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ON A PROPERTY OF ENTIRE FUNCTIONS WITH ALMOST NEGATIVE ZEROS

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We give a generalization of well-known Valiron-Titchmarsh theorem on entire functions with negative zeros. Namely, we prove that $(-1)^p \log P(r) \in ER_{[p,\ p+1]}$, where $P(r)$ denotes the canonical product of an entire function with genus p and almost negative zeros and ER is the class of extended regular variation.

1. INTRODUCTION

We begin with some basic definitions from Karamata's theory of Regular Variation (cf. [**1**], [**2**]).

Definition 1. *A positive measurable function f varies regularly with index ρ, i.e. $f \in R_\rho$, if*

$$\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = \lambda^\rho$$

for each $\lambda > 0$ and some real ρ.

Since $\lim\limits_{x\to\infty} f(\lambda x)/f(x)$ need not always exist, denote

$$f^*(\lambda) := \limsup_{x\to\infty} \frac{f(\lambda x)}{f(x)}, \quad f_*(\lambda) := \liminf_{x\to\infty} \frac{f(\lambda x)}{f(x)} \quad (\lambda > 0).$$

Therefore we have

Definition 2 (cf. [**1**, p. 65]). *A positive measurable function f belongs to the class ER of extended regularly varying functions if*

$$\lambda^a \le f_*(\lambda) \le f^*(\lambda) \le \lambda^b, \quad \forall \lambda \ge 1,$$

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for some constants a, b. More specifically, in this case we write $f \in ER_{[a,\ b]}$.

It is obvious that $R \subset ER$.

The Theory of Regular Variation is very well developed (cf. [**1**], [**2**]) and has many applications in Analysis, Probability Theory, Number Theory, Theory of Distributions etc. One of the brightest examples of this sort is Valiron-Titchmarsh Theorem on entire functions of finite order ρ with negative zeros only (cf. [**1**, pp. 301–308]). Roughly speaking, it asserts that $c_\rho \log P(r) \in R_\rho$ if and only if $n(r) \in R_\rho$, where $P(z)$ is *the canonical product* with negative zeros and $n(r)$ denotes the number of zeros in the circle $|z| \le r$.

In the case of non-integral order one obtain that $c_\rho = \dfrac{1}{\pi} \sin \pi\rho$.

Recall that the canonical product with zeros $z_1, z_2, \dots$ is

$$P(z) = \prod_{n=1}^{\infty} (1 - \frac{z}{z_n}) \exp\left(\sum_{k=1}^{p} (z/z_n)^k / k \right), \tag{1}$$

where p is its *genus* i. e. the least non-negative integer such that $\sum_n |z_n|^{-(p+1)}$ converges.

The Hadamard Factorization Theorem states that an entire function g of finite order ρ, $p \le \rho \le p+1$, with zeros $z_1, z_2, \dots$ may be written in the form

$$g(z) = z^m P(z) \exp\bigl(Q(z)\bigr),$$

where m is the order of $z = 0$ as a zero of g and Q is a polynomial of degree $\le \rho$.

It will be proved here that for the canonical product P with a genus p and negative zeros, $(-1)^p \log P(r) \in ER_{[p,p+1]}$ without any assumption on the distribution of zeros.

2. RESULTS

If $z_1, z_2, \dots$ are the zeros of $P(z)$ with genus $p \ge 0$, then there exist positive constants $C_{n,p}$ such that

$$|\pi - \arg z_n| \le C_{n,p}, \quad n \in \mathbb{N}.$$

If $\lim_{n\to\infty} C_{n,p} = 0$, the zeros are *oriented* and for this class results similar to Valiron-Titchmarsh Theorem are obtained by Bowen [**3**].

Definition 3. *The zeros $z_1, z_2, \dots$ are almost negative if the relation*

$$|\pi - \arg z_n| \le C_p \tag{2}$$

holds for some constant C_p, $0 < C_p < \pi/2$ and each $n \in \mathbb{N}$.

Hence, almost negative zeros belong to some angle in the left complex half-plane including the negative part of the real axis.

Denote by A_p the class of zeros satisfying (2) with $C_p = \dfrac{\pi}{2p+4}$. Then the following assertion holds.

Theorem A. *If the canonical product $P(z)$ is formed by the zeros from the class A_p and is real on the real axis, then*

$$(-1)^p \log P(r) \in ER_{[p,p+1]}.$$

Proof. Taking the logarithmic derivative in (1) we obtain

$$\widetilde{P}(z) := z\frac{P'(z)}{P(z)} = z^{p+1} \sum_{a \in A_p} \frac{1}{a^p(z-a)}. \tag{3}$$

Hence, for $\lambda > 1$ we get

$$\lambda^{p+1}\widetilde{P}(z) - \widetilde{P}(\lambda z) = \lambda^{p+1}(\lambda-1)z^{p+2} \sum_{a \in A_p} \frac{1}{a^p(z-a)(\lambda z - a)}, \tag{4}$$

and

$$\lambda^{p}\widetilde{P}(z) - \widetilde{P}(\lambda z) = \lambda^{p}(\lambda-1)z^{p+1} \sum_{a \in A_p} \frac{1}{a^{p-1}(z-a)(\lambda z - a)}. \tag{5}$$

For $a \in A_p$ and $n \le p+2$, we have $\Re(a^n) = (-1)^n b_n$ with $b_n \ge 0$. Since $\Im\widetilde{P}(r) = 0$, from (3), (4) and (5), after some calculation we obtain

$$(-1)^p\widetilde{P}(r) > 0; \quad (-1)^p(\lambda^{p+1}\widetilde{P}(r) - \widetilde{P}(\lambda r)) > 0; \quad (-1)^p(\lambda^{p}\widetilde{P}(r) - \widetilde{P}(\lambda r)) < 0, \tag{6}$$

for each $r > 0$.

Hence

$$(-1)^p\widetilde{P}(r) \in ER_{[p,p+1]}.$$

Now, since $P(0) = 1$ and $\widetilde{P}(r)$ is real and continuous on the positive part of real axis, by (6) we get

$$\begin{aligned}(-1)^p \log P(\lambda r) &= (-1)^p \int_0^{\lambda r} \widetilde{P}(t)\,\frac{\mathrm{d}t}{t} = (-1)^p \int_0^{r} \widetilde{P}(\lambda t)\,\frac{\mathrm{d}t}{t} \\ &\le (-1)^p \lambda^{p+1} \int_0^{r} \widetilde{P}(t)\,\frac{\mathrm{d}t}{t} = (-1)^p\lambda^{p+1}\log P(r).\end{aligned}$$

Analogously by the second part of (6), for $r > 0$ we obtain

$$(-1)^p \log P(\lambda r) \ge (-1)^p \lambda^p \log P(r).$$

Hence

$$(-1)^p \log P(r) \in ER_{[p,p+1]},$$

and the proof is done. □

Remark 1. The statement of Theorem A is of global nature and allows the use of the tools of Extended Variation Theory (cf. [**1**, pp. 61–81]) in the case when the growth of $n(r)$ is not specified.

Remark 2. Inequalities (6) hold for all $\lambda > 1$ and $r > 0$; hence (6) is stronger result than the result of the theorem (which is stated in terms of lim inf and lim sup).

Corollary 1. *If the canonical product $P(z)$ with genus p is formed by negative zeros only, then*

$$(-1)^p \widetilde{P}(r) \in ER_{[p,p+1]}; \qquad (-1)^p \log P(r) \in ER_{[p,p+1]},$$

without any assumption on the distribution of zeros.

We illustrate this point by an example.

According to the Hadamard Factorization Theorem, the class A of entire functions with negative zeros and genus zero is represented by

$$\prod_1^\infty (1 + z/a_k),$$

where $\{a_k\}_1^\infty$ is a sequence of positive numbers with $\sum_1^\infty 1/a_k < \infty$. In particular,

$$f_b(z) := \prod_{k \in N} (1 + z/k^b), \quad b > 1,$$

belongs to the class A. Since in this case

$$n(r) \sim r^{1/b} \in R_{1/b} \quad (r \to \infty),$$

the Valiron-Titchmarsh Theorem gives $\log f_b(r) \in R_{1/b}$ and

$$\log f_b(r) \sim \frac{\pi}{\sin(\pi/b)} r^{1/b} \quad (r \to \infty).$$

Consider now the function $f_K(z)$ defined by

$$f_K(z) := \prod_{k \in K} (1 + z/k^b),$$

where K is any subset of N. Since $b > 1$, we have that

$$\sum_{k \in K} 1/k^b \le \sum_{k \in N} 1/k^b < \infty.$$

Hence $f_K(z) \in A$ and Corollary 1 states that $\log f_K(r) \in ER_{[0,1]}$ independently of K. Moreover, Theorem A asserts that for

$$f_K^*(z) := \prod_{k\in K}\Big(1+\frac{z}{k^b e^{i\phi_k}}\Big),$$

we have

$$\log f_K^*(r) \in ER_{[0,1]},$$

providing that $|\phi_k| \le \pi/4,\ k \in K$.

Remark 3. The referee posed two interesting questions.

1. *In the regularly varying case there is a relation between the asymptotic behavior of* $\log P(z)$ *and that of the number of zeros* $n(r)$ *in the circle with radius* r. *Is it possible to transfer the main result to obtain also a new result concerning* $n(r)$?

2. *Using the technique of the paper, perhaps it is possible to prove not only that*

$$(-1)^p \log P(r) \in ER,$$

but also that all derivatives of $(-1)^p \log P(r)$ *belong to ER.*

The answer to the first question should be negative; from the above example, since the set of zeros of $f_K(z)$ is arbitrary, it is seen that the growth of $n(r)$ does not affect the statement of Theorem A. Therefore, one cannot expect that the main result could produce a new result concerning $n(r)$.

The second question is more complex and difficult. Although its assertion formally is not true (note that for $r > 0$, $\big(\log f_K(r)\big)'' < 0$, hence $\notin ER$), we are able to give just a partial answer in the simplest case i.e. when all zeros of $P(z)$ are negative.

Proposition 1. *If the canonical product* $P(z)$ *of genus* $p \ge 0$, *have all its zeros negative, then*

$$1.\ \big((-1)^p \log P(r)\big)' \in ER_{[p-1,p]};\quad 2.\ \big((-1)^p \log P(r)\big)'' \in ER_{[p-2,p-1]},\ p \ge 2,$$

but,

$$\big(-\log P(r)\big)'' \in ER_{[-2,0]}, p=0;\ \big(\log P(r)\big)'' \in ER_{[-2,0]}, p=1.$$

The part 1 is a consequence of Theorem 1. The part 2 shows irregularities concerning parameter p.

Proposition 2. *Under the conditions of Proposition* 1 *and for all* $m > p$, *we have that*

$$1.\quad \big((-1)^{p+m+1} \log P(r)\big)^{(m)} \in ER_{[-m,0]},$$

but, for $m = p$ *it follows that*

$$2.\quad \big((-1)^p \log P(r)\big)^{(p)} \in ER_{[0,1]}.$$

Proof. From (3) we obtain that

$$\big((-1)^p \log P(r)\big)' = \sum_{a\in A} \frac{r^p}{a^p(r+a)}\,, \tag{7}$$

where A is a set of positive numbers satisfying

$$\sum_{a\in A} 1/a^{p+1} < \infty.$$

Hence for $p \ge 1$,

$$\big((-1)^p \log P(r)\big)'' = \sum_{a\in A} r^{p-1}\,\frac{(p-1)r + ap}{a^p(r+a)^2}\,,$$

and, applying the method from the proof of Theorem A, after some calculation we obtain the assertion from Proposition 1.

To get the proof of Proposition 2, note that for $p \ge 1$,

$$\frac{r^p}{a^p(r+a)} = (r^{p-1}/a^p + \cdots) + \frac{(-1)^p}{r+a}\,,$$

where the expression in the brackets is a polynomial of the degree $p-1$.

Therefore from (7), for $m > p$ we obtain

$$\big((-1)^{p+m+1} \log P(r)\big)^{(m)} = (m-1)! \sum_{a\in A} \frac{1}{(r+a)^m}\,,$$

and for $\lambda > 1$,

$$\begin{aligned}\big((-1)^{p+m+1} \log P(\lambda r)\big)^{(m)} &- \big((-1)^{p+m+1} \log P(r)\big)^{(m)}\\ &= (m-1)! \sum_{a\in A} \Big(\frac{1}{(\lambda r+a)^m} - \frac{1}{(r+a)^m}\Big) < 0,\end{aligned}$$

i.e.

$$\begin{aligned}\big((-1)^{p+m+1} \log P(\lambda r)\big)^{(m)} &- (1/\lambda^m)\big((-1)^{p+m+1} \log P(r)\big)^{(m)}\\ &= (m-1)! \sum_{a\in A} \Big(\frac{1}{(\lambda r+a)^m} - \frac{1}{(\lambda r+\lambda a)^m}\Big) > 0.\end{aligned}$$

Hence the result follows. For $m = p$, from (7) we get

$$\big((-1)^{p} \log P(r)\big)^{(p)} = (p-1)! \sum_{a\in A} \Big(\frac{1}{a^p} - \frac{1}{(r+a)^p}\Big),$$

and the result from Proposition 2, part 2, follows analogously.

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