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STABILITY OF FUNCTIONAL EQUATIONS IN NON-ARCHIMEDEAN SPACES

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We prove the generalized Hyers–Ulam stability of the Cauchy functional equation $f(x+y) = f(x)+f(y)$ and the quadratic functional equation $f(x+y)+f(x-y) = 2f(x)+2f(y)$ in non-Archimedean normed spaces.

1. INTRODUCTION AND PRELIMINARIES

A classical question in the theory of functional equations is the following: "When is it true that a function which approximately satisfies a functional equation $\mathcal{E}$ must be close to an exact solution of $\mathcal{E}$?"

If the problem accepts a solution, we say that the equation $\mathcal{E}$ is stable. The first stability problem concerning group homomorphisms was raised by Ulam [**30**] in 1940.

We are given a group G and a metric group G' with metric $\rho(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $\rho\big(f(xy), f(x)f(y)\big) < \delta$ for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $\rho\big(f(x), h(x)\big) < \epsilon$ for all $x \in G$?

Ulam's problem was partially solved by Hyers [**12**] in 1941.

Let E_1 be a normed space, E_2 a Banach space and suppose that the mapping $f : E_1 \to E_2$ satisfies the inequality

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon \qquad (x, y \in E_1),$$

where $\epsilon > 0$ is a constant. Then the limit $T(x) = \lim 2^{-n} f(2^n x)$ exists for each $x \in E_1$ and T is the unique additive mapping satisfying

$$\|f(x) - T(x)\| \le \epsilon \qquad (x \in E_1). \tag{1.1}$$

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Also, if for each x the function $t \mapsto f(tx)$ from $\mathbb{R}$ to E_2 is continuous on $\mathbb{R}$, then T is linear. If f is continuous at a single point of E_1, then T is continuous everywhere in E_1. Moreover (1.1) *is sharp.*

In 1978, Th. M. Rassias [**23**] formulated and proved the following theorem, which implies Hyers's Theorem as a special case.

Suppose that E and F are real normed spaces with F a complete normed space, $f : E \to F$ is a mapping such that for each fixed $x \in E$ the mapping $t \mapsto f(tx)$ is continuous on $\mathbb{R}$, and let there exist $\varepsilon \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p) \quad (x, y \in E). \tag{1.2}$$

Then there exists a unique linear mapping $T : E \to F$ such that

$$\|f(x) - T(x)\| \leq \varepsilon\|x\|^p/(1 - 2^{p-1}) \quad (x \in E).$$

The case of the existence of a unique additive mapping had been obtained by T. Aoki [**1**], as it is recently noticed by Lech Maligranda. However Aoki [**1**] had claimed the existence of a unique linear mapping, that is not true because he did not allow the mapping f to satisfy some continuity assumption. D. G. Bourgin, in his review of Aoki's paper which was written for Mathematical Reviews in 1951, wrote that "there is a unique additive transformation". From this it clearly follows that for the mathematicians who were writing mathematical papers in the period around 1950's the distinction between "linearity" and "additivity" was known. Thus Th. M. Rassias [**23**], who independently introduced the unbounded Cauchy difference was the first to prove that there exists a unique linear mapping T satisfying

$$\|f(x) - T(x)\| \leq \varepsilon\|x\|^p/(1 - 2^{p-1}) \quad (x \in E).$$

In 1990, Th. M. Rassias [**24**] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. In 1991, Z. Gajda [**8**] following the same approach as in Th. M. Rassias [**23**], gave an affirmative solution to this question for $p > 1$. It was proved by Z. Gajda [**8**] , as well as by Th. M. Rassias and P. Semrl [**27**] that one cannot prove a Th. M. Rassias' type theorem when $p = 1$. In 1994, P. Găvruta **9**] provided a further generalization of Th. M. Rassias' theorem in which he replaced the bound $\varepsilon(\|x\|^p + \|y\|^p)$ in (1.2) by a general control function $\varphi(x, y)$ for the existence of a unique linear mapping.

A generalized version of Cauchy equation is the equation of Pexider type $f_1(x + y) = f_2(x) + f_3(y)$. Lee, Jun, Shin and Kim obtained the generalized Hyers–Ulam stability of this Pexiderized equation; cf. [**14**, **18**]. In addition, the stability of the linear mappings in Banach modules were studied by M. S. Moslehian [**19**] and C. Park [**21**]. The functional equation $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ is called the *quadratic functional equation.* In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*, see [**25**]. A generalized Hyers–Ulam stability problem for the quadratic functional equation was proved by Skof [**29**] for mappings $f : X \to Y$, where X is a normed space and Y is a BANACH space. CHOLEWA [**5**] noticed that the theorem of SKOF is still true if the relevant domain X is replaced by an Abelian group. In [**6**], CZERWIK proved the generalized HYERS–ULAM stability of the quadratic functional equation. BORELLI and FORTI [**4**] generalized the stability result as follows (cf. [**15**]):

Let G be an Abelian group, and X a Banach space. Assume that a mapping $f : G \to X$ satisfies the functional inequality

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \le \varphi(x,y)$$

for all $x, y \in G$, and $\varphi : G \times G \to [0,\infty)$ is a function such that

$$\Phi(x,y) := \sum_{i=0}^{\infty} \frac{1}{4^{i+1}}\, \varphi(2^i x, 2^i y) < \infty$$

for all $x, y \in G$. Then there exists a unique quadratic mapping $Q : G \to X$ with the properties $\|f(x) - Q(x)\| \le \Phi(x,x)$ for all $x \in G$.

Here, we cannot fail to notice that S.-M. JUNG [**15**] dealt with stability problems for the quadratic functional equation of PEXIDER type $f_1(x+y) + f_2(x-y) = f_3(x) + f_4(y)$. In addition, the conditional stability of quadratic equation and stability of the quadratic mappings in BANACH modules were studied by M. S. MOSLEHIAN [**20**] and C. PARK [**22**].

During the last decades several stability problems of functional equations have been investigated by a number of mathematicians; see [**3**, **7**, **13**, **16**, **26**] and references therein for more detailed information.

By a *non-Archimedean field* we mean a field K equipped with a function (valuation) $|\cdot|$ from K into $[0,\infty)$ such that $|r| = 0$ if and only if $r = 0$, $|rs| = |r|\,|s|$, and $|r+s| \le \max\{|r|,|s|\}$ for all $r, s \in K$. Clearly $|1| = |-1| = 1$ and $|n| \le 1$ for all $n \in \mathbb{N}$.

Let X be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $\|\cdot\| : X \to \mathbb{R}$ is a *non-Archimedean norm* (*valuation*) if it satisfies the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|rx\| = |r|\|x\| \quad (r \in \mathbb{K}, x \in X)$;

(iii) the strong triangle inequality (ultrametric); namely,

$$\|x+y\| \le \max\{\|x\|, \|y\|\} \quad (x, y \in X).$$

Then $(X, \|\cdot\|)$ is called a non-Archimedean space. Due to the fact that

$$\|x_n - x_m\| \le \max\{\|x_{j+1} - x_j\| : m \le j \le n-1\} \qquad (n > m)$$

a sequence $\{x_n\}$ is CAUCHY if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean space. By a complete non-Archimedean space we mean one in which every CAUCHY sequence is convergent.

In 1897, HENSEL [**11**] discovered the p-adic numbers as a number theoretical analogue of power series in complex analysis. Fix a prime number p. For any nonzero rational number x, there exists a unique integer $n_x \in \mathbb{Z}$ such that $x = \frac{a}{b} p^{n_x}$, where a and b are integers not divisible by p. Then $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x,y) = |x-y|_p$ is denoted by $\mathbb{Q}_p$ which is called the *p-adic number field*. In fact, $\mathbb{Q}_p$ is the set of all formal series $x = \sum\limits_{k \geq n_x}^{\infty} a_k p^k$, where $|a_k| \leq p-1$ are integers. The addition and multiplication between any two elements of $\mathbb{Q}_p$ are defined naturally. The norm $\Big| \sum\limits_{k \geq n_x}^{\infty} a_k p^k \Big|_p = p^{-n_x}$ is a non-Archimedean norm on $\mathbb{Q}_p$ and it makes $\mathbb{Q}_p$ a locally compact field; see [**10**, **28**].

During the last three decades p-adic numbers have gained the interest of physicists for their research in particular in problems coming from quantum physics, p-adic strings and superstrings (cf. [**17**]). A key property of p-adic numbers is that they do not satisfy the Archimedean axiom: for all $x, y > 0$, there exists an integer n such that $x < ny$. “It is very difficult to imagine a situation where this axiom does not hold, but in fact the very space and time we inhabit have both been shown by 20th century science to be unequivocally non-Archimedean: The Archimedean axiom breaks down at the Planck scale, that is, for distances less than 1.6×10^{-33} metres and durations less than 5.4×10^{-44} seconds. Despite our entrenched belief that space and time are continuous, homogeneous, infinitely divisible quantities, we are now confronted with the fact that below this scale, distances and durations cannot scaled up in order to produce macroscopic distances and durations. Equivalently, we cannot meaningfully measure distances or durations below this scale. So a suggestion emerges to abandon the Archimedean axiom at very small distances. This leads to a non-Euclidean and non-Riemannian geometry of space at small distances”; cf. [**31**].

In [**2**], the authors investigated stability of approximate additive mappings $f : \mathbb{Q}_p \to \mathbb{R}$. They showed that if $f : \mathbb{Q}_p \to \mathbb{R}$ is a continuous mapping for which there exists a fixed ϵ such that $|f(x+y) - f(x) - f(y)| \leq \epsilon$ for all $x, y \in \mathbb{Q}_p$, then there exists a unique additive mapping $T : \mathbb{Q}_p \to \mathbb{R}$ such that $|f(x) - T(x)| \leq \epsilon$ for all $x \in \mathbb{Q}_p$. In this paper, we solve the stability problem for Cauchy and quadratic functional equations when the unknown function is one with values in a non-Archimedean space, in particular in the field of p-adic numbers.

2. STABILITY OF THE CAUCHY FUNCTIONAL EQUATION

In this section, we prove the generalized Hyers–Ulam stability of the Cauchy functional equation. Throughout this section, we assume that H is an additive semigroup and X is a complete non-Archimedean space.

Theorem 2.1. *Let $\varphi : H \times H \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{|2|^n} = 0 \qquad (x, y \in H) \tag{2.1}$$

and let for each $x \in H$ *the limit*

$$\lim_{n\to\infty} \max\Big\{\frac{\varphi(2^j x, 2^j x)}{|2|^j} : 0 \le j < n\Big\}, \tag{2.2}$$

denoted by $\widetilde{\varphi}(x)$*, exists. Suppose that* $f : H \to X$ *is a mapping satisfying*

$$\|f(x+y) - f(x) - f(y)\| \le \varphi(x,y) \qquad (x, y \in H). \tag{2.3}$$

Then there exists an additive mapping $T : H \to X$ *such that*

$$\|f(x) - T(x)\| \le \frac{1}{|2|}\,\widetilde{\varphi}(x) \qquad (x \in H). \tag{2.4}$$

Moreover, if

$$\lim_{k\to\infty}\lim_{n\to\infty} \max\Big\{\frac{\varphi(2^j x, 2^j x)}{|2|^j} : k \le j < n+k\Big\} = 0$$

then T *is the unique additive mapping satisfying* (2.4)*.*

Proof. Putting $y = x$ in (2.3), we get

$$\|f(2x) - 2f(x)\| \le \varphi(x,x) \qquad (x \in H). \tag{2.5}$$

Let $x \in H$. Replacing x by $2^{n-1}x$ in (2.5) we get

$$\Big\|\frac{f(2^n x)}{2^n} - \frac{f(2^{n-1}x)}{2^{n-1}}\Big\| \le \frac{\varphi(2^{n-1}x, 2^{n-1}x)}{|2|^n} \qquad (x \in H). \tag{2.6}$$

It follows from (2.6) and (2.1) that the sequence $\Big\{\frac{f(2^n x)}{2^n}\Big\}$ is Cauchy. Since X is complete, we conclude that $\Big\{\frac{f(2^n x)}{2^n}\Big\}$ is convergent. Set $T(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$.

Using induction one can show that

$$\Big\|\frac{f(2^n x)}{2^n} - f(x)\Big\| \le \frac{1}{|2|} \max\Big\{\frac{\varphi(2^k x, 2^k x)}{|2|^k} : 0 \le k < n\Big\} \tag{2.7}$$

for all $n \in \mathbb{N}$ and all $x \in H$. By taking n to approach infinity in (2.7) and using (2.2) one obtains (2.4). Replacing x and y by $2^n x$ and $2^n y$, respectively, in (2.3) we get

$$\Big\|\frac{f(2^n(x+y))}{2^n} - \frac{f(2^n x)}{2^n} - \frac{f(2^n y)}{2^n}\Big\| \le \frac{\varphi(2^n x, 2^n y)}{|2|^n} \qquad (x, y \in H).$$

Taking the limit as $n \to \infty$ and using (2.1) we get $T(x+y) = T(x) + T(y) \quad (x, y \in H)$.

If T' is another additive mapping satisfying (2.4), then

$$\begin{aligned}\|T(x)-T'(x)\| &= \lim_{k\to\infty}|2|^{-k}\|T(2^kx)-T'(2^kx)\|\\ &\le \lim_{k\to\infty}|2|^{-k}\max\{\|T(2^kx)-f(2^kx)\|,\|f(2^kx)-T'(2^kx)\|\}\\ &\le \frac{1}{|2|}\lim_{k\to\infty}\lim_{n\to\infty}\max\{\frac{\varphi(2^jx,2^jx)}{|2|^j}:k\le j<n+k\}\\ &=0\quad (x\in H).\end{aligned}$$

Therefore $T=T'$. This completes the proof of the uniqueness of T. □

Corollary 2.2. *Let $\rho:[0,\infty)\to[0,\infty)$ be a function satisfying*

$$\rho(|2|t)\le\rho(|2|)\rho(t)\quad (t\ge 0),\qquad \rho(|2|)<|2|.$$

Let $\delta>0$, let H be a normed space and let $f:H\to X$ fulfill the inequality

$$\|f(x+y)-f(x)-f(y)\|\le\delta\big(\rho(\|x\|)+\rho(\|y\|)\big)\quad (x,y\in H).$$

Then there exists a unique additive mapping $T:H\to X$ such that

$$\|f(x)-T(x)\|\le\frac{2}{|2|}\,\delta\rho(\|x\|)\quad (x\in H). \tag{2.8}$$

Proof. Defining $\varphi:H\times H\to[0,\infty)$ by $\varphi(x,y):=\delta\big(\rho(\|x\|)+\rho(\|y\|)\big)$ we have

$$\lim_{n\to\infty}\frac{\varphi(2^nx,2^ny)}{|2|^n}\le\lim_{n\to\infty}\left(\frac{\rho(|2|)}{|2|}\right)^n\varphi(x,y)=0\qquad (x,y\in H)$$

$$\widetilde{\varphi}(x)=\lim_{n\to\infty}\max\left\{\frac{\varphi(2^jx,2^jx)}{|2|^j}:0\le j<n\right\}=\varphi(x,x).$$

$$\lim_{k\to\infty}\lim_{n\to\infty}\max\left\{\frac{\varphi(2^jx,2^jx)}{|2|^j}:k\le j<n+k\right\}=\lim_{k\to\infty}\frac{\varphi(2^kx,2^ky)}{|2|^k}=0.$$

Applying Theorem 2.1 we conclude the required result. □

REMARK 2.3. The classical example of the function ρ is the mapping $\rho(t)=t^p\quad(t\in[0,\infty))$, where $p>1$ with the further assumption that $|2|<1$.

3. STABILITY OF THE QUADRATIC FUNCTIONAL EQUATION

In this section, we prove the generalized HYERS–ULAM stability of the quadratic functional equation. Throughout this section, we assume that G is an additive group and X is a complete non-Archimedean space.

Theorem 3.1. *Let $\psi : G \times G \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\psi(2^n x, 2^n y)}{|4|^n} = 0 \qquad (x, y \in G)$$

and let for each $x \in G$ the limit

$$\lim_{n\to\infty} \max\Big\{\frac{\psi(2^j x, 2^j x)}{|4|^j} : 0 \le j < n\Big\},$$

denoted by $\widetilde{\psi}(x)$, exists. Suppose that $f : G \to X$ is a mapping satisfying $f(0) = 0$ and

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \le \psi(x, y) \qquad (x, y \in G). \tag{3.1}$$

Then there exists a quadratic mapping $Q : G \to X$ such that

$$\|f(x) - Q(x)\| \le \frac{1}{|4|}\,\widetilde{\psi}(x) \qquad (x \in G). \tag{3.2}$$

Moreover, if

$$\lim_{k\to\infty} \lim_{n\to\infty} \max\Big\{\frac{\psi(2^j x, 2^j x)}{|4|^j} : k \le j < n + k\Big\} = 0$$

then Q is the unique quadratic mapping satisfying (3.2).

Proof. Putting $y = x$ in (3.1), we get

$$\|f(2x) - 4f(x)\| \le \psi(x, x) \quad (x \in G). \tag{3.3}$$

Replacing x by $2^{n-1}x$ in (3.3) we get

$$\Big\|\frac{f(2^n x)}{4^n} - \frac{f(2^{n-1}x)}{4^{n-1}}\Big\| \le \frac{\psi(2^{n-1}x, 2^{n-1}x)}{|4|^n} \qquad (x \in G).$$

Hence the sequence $\Big\{\frac{f(2^n x)}{4^n}\Big\}$ is Cauchy. Using the same method as in the proof of Theorem 2.1 we conclude that $Q(x) := \lim_{n\to\infty} \frac{f(2^n x)}{4^n}$ defines a quadratic mapping satisfying (3.2). □

Corollary 3.2. *Let $\tau : [0, \infty) \to [0, \infty)$ be a function satisfying*

$$\tau(|2|t) \le \tau(|2|)\tau(t) \quad (t \ge 0), \qquad \tau(|2|) < |2|.$$

Let $\delta > 0$, let G be a normed space and let $f : G \to X$ fulfill $f(0) = 0$ and the inequality

$$\|f(x+y) - f(x-y) - 2f(x) - 2f(y)\| \le \delta\big(\tau(\|x\|)\tau(\|y\|)\big) \quad (x, y \in G).$$

Then there exists a unique quadratic mapping $Q : G \to X$ such that

$$\|f(x) - Q(x)\| \leq \frac{\delta}{|4|}\,\tau(\|x\|)^2 \quad (x \in G).$$

Proof. Defining $\psi : G \times G \to [0, \infty)$ by $\psi(x, y) := \delta\big(\tau(\|x\|)\tau(\|y\|)\big)$ we have

$$\lim_{n\to\infty} \frac{\psi(2^n x, 2^n y)}{|4|^n} \leq \lim_{n\to\infty} \left(\frac{\tau(|2|)}{|2|}\right)^{2n} \psi(x, y) = 0 \qquad (x, y \in G)$$

$$\widetilde{\psi}(x) = \lim_{n\to\infty} \max\left\{\frac{\psi(2^j x, 2^j x)}{|4|^j} : 0 \leq j < n\right\} = \psi(x, x).$$

$$\lim_{k\to\infty} \lim_{n\to\infty} \max\left\{\frac{\psi(2^j x, 2^j x)}{|4|^j} : k \leq j < n + k\right\} = \lim_{k\to\infty} \frac{\psi(2^k x, 2^k y)}{|4|^k} = 0$$

Applying Theorem 3.1 we conclude the required result. □

Remark 3.3. The classical example of the function τ is the mapping $\tau(x) = t^p \quad (t \in [0, \infty))$, where $p > 1$ with the further assumption that $|2| < 1$.

Remark 3.4. We can formulate similar statements to Theorem 2.1 and Theorem 3.1 in which we can define the sequences $T(x) := \lim\limits_{n\to\infty} 2^n f\left(\dfrac{x}{2^n}\right)$ and $Q(x) := \lim\limits_{n\to\infty} 4^n f\left(\dfrac{x}{2^n}\right)$, respectively under suitable conditions on the functions φ and ψ and then obtain similar results to Corollary 2.2 and Corollary 3.2 for $p < 1$.

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