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ADDITIONAL ANALYSIS OF BINOMIAL RECURRENCE COEFFICIENTS

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This paper involves an investigation of $\big(f(n)\big)_{n=1}^{\infty}$, where $f(n)$ is defined by

$$f(n+1) = \sum_{k=1}^{n} \binom{n}{k} f(k), \qquad n \geq 1. \tag{0.1}$$

Through successive iterations of (0.1), it is shown that

$$f(n+r) = \sum_{k=1}^{n} f(k) \sum_{j=0}^{r-1} A_j^r(n) \binom{n+j}{k}, \qquad r \geq 1,\ n \geq 1. \tag{0.2}$$

The $A_j^r(n)$ of (0.2) are the *binomial recurrence coefficients.* The main result of this paper is a recurrence formula for the $A_j^r(n)$, namely,

$$\sum_{j=k}^{r-1} \binom{j}{k} A_j^r = A_{k-1}^r, \tag{0.3}$$

where $A_j^r \equiv A_j^r(0)$. This paper then provides two applications involving (0.3). The first involves series inversion while the second involves polynomials whose general term has the form $A_j^r x^j$.

1. INTRODUCTION

This paper is a continuation of the work done in [**1**]. In that paper, we investigated the general binomial recurrence

$$f(n+1) = \sum_{k=0}^{n} \binom{n}{k} f(k), \qquad n \geq 1. \tag{1.1}$$

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Through successive iterations of (1.1), we obtained the linear recurrence formula

$$f(n+r) = \sum_{k=0}^{n} f(k) \sum_{j=0}^{r-1} A_j^r(n) \binom{n+j}{k}, \qquad r \geq 1,\, n \geq 1, \tag{1.2}$$

where the $A_j^r(n)$ satisfy the recurrence relation

$$A_j^{r+1}(n) = \sum_{i=0}^{r-j-1} \binom{n+r}{i} A_j^{r-i}(n), \qquad 0 \leq j \leq r-1, \tag{1.3}$$

with $A_r^{r+1}(n) = 1$. We assume $A_j^r(n) = 0$ for $j < 0$ or $j > r-1$. In [**1**], we called the $A_j^r(n)$ **binomial recurrence coefficients.**

We are particularly interested in the quantity $A_j^r(0) \equiv A_j^r$, since $A_j^r(n)$ can be determined directly from A_j^r by using the recursive formula

$$A_{j+1}^{r+1}(n) = A_j^r(n+1), \qquad j \geq 1,\, r \geq 0. \tag{1.4}$$

Thus, a useful reformulation of (1.3) is

$$A_j^{r+1} = \sum_{i=0}^{r-j-1} \binom{r}{i} A_j^{r-i}, \qquad 0 \leq j \leq r-1. \tag{1.5}$$

Notice that in (1.3) and (1.5), the recurrence formula involves summation over the upper index of the binomial recurrence coefficient. The main result of this paper, Theorem 2.1, involves a new recurrence formula of the A_j^r. In this new formula, we sum on the *lower* index of the binomial recurrence coefficient. We then discuss two applications of Theorem 2.1 The first application is the proof of the inversion theorem, Theorem 3.1, which was stated in [**1**] without proof. We also use Theorem 2.1 to obtain a closed form for the $\sum_{j=0}^{r-1} A_j^r x^j$.

2. ANOTHER RECURSION FOR A_j^r

The discovery of Theorem 2.1 comes from analyzing the series

$$\sum_{j=1}^{r-1} j^p A_j^r, \qquad p \geq 0. \tag{2.1}$$

In [**1**], we proved that

$$\sum_{j=1}^{r-1} j A_j^r = A_0^r, \qquad r \geq 2, \tag{2.2}$$

and stated that

$$\sum_{j=1}^{r-1} j^2 A_j^r = A_0^r + 2A_1^r, \qquad r \geq 3. \tag{2.3}$$

We now provide a short proof of (2.3). In order to prove (2.3), recall, from [**1**], that

$$f(n+r) = \sum_{k=0}^{n} f(k) \sum_{j=0}^{r-1} A_j^r(n) \binom{n+j}{k}, \qquad r \geq 1,\, n \geq 1. \tag{2.4}$$

Let $n = 2$ in (2.4). We then obtain

$$\begin{aligned} f(r+2) = f(0) \sum_{j=0}^{r-1} A_j^r(2) + f(1) \sum_{j=0}^{r-1} (2+j) A_j^r(2) \\ + f(2) \sum_{j=0}^{r-1} \frac{(j+2)(j+1)}{2} A_j^r(2). \end{aligned} \tag{2.5}$$

In [**1**], we show that

$$A_{j+1}^{r+1}(n) = A_j^r(n+1), \qquad j \geq 1,\, r \geq 0. \tag{2.6}$$

By substituting (2.6) into the right hand sums of (2.5), we obtain

$$\begin{aligned} f(r+2) &= f(0) \sum_{j=0}^{r-1} A_{j+2}^{r+2} + f(1) \sum_{j=0}^{r-1} (2+j) A_{j+2}^{r+2} + f(2) \sum_{j=0}^{r-1} \frac{(j+2)(j+1)}{2} A_{j+2}^{r+2} \\ &= f(0) \sum_{j=1}^{r} A_{j+1}^{r+2} + f(1) \sum_{j=1}^{r} (1+j) A_{j+1}^{r+2} + f(2) \sum_{j=1}^{r} \frac{j(j+1)}{2} A_{j+1}^{r+2}. \end{aligned} \tag{2.7}$$

In [**1**], we showed that

$$\mathcal{B}(r) = \sum_{j=0}^{r-1} A_j^r, \qquad r \geq 1, \tag{2.8}$$

where $\mathcal{B}(r)$ is the r^{th} Bell number.

Substitute (2.8) into (2.7) and obtain

$$\begin{aligned} f(r+2) = f(0)(\mathcal{B}(r+2) - A_0^{r+2} - A_1^{r+2}) + f(1) \sum_{j=1}^{r} (1+j) A_{j+1}^{r+2} \\ + f(2) \sum_{j=1}^{r} \frac{j(j+1)}{2} A_{j+1}^{r+2}. \end{aligned} \tag{2.9}$$

In (2.9), we let $f(r) = \mathcal{B}(r)$, $f(0) = \mathcal{B}(0) = 1$, $f(1) = \mathcal{B}(1) = 1$, and $f(2) = \mathcal{B}(2) = 2$. After making these substitutions and simplifying the following result, we obtain

$$A_0^{r+2} + A_1^{r+2} = \sum_{j=1}^{r} (j+1)^2 A_{j+1}^{r+2}. \tag{2.10}$$

In (2.10), replace $j+1$ with j and $r+2$ with r. We then obtain

$$A_0^r + A_1^r = \sum_{j=1}^{r-1} j^2 A_j^r - A_1^r. \tag{2.11}$$

Clearly (2.11) is equivalent to (2.3).

By assuming (2.2) and (2.3), we can inductively iterate (2.4) to obtain the following results.

$$\sum_{j=1}^{r-1} j^3 A_j^r = A_0^r + 6A_1^r + 6A_2^r, \qquad r \geq 4 \tag{2.12}$$

$$\sum_{j=1}^{r-1} j^4 A_j^r = A_0^r + 14A_1^r + 36A_2^r + 24A_3^r, \qquad r \geq 5 \tag{2.13}$$

$$\sum_{j=1}^{r-1} j^5 A_j^r = A_0^r + 30A_1^r + 150A_2^r + 240A_3^r + 120A_4^r, \qquad r \geq 6. \tag{2.14}$$

Inspection of Equations (2.2), (2.3), (2.12), (2.13) and (2.14) allows us to form the following conjecture.

Conjecture 2.1.

$$\sum_{j=0}^{r-1} j^p A_j^r = \sum_{k=1}^{p} k! \left\{ {p \atop k} \right\} A_{k-1}^r. \tag{2.15}$$

where $\left\{ {p \atop k} \right\}$ *is the appropriate Stirling number of the second kind.*

It is well known that [**2**, p. 70]

$$x^p = \sum_{k=0}^{p} k! \binom{x}{k} \left\{ {p \atop k} \right\}. \tag{2.16}$$

In fact, this is the definition of $\left\{ {p \atop k} \right\}$.

If we let $x = j$, (2.16) implies

$$\sum_{j=0}^{r-1} j^p A_j^r = \sum_{j=0}^{r-1} \sum_{k=0}^{p} k! \binom{j}{k} \left\{ {p \atop k} \right\} A_j^r = \sum_{k=0}^{p} k! \left\{ {p \atop k} \right\} \sum_{j=0}^{r-1} \binom{j}{k} A_j^r. \tag{2.17}$$

By comparing the coefficient of $k!\left\{ p \atop k \right\}$ in (2.15) versus (2.17), we obtain the following conjecture.

Conjecture 2.2.

$$\sum_{j=k}^{r-1} \binom{j}{k} A_j^r = A_{k-1}^r. \tag{2.18}$$

Remark 2.1. An alternative way of showing the equivalence between Conjectures 2.1 and 2.2 uses the Stirling Inversion Formula. Recall that [**2**, p. 94],

$$f(n) = \sum_{k=0}^{n} \left\{ n \atop k \right\} g(k)$$

if and only if

$$g(n) = \sum_{k=0}^{n} (-1)^{n-k} \left[n \atop k \right] f(k), \tag{2.19}$$

where $(-1)^{n-k}\left[n \atop k \right]$ is the appropriate Stirling number of the first kind.

In our case, let $g(k) = k!A_{k-1}^r$ and $f(k) = \sum\limits_{j=0}^{r-1} j^k A_j^r$. Then (2.19) becomes,

$$\begin{aligned} p!A_{p-1}^r &= \sum_{k=0}^{p} (-1)^{p-k} \left[p \atop k \right] \sum_{j=0}^{r-1} j^k A_j^r \\ &= \sum_{j=0}^{r-1} A_j^r \sum_{k=0}^{p} (-1)^{p-k} \left[p \atop k \right] j^k = \sum_{j=0}^{r-1} p!A_j^r \binom{j}{p}. \end{aligned} \tag{2.20}$$

Note that the equivalence between the two sums in (2.20) is simply the definition of $(-1)^{n-k}\left[n \atop k \right]$ [**2**, p. 70]. The above calculations show that

$$p!A_{p-1}^r = \sum_{j=0}^{r-1} p!A_j^r \binom{j}{p}. \tag{2.21}$$

By dividing both sides of (2.21) by $p!$, we obtain Conjecture 2.2.

Conjecture 2.2 provides a new recursive formula for the A_k^r. In terms of Table 1, (2.18) implies that a particular value of A_k^r can be obtained by summing the row entries that lie to the right of the value we are trying to find. In [**1**], we obtain a different recursive formula for the A_k^r, namely,

$$\sum_{i=0}^{r-1-k} \binom{r-1}{i} A_{k-1}^{r-1-i} = A_{k-1}^r. \tag{2.22}$$

In terms of Table 1, (2.22) implies that a particular value of A_k^r can be obtained by summing the vertical entries that lie above the value we are trying to find. By using (2.22) and induction on r, we are able to prove (2.18). Thus, we have the following theorem.

Theorem 2.1.

$$\sum_{j=k}^{r-1} \binom{j}{k} A_j^r = A_{k-1}^r. \tag{2.23}$$

Proof. We use mathematical induction on r. It is easily shown that (2.23) is true for $r = 2$ since

$$\sum_{j=1}^{1} \binom{j}{k} A_j^2 = A_1^2 = 1 = A_0^2.$$

We now assume (2.23) is true for the first $r-1$ rows of Table 1. We look at A_{k-1}^r. By (2.22) we know

$$A_{k-1}^r = \sum_{i=0}^{r-1-k} \binom{r-1}{i} A_{k-1}^{r-1-i}. \tag{2.24}$$

By our induction hypothesis, we assume (2.23) is true for each A_{k-1}^{r-1-i} that occurs in the right hand sum of (2.24). Thus, we can write (2.24) as the following double sum.

$$A_{k-1}^r = \sum_{i=0}^{r-1-k} \binom{r-1}{i} \sum_{j=k}^{r-2-i} \binom{j}{k} A_j^{r-1-i}. \tag{2.25}$$

Interchanging the order of summation in (2.25) gives us

$$A_{k-1}^r = \sum_{j=k}^{r-2} \binom{j}{k} \sum_{i=0}^{r-2-j} \binom{r-1}{i} A_j^{r-1-i} + \binom{r-1}{r-1-k} A_{k-1}^k. \tag{2.26}$$

The inner sum on the right hand side of (2.26) is a special case of (2.22). Also, in [**1**], we showed that $A_{k-1}^k = A_{r-1}^r = 1$. Thus, (2.26) becomes,

$$A_{k-1}^r = \sum_{j=k}^{r-2} \binom{j}{k} A_j^r + \binom{r-1}{k} A_{r-1}^r = \sum_{j=k}^{r-1} \binom{j}{k} A_j^r. \qquad \square$$

By applying (2.6) to (2.23), we can easily prove the following lemma.

Lemma 2.1. *Let n be a non-negative integer.*

$$\sum_{j=k+n}^{r+n-1} \binom{j}{k+n} A_{j-n}^r(n) = A_{k-1}^r(n). \tag{2.27}$$

Remark 2.2. We can extend (2.27) for arbitrary integers if we use (2.27) to obtain a polynomial in n, and then assign n the value of the desired negative integer.

3. A BASIC INVERSION THEOREM

Theorem 2.1 allows us to offer an elegant proof of the following inversion theorem. This theorem was stated without proof in [**1**].

Before we prove this inversion theorem, we need the following lemma.

Lemma 3.1.

$$\sum_{j=k+1}^{r} \binom{r}{j} A_k^j = A_k^{r+1}.$$

Proof. Using (2.19), we see that

$$\sum_{j=k+1}^{r} \binom{r}{j} A_k^j = \sum_{j=k+1}^{r} \binom{r}{r-1} A_k^j = \sum_{J=0}^{r-k-1} \binom{r}{J} A_k^{r-J} = A_k^{r+1}. \qquad \square$$

Theorem 3.1. (Inversion Theorem)

$$f(r) = \sum_{j=0}^{r-1} A_j^r g(j), \qquad r \geq 1 \tag{3.1}$$

if and only if

$$g(r) = f(r+1) - \sum_{j=1}^{r} \binom{r}{j} f(j), \quad \textit{with} \quad g(0) = f(1). \tag{3.2}$$

Proof. We begin by substituting (3.2) into the right hand side of (3.1).

$$\begin{aligned} f(r) &= \sum_{j=0}^{r-1} A_j^r g(j) = \sum_{j=0}^{r-1} A_j^r f(j+1) - \sum_{j=1}^{r-1} \sum_{k=1}^{j} A_j^r \binom{j}{k} f(k) \\ &= \sum_{j=0}^{r-1} A_j^r f(j+1) - \sum_{k=1}^{r-1} f(k) \sum_{j=k}^{r-1} A_j^r \binom{j}{k}. \end{aligned}$$

Notice that the inner sum of the second term is exactly (2.18), which we know is true by Theorem 2.1. Hence, the preceding line becomes

$$\begin{aligned} f(r) &= \sum_{j=0}^{r-1} A_j^r f(j+1) - \sum_{k=1}^{r-1} f(k) A_{k-1}^r \\ &= A_{r-1}^r f(r) + \sum_{j=0}^{r-2} A_j^r f(j+1) - \sum_{k=0}^{r-2} A_k^r f(k+1) = A_{r-1}^r f(r) = f(r), \end{aligned}$$

since $A_{r-1}^r = 1$.

Next, we substitute (3.1) into (3.2).

$$\begin{aligned} g(r) &= f(r+1) - \sum_{j=1}^{r}\binom{r}{j}f(j) = \sum_{j=0}^{r}A_j^{r+1}g(j) - \sum_{j=1}^{r}\binom{r}{j}\sum_{k=0}^{j-1}A_k^j g(k) \\ &= \sum_{j=0}^{r}A_j^{r+1}g(j) - \sum_{k=0}^{r-1}g(k)\sum_{j=k+1}^{r}\binom{r}{j}A_k^j \\ &= g(r)A_r^{r+1} + \sum_{j=0}^{r-1}A_j^{r+1}g(j) - \sum_{k=0}^{r-1}g(k)A_k^{r+1} = g(r)A_r^{r+1} = g(r) \end{aligned}$$

since $A_r^{r+1} = 1$. The equality connecting lines 2 and 3 comes from Lemma 3.1. □

4. SUMS INVOLVING A_j^r AND AN ARBITRARY POWER OF x

In [**1**], Section 7., we found an integral exponential generating function for A_0^n, namely,

$$\sum_{n=0}^{\infty}A_0^n\frac{x^n}{n!} = e^{e^x-1}\int_0^x e^{1-e^t}\,\mathrm{d}t.$$

The goal of this section is to use (2.18) to define a new family of generating functions involving the A_k^r. This new family is denoted $\big(S_r(x)\big)_{r=1}^{\infty}$, where,

$$S_r(x) = \sum_{j=0}^{r-1}A_j^r x^j.$$

We can find a functional equation involving $S_r(x)$ by multiplying each side of (2.18) by x^k, and summing over k. In particular,

$$\begin{aligned} \sum_{k=1}^{r-1}A_{k-1}^r x^k &= \sum_{k=1}^{r-1}x^k\sum_{j=k}^{r-1}\binom{j}{k}A_j^r \\ &= \sum_{j=1}^{r-1}\sum_{k=1}^{j}x^k\binom{j}{k}A_j^r = \sum_{j=1}^{r-1}A_j^r\bigg(\sum_{k=0}^{j}x^k\binom{j}{k} - 1\bigg) \\ &= \sum_{j=1}^{r-1}A_j^r\big((1+x)^j - 1\big) = \sum_{j=1}^{r-1}A_j^r(1+x)^j - \sum_{j=1}^{r-1}A_j^r. \end{aligned}$$

Substituting (2.8) into the previous line proves the following lemma.

Lemma 4.1.

$$\sum_{j=0}^{r-1}A_j^r(1+x)^j - \mathcal{B}(r) = \sum_{j=1}^{r-1}A_{j-1}^r x^j. \tag{4.1}$$

We now do some basic manipulations on (4.1) in order to form an equational relationship that will allow us to iteratively compute values for $\sum_{j=0}^{r-1} A_j^r x^j$. In particular, (4.1) implies

$$\sum_{j=0}^{r-1} A_j^r (1+x)^j = \mathcal{B}(r) + \sum_{j=0}^{r-2} A_j^r x^{j+1} = \mathcal{B}(r) + \sum_{j=0}^{r-1} A_j^r x^{j+1} - A_{r-1}^r x^r.$$

Since $A_{r-1}^r = 1$, the above line becomes

$$\sum_{j=0}^{r-1} A_j^r (1+x)^j = \mathcal{B}(r) + x\sum_{j=0}^{r-1} A_j^r x^j - x^r. \tag{4.2}$$

Thus, (4.2) becomes

$$S_r(1+x) = \mathcal{B}(r) + xS_r(x) - x^r. \tag{4.3}$$

Our goal is to calculate the value of $S_r(x)$ for x an arbitrary nonnegative integer. If $x = 0$, (4.3) implies $S_r(1) = \mathcal{B}(r)$. When $x = 1$, (4.3) implies

$$S_r(2) = \mathcal{B}(r) + S_r(1) - 1 = 2\mathcal{B}(r) - 1.$$

By successively substituting various integer values of x into (4.3), we obtain the following results.

$$\begin{aligned}
S_r(3) &= 5\mathcal{B}(r) - 2 - 2^r,\\
S_r(4) &= 16\mathcal{B}(r) - 6 - 3(2^r) - 3^r,\\
S_r(5) &= 65\mathcal{B}(r) - 24 - 4(3)2^r - 4(3^r) - 4^r,\\
S_r(6) &= 326\mathcal{B}(r) - 120 - 5(4)(3)2^r - 5(4)3^r - 5(4^r) - 5^r.
\end{aligned}$$

Inspection of these equations implies the following lemma.

Lemma 4.2. *Let* $S_r(x) = \sum_{j=0}^{r-1} A_j^r x^j$. *Let* k *be a positive integer.*

$$S_r(k) = a_k\mathcal{B}(r) - \sum_{j=2}^{k} \frac{(k-1)!}{(j-1)!}(j-1)^r, \tag{4.4}$$

where $a_1 = 1$ *and* $a_k = 1 + (k-1)a_{k-1}$.

Proof. Apply mathematical induction on k to (4.4). We leave the details of this straightforward induction to the reader. □

By substituting the expression $a_k = 1 + (k-1)a_{k-1}$ into itself $k-1$ times, we can easily obtain the following formula for a_n, namely,

$$a_n = \sum_{k=0}^{n-1} P(n-1,k) = \sum_{k=0}^{n-1} \frac{(n-1)!}{(n-1-k)!}. \tag{4.5}$$

From (4.5), we are able to obtain an exponential generating function for the $(a_n)_{n=1}^{\infty}$. In particular, define $A(x) = \sum\limits_{n=0}^{\infty} a_{n+1} \dfrac{x^n}{n!}$. Then $A(x) = \dfrac{e^x}{1-x}$.

5. OPEN QUESTIONS

Clearly, the binomial recurrence coefficients $A_j^r(n)$ provide a vast array of results. In particular, we have shown that the A_j^r obey two different recurrence relations

$$A_j^{r+1} = \sum_{i=0}^{r-j-1} \binom{r}{i} A_j^{r-i}, \qquad 0 \le j \le r-1, \tag{5.1}$$

and

$$A_{k-1}^r = \sum_{j=k}^{r-1} \binom{j}{k} A_j^r. \tag{5.2}$$

An open research question is to investigate the connection between these two seemingly different recurrence relations. We also leave the combinatorial meaning of the A_j^r as fodder for future research.

Another open question involves the definition of $\big(f(n)\big)_{n=1}^{\infty}$. Instead of defining $f(n)$ by (1.1), we define $f(n)$ by the nonlinear recurrence

$$f(n+1) = \sum_{k=0}^{n} \binom{n}{k} f(k) f(n-k), \qquad n \ge a, \tag{5.3}$$

where a is a nonnegative integer. In [**3**], we investigated (5.3) for the case of $a = 0$ and showed that such sequences have connections to various aspects of cell growth [**4**], [**5**]. Future research possibilities involve investigating (5.3) for $a \ge 1$.

r/j	0	1	2	3	4	4	6	7	8	9
1	1									
2	1	1								
3	3	1	1							
4	9	4	1	1						
5	31	14	5	1	1					
6	121	54	20	6	1	1				
7	523	233	85	27	7	1	1			
8	2468	1101	400	125	35	8	1	1		
9	12611	5625	2046	635	175	44	9	1	1	
10	69161	30846	11226	3488	952	236	54	10	1	1

Table 1: The Binomial Recurrence Coefficients A_j^r. The r runs vertically while the j runs horizontally.

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