{mt} + be^{\bar{m}t},
\end{aligned}$$

where

$$\begin{aligned}
w^*(1+\hat{\ell}_2)e^{\bar{\zeta}_1(-T_n+\delta)} = \frac{1+k_2}{\beta_2} - ae^{m(-T_n+\delta)},\ m > \frac{1+\sqrt{1+\frac{4D\alpha}{c_2^2}}}{2} = r_1,\\
\frac{\kappa_1}{\kappa_1+\varepsilon_2} < \hat{\ell}_2 e^{-s(T_n-\delta)},\\
am = b\bar{m},\ (\varphi^*)'(0) = 0,\ 0 < \bar{m} < \bar{\zeta}_1 < r_1 < m,\ a > 0, b > 0,
\end{aligned}$$

Note that

$$\begin{aligned}
&\frac{D}{c_2^2}\varphi^{*\prime\prime} - \varphi^{*\prime} + \alpha\varphi^*\left(1 - \frac{\beta_2\varphi^*}{1+k_2}\right)\\
&= \frac{D}{c_2^2}\varphi^{*\prime\prime} - \varphi^{*\prime} + \frac{\alpha\varphi^*\beta_2}{1+k_2}(ae^{mt} - be^{\bar{m}t}) \leq \frac{D}{c_2^2}\varphi^{*\prime\prime} - \varphi^{*\prime} + \alpha a e^{mt}\\
&= (-a)e^{mt}\left[\frac{D}{c_2^2}m^2 - m - \alpha\right] + be^{\bar{m}t}\left[\frac{D}{c_2^2}\bar{m}^2 - \bar{m} - \alpha\right] < 0,
\end{aligned}$$

and there exist $\tilde{t}_1 < 0$ and $\tilde{t}_2 > 0$ such that

$$\begin{aligned}
&-ame^{m\tilde{t}_1} + b\bar{m}e^{\bar{m}\tilde{t}_1} > w^*(1+\hat{\ell}_2)\bar{\zeta}_1 e^{\bar{\zeta}_1\tilde{t}_1},\\
&-ame^{m\tilde{t}_2} + b\bar{m}e^{\bar{m}\tilde{t}_2} < -sw^*\hat{l}_2 e^{-s\tilde{t}_2}.
\end{aligned}\tag{28}$$

Define

(29)
$$\bar{\phi}_1^{(n)}=\begin{cases} u^*(1+\hat{\ell}_1 e^{-st}), & t>0,\\ u^*(1+\hat{\ell}_1)e^{t}, & t\le 0,\end{cases}$$
$$\bar{\phi}_2^{(n)}=\begin{cases} \varphi_1^{(n)}(t), & t\in\{t>T_n-\delta\}\cap\{t\in\Omega\},\\ w^*(1+\hat{\ell}_2 e^{-st}), & \tilde{t}_2<t<T_n-\delta,\\ \varphi^*(t), & \tilde{t}_1<t<\tilde{t}_2,\\ w^*(1+\hat{\ell}_2)e^{\bar{\zeta}_1 t}, & -T_n+\delta\le t<\tilde{t}_1,\\ \varphi_2(t), & t\in\{t<-T_n+\delta\}\cap\{t\in\Omega\},\end{cases}$$

and

(30)
$$\underline{\phi}_1^{(n)}=\begin{cases} u^*(1-\ell_1 e^{-st}), & t>0,\\ u^*(1-\ell_1 e^{-s})e^{(1-\varepsilon_1)t}, & t\le 0,\end{cases}$$
$$\underline{\phi}_2^{(n)}=\begin{cases} \varphi_3^{(n)}(t), & t\in\{t>T_n-\delta\}\cap\{t\in\Omega\},\\ w^*(1-\ell_2 e^{-st}), & 0<t<T_n-\delta,\\ w^*(1-\ell_2 e^{\varepsilon_2 t})e^{\kappa_1 t}, & t\le 0.\end{cases}$$

Here

$$\Sigma_1=\{(t,y)|t=T_n-\delta, y\in(-L,L)\},\qquad \Sigma_2=\{(t,y)|t=\tilde{t}_2, y\in(-L,L)\},$$
$$\Sigma_3=\{(t,y)|t=-T_n+\delta, y\in(-L,L)\},\quad \Sigma_4=\{(t,y)|t=\tilde{t}_1, y\in(-L,L)\}.$$

By simple calculation, we have

$$\frac{\partial\bar{\phi}_2}{\partial\ell_i^+}+\frac{\partial\bar{\phi}_2}{\partial\ell_i^-}\ge 0,\ (t,y)\in\Sigma_i,\ i=1,2,3,4,$$
$$\frac{\partial\underline{\phi}_2}{\partial\ell_i^+}+\frac{\partial\underline{\phi}_2}{\partial\ell_i^-}\le 0,\ (t,y)\in\Sigma_i,\ i=1,2,3,4.$$

Together with

$$\begin{aligned}&\frac{D}{c_2^2}\varphi_2''-\varphi_2'+\alpha\varphi_2\left(1-\frac{\beta_2\varphi_2}{1+k_2}\right)\\ &=\frac{D}{c_2^2}\varphi_2''-\varphi_2'+\frac{\alpha\varphi_2\beta_2}{1+k_2}(ae^{mt})\le\frac{D}{c_2^2}\varphi_2''-\varphi_1'+\alpha ae^{mt}\\ &=(-a)e^{mt}\left[\frac{D}{c_2^2}m^2-m-\alpha\right]<0,\end{aligned}$$

and $\frac{\partial\varphi_1^{(n)}}{\partial\mathbf{n}}|_{\partial\Omega_n}>0$, $\frac{\partial\varphi_2}{\partial\mathbf{n}}|_{\partial\Omega_n}>0$, $\frac{\partial\varphi_3^{(n)}}{\partial\mathbf{n}}|_{\partial\Omega_n}<0$, it follows that $(\bar{\phi}_1^{(n)},\ \bar{\phi}_2^{(n)})-(\underline{\phi}_1^{(n)},\underline{\phi}_1^{(n)})$ are the weak coupled upper-lower solutions of (19). In view of

$$\lim_{n\to+\infty}\underline{\phi}_1^{(n)}(T_n)=\lim_{n\to+\infty}\bar{\phi}_1^{(n)}(T_n)=u^*,$$
$$\lim_{n\to+\infty}\underline{\phi}_2^{(n)}(T_n)=\lim_{n\to+\infty}\bar{\phi}_2^{(n)}(T_n)=w^*,$$

$$\lim_{n\to+\infty} \underline{\phi}_1^{(n)}(-T_n) = \lim_{n\to+\infty} \bar{\phi}_1^{(n)}(-T_n) = 0,$$
$$\lim_{n\to+\infty} \underline{\phi}_2^{(n)}(-T_n) = \lim_{n\to+\infty} \bar{\phi}_2^{(n)}(-T_n) = 0,$$

by Lemma 8, $(U, W)$ is the solution of (16) where $(U(-\infty), W(-\infty)) = (0, 0)$, $(U(+\infty), W(+\infty)) = (u^*, w^*)$. Hence $(U, W)$ is the type IB solution of (19) with respect to $y$. Similarly, type IIB-type IVB solutions can be obtained by doing similar modifications on the corresponding weak coupled upper-lower solutions in section 2. Thus we have the following theorem.

**Theorem 5.** *For $c_1, c_2 > 0$, the following statements are valid.*

(i) *Suppose that* (H1) *holds, then type IB-IIB wave solutions exist if $c_2 \geq 2\sqrt{\alpha}$, neither of them exists if $c_2 < 2\sqrt{\alpha}$;*

(ii) *Suppose that* (H1) *and $u^* < \beta_2$ hold, then type IIIB wave solution exists if $c_1 > 0$.*

(iii) *Suppose that* (H2) *holds, then type IVB wave solution exists if $c_2 \geq 2\sqrt{\alpha}$ and does not exist if $c_2 < 2\sqrt{\alpha}$.*

## 4. Discussion, conclusion and simulation

**1. $k_1/\beta_1$ and $k_2/\beta_2$**

There are four possible equilibria of the system (1) and their meaning is analyzed in [**3**] and [**13**]. One can see that $(0, 0)$ and $(1, 0)$ are unstable, while $(u^*, w^*)$ and $(0, k_2/\beta_2)$ are stable under different conditions. This causes a change in the long time behavior of the system when moving to either $(u^*, w^*)$ or $(0, k_2/\beta_2)$. From Proposition 1, one can see that $k_1/\beta_1$ and $k_2/\beta_2$ can be used to describe the survivability of the prey $u$ and the predator $w$, respectively. For more details, we refer the reader to [**13**]. For $k_1/\beta_1 > k_2/\beta_2$, the prey has strong survivability, so it won't vanish, and thus a co-existing equilibrium $(u^*, w^*)$ exists. For $k_1/\beta_1 < k_2/\beta_2$, the prey has weak survivability and will be extinct, thus the co-existing equilibrium doesn't exist. However the predator can still survive due to its food diversity.

**2. Traveling wave solutions and invasion speeds**

The wave solutions exhibit many kinds of propagation phenomena in ecology. In a linear habitat initially not populated by neither predators nor prey, the introduction of species at one end may result in invasion of prey or predators under combined effects of both diffusion and population growth. A type I wave solution means a zone of transition from the absence of both species to a globally stable state of co-existence across the habitat. A type II wave solution means a zone of transition from absence of predators and a large amount of prey to $(u^*, w^*)$, which means that predators are successful in invasion. Similarly, a type III solution means the successful invasion of the prey. As for a type IV wave solution, it means the successful invasion of predators and the extinction of prey due to its weak survivability.

### 3. The significance of the results

Using Theorem 3 as an example, we illustrate the significance of our results. First, we point out that the invasion speed of the predator is $2\sqrt{\alpha}$ (refer to (i)-(ii) of Theorem 3). Second, condition (H1) implies the prey has strong survivability, thus the wave solution shifts towards the co-existing equilibrium $(u^*, w^*)$ if and only if the invasion speed of the predator is not less than $2\sqrt{\alpha}$, see Theorem 3 (i). As for the decrease of prey in the type (II) solution, it is caused by interspecific competition of the prey itself. Condition $u^* < \beta_2$ in (ii) of Theorem 3 means the prey needs strong survivability when the initial predator is $k_2/\beta_2$ and not 0, which is reasonable. However, (H2) in (iii) of Theorem 3 means the survivability of the prey is weak, thus it will be extinct. Therefore the predator is successful in invading if the speed of predator is not less than $2\sqrt{\alpha}$.

To conclude, the system (1) exhibits specific biological significance, and the theorems provided in this paper comply with the natural rules of ecology.

In what follows we provide the simulations of some results. Figures 1 and 2 are the traveling wave solutions in $\mathbf{R}^2$, while Figure 3 shows the graphs of the traveling wave solutions in a 2-D strip. The first two graphs of Figures 1–2 are the sectional view of the solutions of $u(t, x_1, x_2) = U(e_1x_1 + e_2x_2 + c_1t)$, $w(t, x_1, x_2) = W(e_1x_1 + e_2x_2 + c_2t)$ at $x_2 = 10$, whereas the last two graphs of show the graphs of $U(s_1), W(s_2)$, where $s_i = e_1x_1 + e_2x_2 + c_it$. Likewise, the first two graphs of Figure 3 show the section view of the solutions of $u(t, x, y) = U(x + c_1t, y)$, $W(t, x, y) = U(x+c_2t, y)$ for $y = 10$ and last two graphs are the graphs of $U(s_1, y)$, $W(s_2, y)$, where $s_1 = x + c_1t$, $s_2 = x + c_2t$. We select the parameters as follows: $k_1 = 3, k_2 = 1, \beta_1 = 1, \beta_2 = 1, D = 1, \alpha = 2, c_1 = 3, c_2 = 4$.

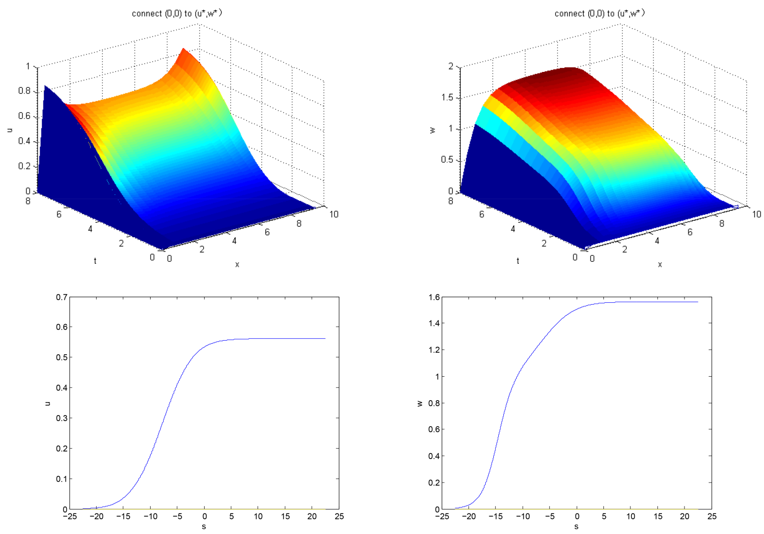


Figure 1.

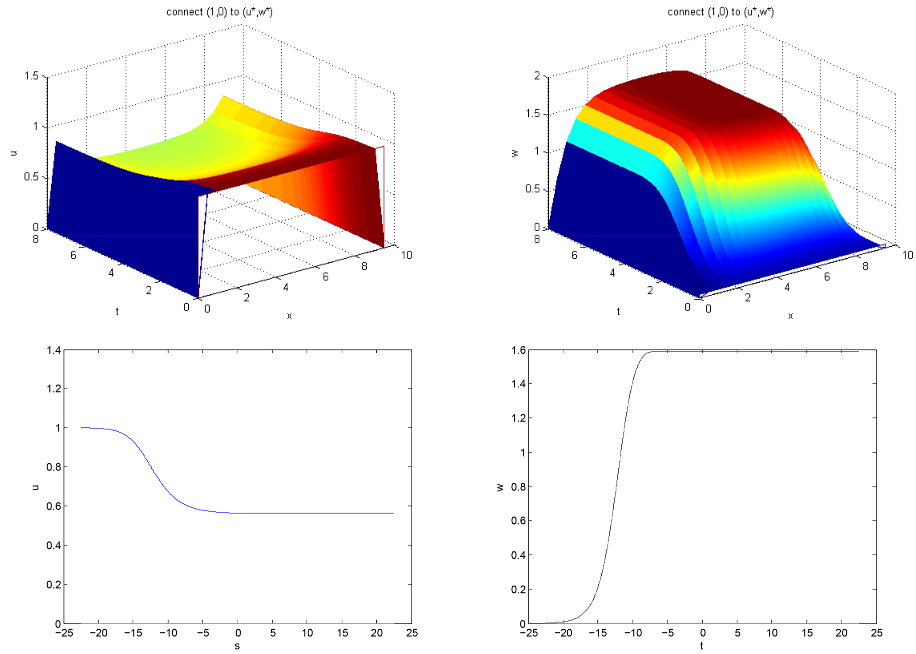


Figure 2.

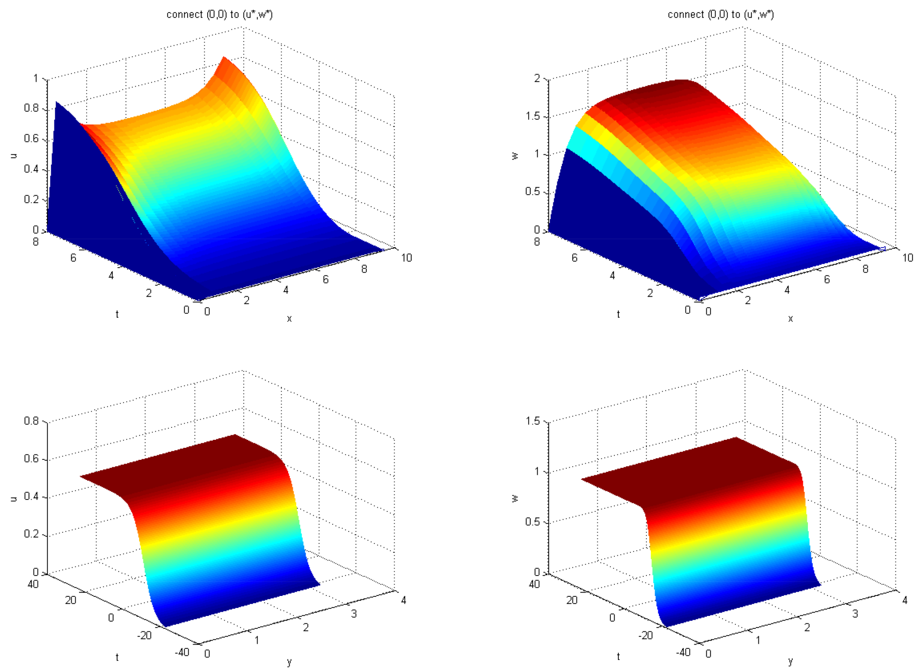


Figure 3.

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