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THE CONJUGATE FORMAL PRODUCT OF A GRAPH

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Let G be a simple graph of order n and let $V(G)$ be its vertex set. Let $c = a + b\sqrt{m}$ and $\overline{c} = a - b\sqrt{m}$, where a and b are two nonzero integers and m is a positive integer such that m is not a perfect square. We say that $A^c = [c_{ij}]$ is the conjugate adjacency matrix of the graph G if $c_{ij} = c$ for any two adjacent vertices i and j, $c_{ij} = \overline{c}$ for any two nonadjacent vertices i and j, and $c_{ij} = 0$ if $i = j$. Let $P_G^c(\lambda) = |\lambda I - A^c|$ denote the conjugate characteristic polynomial of G and let $[\mathbf{A^c_{ij}}] = \{\lambda I - A^c\}$, where $\{M\}$ denotes the adjoint matrix of a square matrix M. For any two subsets $X, Y \subseteq V(G)$ define $\langle X, Y\rangle^c = \sum\limits_{i\in X}\sum\limits_{j\in Y} \mathbf{A^c_{ij}}$. The expression $\langle X, Y\rangle^c$ is called the conjugate formal product of the sets X and Y, associated with the graph G. Using the conjugate formal product we continue our previous investigations of some properties of the conjugate characteristic polynomial of G.

1. INTRODUCTION

Let G be a simple graph of order n and let $V(G)$ be its vertex set. The spectrum of the graph G consists of the eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ of its (0,1) adjacency matrix $A = A(G)$ and is denoted by $\sigma(G)$. The Seidel spectrum of G consists of the eigenvalues $\lambda_1^* \geq \lambda_2^* \geq \cdots \geq \lambda_n^*$ of its $(0, -1, 1)$ adjacency matrix $A^* = A^*(G)$ and is denoted by $\sigma^*(G)$. Let $P_G(\lambda) = |\lambda I - A|$ and $P_G^*(\lambda) = |\lambda I - A^*|$ denote the characteristic polynomial and the Seidel characteristic polynomial, respectively. Let $c = a + b\sqrt{m}$ and $\overline{c} = a - b\sqrt{m}$ where a and b are two nonzero integers and m is a positive integer such that m is not a perfect square. We say that $A^c = [c_{ij}]$ is the conjugate adjacency matrix of G if $c_{ij} = c$ for any two adjacent vertices i and j, $c_{ij} = \overline{c}$ for any two nonadjacent vertices i and j, and $c_{ij} = 0$ if $i = j$. The conjugate spectrum of G is the set of the eigenvalues $\lambda_1^c \geq \lambda_2^c \geq \cdots \geq \lambda_n^c$ of its conjugate adjacency matrix $A^c = A^c(G)$ and is denoted by $\sigma^c(G)$. Let $P_G^c(\lambda) = |\lambda I - A^c|$ denote the conjugate characteristic polynomial of G.

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The concept of conjugate adjacency matrices has been defined in [**7**]. In that paper we have proved some elementary results related to the conjugate characteristic polynomial. In particular, we have proved the following results (i) the conjugate characteristic polynomial of its complement $\overline{G}$ can be determined by the conjugate characteristic polynomial of G; (ii) the characteristic polynomial of G with respect to the ordinary adjacency matrix can be determined by its conjugate characteristic polynomial; (iii) the conjugate characteristic polynomial of G is uniquely determined by the conjugate characteristic polynomials of its vertex-deleted subgraphs.

In this work, in order to provide more information on conjugate characteristic polynomial, we define the conjugate formal product associated with the graph G, as follows.

2. THE CONJUGATE FORMAL PRODUCT

For a square matrix M denote by $\{M\}$ the adjoint of M and for any two subsets $X, Y \subseteq V(G)$ define $\langle X, Y\rangle = \sum\limits_{i\in X}\sum\limits_{j\in Y} \mathbf{A_{ij}}$, where $[\mathbf{A_{ij}}] = \{\lambda I - A\}$. According to [**3**], the expression $\langle X, Y\rangle$ is called the formal product of the sets X and Y, associated with a graph G. In this work $\langle X, Y\rangle^c = \sum\limits_{i\in X}\sum\limits_{j\in Y} \mathbf{A^c_{ij}}$ is called the conjugate formal product of the sets X and Y, associated with the graph G, where $[\mathbf{A^c_{ij}}] = \{\lambda I - A^c\}$.

For any two disjoint subsets $X, Y \subseteq V(G)$ let $X + Y$ denote the union of X and Y. We note that $\langle X + Y, Z\rangle^c = \langle X, Z\rangle^c + \langle Y, Z\rangle^c$ for any $Z \subseteq V(G)$ and $\langle X, Y\rangle^c = \langle Y, X\rangle^c$ for any (not necessarily disjoint) $X, Y \subseteq V(G)$.

Further, let S be any (possibly empty) subset of the vertex set $V(G)$ and let G_S be the graph obtained from the graph G by adding a new vertex x ($x \notin V(G)$), which is adjacent exactly to the vertices from S. According to [**3**],

$$P_{G_S}(\lambda) = \lambda P_G(\lambda) - \langle S, S\rangle \,. \tag{1}$$

Using the method applied in [**3**] for getting relation (1), one can easily see that the conjugate characteristic polynomial of G_S is

$$P^c_{G_S}(\lambda) = \lambda P^c_G(\lambda) - c^2\,\langle S, S\rangle^c - \overline{c}^{\,2}\,\langle T, T\rangle^c - 2c\,\overline{c}\,\langle S, T\rangle^c\,, \tag{2}$$

where $T = V(G) \setminus S$.

Let G be an arbitrary connected graph of order n. We say that two vertices $x, y \in V(G)$ are equivalent and write $x \sim y$ if x is non-adjacent to y, and x and y have the same neighbors in G. Relation $\sim$ is an equivalence relation on the vertex set $V(G)$. The corresponding quotient graph is denoted by $\tilde{G}$ and is called the canonical graph of G.

Let $\tilde{G}$ be the canonical graph of G, $|\tilde{G}| = k$, and $N_1, N_2, \dots, N_k$ be the corresponding sets of equivalent vertices of G. Then we write $G = \tilde{G}(n_1, n_2, \dots, n_k)$, where $|N_i| = n_i$ $(i = 1, 2, \dots, k)$, understanding that $\tilde{G}$ is a labelled graph.

It was proved in [**9**] that the characteristic polynomial $P_G(\lambda)$ of the graph G takes the form

$$(3)\qquad P_G(\lambda) = n_1 \cdot n_2 \cdot \ldots \cdot n_k \, \lambda^{n-k} \begin{vmatrix} \dfrac{\lambda}{n_1} & -\tilde{a}_{12} & \cdots & -\tilde{a}_{1k} \\ -\tilde{a}_{21} & \dfrac{\lambda}{n_2} & \cdots & -\tilde{a}_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ -\tilde{a}_{k1} & -\tilde{a}_{k2} & \cdots & \dfrac{\lambda}{n_k} \end{vmatrix},$$

where $\tilde{A} = [\tilde{a}_{ij}]$ is the adjacency matrix of the canonical graph $\tilde{G}$.

Using the same method as in [**9**] for obtaining relation (3), we can easily see that the conjugate characteristic polynomial $P^c_G(\lambda)$ of the graph G is

$$(4)\quad P^c_G(\lambda) = (\lambda + \overline{c})^{n-k} \begin{vmatrix} \lambda - (n_1 - 1)\,\overline{c} & -n_1\,\tilde{c}_{12} & \cdots & -n_1\,\tilde{c}_{1k} \\ -n_2\,\tilde{c}_{21} & \lambda - (n_2 - 1)\,\overline{c} & \cdots & -n_2\,\tilde{c}_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ -n_k\,\tilde{c}_{k1} & -n_k\,\tilde{c}_{k2} & \cdots & \lambda - (n_k - 1)\,\overline{c} \end{vmatrix},$$

where $\tilde{A}^c = [\tilde{c}_{ij}]$ is the conjugate adjacency matrix of the canonical graph $\tilde{G}$.

Let G be any (not necessary canonical) graph of order n. Let $G_{x_1,x_2,\ldots,x_k}$ be the overgraph of G obtained by adding new vertices $x_1, x_2, \ldots, x_k$ equivalent to a vertex i of G, say $i = 1$, so that the vertices $x_1, x_2, \ldots, x_k, 1$ are mutually non-adjacent and have the same neighbors in G. In view of (4), applying the same method as in [**9**] for deriving relation (3), we can see that the conjugate characteristic polynomial of $G_{x_1,x_2,\ldots,x_k}$ is

$$(5)\quad P^c_{G_{x_1,x_2,\ldots,x_k}}(\lambda) = (\lambda + \overline{c})^k \begin{vmatrix} \lambda - k\overline{c} & -(k+1)\,c_{12} & \cdots & -(k+1)\,c_{1n} \\ -c_{21} & \lambda & \cdots & -c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -c_{n1} & -c_{n2} & \cdots & \lambda \end{vmatrix},$$

where $A^c = [c_{ij}]$ is the conjugate adjacency matrix of the graph G.

Let S be any subset of $V(G)$ and let G_{2S} be the overgraph of G obtained by adding two new non-adjacent vertices x, y which are both adjacent to the vertices from S. We note that G_{2S} is obtained from G_S by adding a new vertex y which is equivalent to $x \in V(G_S)$. Thus, using (1) we have

$$P^c_{G_{2S}}(\lambda) = \lambda P^c_{G_S}(\lambda) - c^2 \langle S, S\rangle^c - \overline{c}^{\,2} \langle T, T\rangle^c - 2c\,\overline{c}\,\langle S, T\rangle^c,$$

where $\langle X, Y\rangle^c$ is the conjugate formal product associated with G_S.

Proposition 1. *The conjugate characteristic polynomial $P^c_{G_{2S}}(\lambda)$ of the graph G_{2S} reads*

$$P^c_{G_{2S}}(\lambda) = (\lambda + \overline{c}\,)\big((\lambda - \overline{c}\,)P^c_G(\lambda) - 2c^2\,\langle S,S\rangle^c - 2\overline{c}^{\,2}\,\langle T,T\rangle^c - 4c\,\overline{c}\,\langle S,T\rangle^c\big)\,,$$

where $\langle X,Y\rangle^c$ is the conjugate formal product associated with the graph G.

Proof. Without loss of generality we may assume that $S = \{1,2,\ldots,i\}$. Using (5), we get

$$P^c_{G_{2S}}(\lambda) = (\lambda + \overline{c}\,)\begin{vmatrix} \lambda & \cdots & -\,c_{1i} & \cdots & -\,c_{1n} & -\,c \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ -\,c_{i1} & \cdots & \lambda & \cdots & -\,c_{in} & -\,c \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\,c_{n1} & \cdots & -\,c_{ni} & \cdots & \lambda & -\,\overline{c} \\ & & & & & \\ -\,2c & \cdots & -\,2c & \cdots & -\,2\overline{c} & \lambda - \overline{c} \end{vmatrix}.$$

Using the method which is applied in [**3**] for getting (1), by a straight-forward calculation we obtain the required statement. ☐

Let S be any subset of $V(G)$ and let G_{kS} be the overgraph of G obtained by adding k mutually non-adjacent vertices $x_1, x_2, \ldots, x_k$, all adjacent exactly to the vertices in S.

Corollary 1. *The conjugate characteristic polynomial $P^c_{G_{kS}}(\lambda)$ of the graph G_{ks} reads*

$$P^c_{G_{kS}}(\lambda) = (\lambda + \overline{c}\,)^{k-1}\big((\lambda - (k-1)\overline{c})P^c_G(\lambda) - k[S,S]^c\big)\,,$$

where $[S,S]^c = c^2\langle S,S\rangle^c + \overline{c}^{\,2}\langle T,T\rangle^c + 2c\,\overline{c}\,\langle S,T\rangle^c$.

Using (2) we find that $[S,S]^c = \lambda P^c_G(\lambda) - P^c_{G_S}(\lambda)$. Finally, according to Corollary 1 we obtain the following result.

Proposition 2. *The conjugate characteristic polynomial $P^c_{G_{kS}}(\lambda)$ of the graph G_{ks} reads*

$$P^c_{G_{kS}}(\lambda) = (\lambda + \overline{c}\,)^{k-1}\left(kP^c_{G_S}(\lambda) - (k-1)(\lambda + \overline{c}\,)P^c_G(\lambda)\right)$$

for any $S \subseteq V(G)$ and any $k \in \mathbb{N}$.

Corollary 2. *If G_{S_1} and G_{S_2} are two conjugate cospectral graphs then G_{kS_1} and G_{kS_2} are also conjugate cospectral for any $k \in \mathbb{N}$.*

Let $A^k = [a^{(k)}_{ij}]$ for any non-negative integer k. The number W_k of all walks of length k in G equals $\mathbf{sum}\,A^k$, where $\mathbf{sum}\,M$ is the sum of all elements in a matrix M. According to [**1**], the generating function $W_G(t)$ of the numbers W_k of length

k in the graph G is defined by $W_G(t) = \sum\limits_{k=0}^{+\infty} W_k t^k$. Besides [**1**]

$$W_G(t) = \frac{1}{t}\left[\frac{(-1)^n P_{\overline{G}}\big(-\dfrac{t+1}{t}\big)}{P_G\big(\dfrac{1}{t}\big)} - 1\right], \tag{6}$$

where $\overline{G}$ denotes the complement of G. The function $W_G^c(t) = \sum\limits_{k=0}^{+\infty} W_k^c t^k$ is called the conjugate generating function [**7**], where $W_k^c = \mathbf{sum}\,(A^c)^k$ and $(A^c)^k = [c_{ij}^{(k)}]$. According to [**7**],

$$W_G(t) = -\frac{2b\sqrt{m}}{(a-b\sqrt{m}\,)t}\left[\frac{P_G^c\left(\dfrac{2b\sqrt{m}-(a-b\sqrt{m}\,)t}{t}\right)}{\big(2b\sqrt{m}\,\big)^n P_G\big(\dfrac{1}{t}\big)} - 1\right]. \tag{7}$$

Therefore, making use of (6) and (7), by an easy calculation we obtain the following relation

$$P_G^c\big(2b\sqrt{m}\,\lambda - \overline{c}\,\big) = \big(2b\sqrt{m}\,\big)^{n-1}\big(cP_G(\lambda) - (-1)^n\,\overline{c}\,P_{\overline{G}}(-\lambda-1)\big). \tag{8}$$

Proposition 3 (Lepović [**4**]). *Let G be a graph of order n and let $S \subseteq V(G)$. Then*

$$P_{G_S}(\lambda) - P_{G_T}(\lambda) = (-1)^n\big(P_{\overline{G_S}}(-\lambda-1) - P_{\overline{G_T}}(-\lambda-1)\big), \tag{9}$$

where $T = V(G)\setminus S$.

Proposition 4. *Let G be a graph of order n and let $S \subseteq V(G)$. Then*

$$P_{G_S}^c(\lambda) - P_{G_T}^c(\lambda) = (-1)^n\big(P_{\overline{G_S}}^c(-\lambda-2a) - P_{\overline{G_T}}^c(-\lambda-2a)\big),$$

where $T = V(G)\setminus S$.

Proof. First, applying (8) to G_S and G_T and using (9), by a straight-forward calculation we find that

$$P_{G_S}^c\big(2b\sqrt{m}\,\lambda - \overline{c}\,\big) - P_{G_T}^c\big(2b\sqrt{m}\,\lambda - \overline{c}\,\big) = 2a\big(2b\sqrt{m}\,\big)^n\big(P_{G_S}(\lambda) - P_{G_T}(\lambda)\big).$$

Applying the last relation to $\overline{G_S}$ and $\overline{G_T}$ and making use of (9), we easily obtain the statement. $\square$

Definition 1 (Lepović [**5**]). *A graph G of order n is called spectral complementary, if*

$$P_G(\lambda) - P_{\overline{G}}(\lambda) = (-1)^n\big(P_G(-\lambda-1) - P_{\overline{G}}(-\lambda-1)\big). \tag{10}$$

Some elementary results of the spectral complementary graphs have been proved in [**5**] and [**6**]. Among other things, we proved (i) $G\cup\overline{G}$ is spectral complementary for any G; (ii) there is no spectral complementary graph of order $4k+3$ for any non-negative integer k; (iii) G is spectral complementary if and only if its Seidel spectrum $\sigma^*(G)$ is symmetric with respect to the zero point. In this work, combining (8) and (10) we obtain the following result.

Proposition 5. *Let G be a graph of order n. Then G is spectral complementary if and only if*

$$P^c_G(\lambda)-P^c_{\overline{G}}(\lambda)=(-1)^n\left(P^c_G(-\lambda-2a)-P^c_{\overline{G}}(-\lambda-2a)\right).$$

Further, let $[W^c_G(t)]=\sum\limits_{k=0}^{+\infty}(A^c)^k\,t^k$. It is clear that $W^c_G(t)=\mathbf{sum}\,[W^c_G(t)]$. In view of this we find that $[W^c_G(t)]=|I-tA|^{-1}\cdot\{I-tA^c\}$, which results in

$$\frac{1}{\lambda^{n-1}}\,\mathbf{A^c_{ij}}=\frac{1}{\lambda^n}\,P^c_G(\lambda)\sum_{k=0}^{+\infty}c^{(k)}_{ij}\,\frac{1}{\lambda^k}\,,$$

where $[\mathbf{A^c_{ij}}]=\{\lambda I-A^c\}$. Consequently, for any two sets $X,Y\subseteq V(G)$ the following relations is obtained

$$\begin{aligned}\langle X,Y\rangle^c&=\sum_{i\in X}\sum_{j\in Y}\mathbf{A^c_{ij}}=\frac{P^c_G(\lambda)}{\lambda}\sum_{i\in X}\sum_{j\in Y}\left[\sum_{k=0}^{+\infty}c^{(k)}_{ij}\,\frac{1}{\lambda^k}\right]\\&=\frac{P^c_G(\lambda)}{\lambda}\sum_{k=0}^{+\infty}\left[\sum_{i\in X}\sum_{j\in Y}c^{(k)}_{ij}\right]\frac{1}{\lambda^k}\\&=\frac{P^c_G(\lambda)}{\lambda}\,\mathfrak{F}^c_{X,Y}\Big(\frac{1}{\lambda}\Big)\,.\end{aligned}$$

Proposition 6. *Let X,Y be any two subsets of the vertex set $V(G)$. Then*

$$\langle X,Y\rangle^c=\frac{P^c_G(\lambda)}{\lambda}\,\mathfrak{F}^c_{X,Y}\Big(\frac{1}{\lambda}\Big)\,,$$

where $\mathfrak{F}^c_{X,Y}(t)=\sum\limits_{k=0}^{+\infty}c_kt^k$ and $c_k=\sum\limits_{i\in X}\sum\limits_{j\in Y}c^{(k)}_{ij}$ $(k=0,1,2,\dots)$.

The function $\mathfrak{F}^c_{X,Y}(t)$ is called the formal conjugate generating function, associated with the graph G. In particular, for $Y=X$ we denote the corresponding formal conjugate generating function $\mathfrak{F}^c_{X,Y}(t)$ by $\mathfrak{F}^c_X(t)$.

Let i be a fixed vertex from the vertex set $V(G)$ and let $G_i=G\setminus i$ be its corresponding vertex deleted subgraph.

Proposition 7. *Let G be a graph of order n. Then for any vertex deleted subgraph G_i we have*

$$P^c_{G_i}(\lambda)=\frac{P^c_G(\lambda)}{\lambda}\sum_{k=0}^{+\infty}\frac{c^{(k)}_{ii}}{\lambda^k}\,. \tag{11}$$

Proof. We note that $\mathbf{A^c_{ii}} = P^c_{G_i}(\lambda)$ for $i = 1, 2, \dots, n$. Using Proposition 6 we obtain the proof. $\square$

We note also that $\mathfrak{F}^c_{S^\bullet}(t) = W^c_G(t)$ if $S^\bullet = V(G)$. Using this fact and using the following relation [**7**]

$$W^c_G(t) = \frac{1}{2at}\left[\frac{(-1)^n P^c_{\overline{G}}\left(-\dfrac{2at+1}{t}\right)}{P^c_G\left(\dfrac{1}{t}\right)} - 1\right], \tag{12}$$

we arrive at

Proposition 8. *Let G be a graph of order n and let $G^\bullet = G_{S^\bullet}$, where $S^\bullet = V(G)$. Then*

$$P^c_{G^\bullet}(\lambda) = \left(\lambda + \frac{c^2}{2a}\right) P^c_G(\lambda) - (-1)^n \frac{c^2}{2a} P^c_{\overline{G}}(-\lambda - 2a)\,. \tag{13}$$

Proof. Using (2) we get $P^c_{G^\bullet}(\lambda) = \lambda P^c_G(\lambda) - c^2\langle S^\bullet, S^\bullet\rangle^c$. Using (12) and Proposition 6 we obtain the statement. $\square$

Proposition 9. *Let G be a graph of order n and let $G_\bullet = G_{S_\bullet}$, where $S_\bullet = \emptyset$. Then*

$$P^c_{G_\bullet}(\lambda) = \left(\lambda + \frac{\overline{c}^{\,2}}{2a}\right) P^c_G(\lambda) - (-1)^n \frac{\overline{c}^{\,2}}{2a} P^c_{\overline{G}}(-\lambda - 2a)\,. \tag{14}$$

Proof. Using (2) we get $P^c_{G_\bullet}(\lambda) = \lambda P^c_G(\lambda) - \overline{c}^{\,2}\langle S^\bullet, S^\bullet\rangle^c$. Using (12) and Proposition 6 we obtain the statement. $\square$

Next, replacing λ with $x + y\sqrt{m}$ the conjugate characteristic polynomial $P^c_G(\lambda)$ can be transformed into the form

$$P^c_G(x + y\sqrt{m}) = Q_n(x, y) + \sqrt{m}\, R_n(x, y)\,, \tag{15}$$

where $Q_n(x, y)$ and $R_n(x, y)$ are two polynomials of order n in variables x and y, whose coefficients are integers. Besides, according to [**7**]

$$P^c_{\overline{G}}(x - y\sqrt{m}) = Q_n(x, y) - \sqrt{m}\, R_n(x, y)\,. \tag{16}$$

We demonstrated in [**8**] that the characteristic polynomial and the Seidel characteristic polynomial of G can be expressed by polynomials $Q_n(-a, \lambda)$ and $R_n(-a, \lambda)$. We shall now determine the corresponding polynomials $Q_n(x, y)$ and $R_n(x, y)$ for $G^\bullet$ and $G_\bullet$.

Proposition 10. *Let* $P_G^c(x+y\sqrt{m}\,) = Q_n(x,y) + \sqrt{m}\,R_n(x,y)$. *Then we have*:

$$Q_{n+1}^{(k)}(x,y) = \frac{a^2+mb^2}{2a}\big(Q_n(x,y) - (-1)^n\,Q_n(-x-2a,y)\big) \pm mb\,\big(R_n(x,y) + (-1)^n\,R_n(-x-2a,y)\big) + x\,Q_n(x,y) + myR_n(x,y)\,; \tag{17}$$

$$R_{n+1}^{(k)}(x,y) = \frac{a^2+mb^2}{2a}\big(R_n(x,y) + (-1)^n\,R_n(-x-2a,y)\big) \pm b\,\big(Q_n(x,y) - (-1)^n\,Q_n(-x-2a,y)\big) + x\,R_n(x,y) + y\,Q_n(x,y)\,, \tag{18}$$

where $Q_{n+1}^{(k)}(x,y)$ *and* $R_{n+1}^{(k)}(x,y)$ *for* $k=1,2$ *are related to* $G^{\bullet}$ *and* $G_{\bullet}$, *respectively. The symbol '*$\pm$*' is related to '*$+$*' if* $k=1$ *and '*$\pm$*' is related to '*$-$*' if* $k=2$.

Proof. Applying (13) and (14) to $G^{\bullet}$ and $G_{\bullet}$ and to their complements $\overline{G}_{\bullet}$ and $\overline{G}^{\bullet}$, and making use of (15) and (16), we arrive at (17) and (18). □

Corollary 3. *Let* $P_G^c(x+y\sqrt{m}\,) = Q_n(x,y) + \sqrt{m}\,R_n(x,y)$. *Then we have*:

- $Q_{n+1}^{(k)}(-a,\lambda) = -a\,Q_n(-a,\lambda) + m(\lambda \pm 2b)\,R_n(-a,\lambda)$ *if* n *is even*;
- $Q_{n+1}^{(k)}(-a,\lambda) = \dfrac{mb^2}{a}\,Q_n(-a,\lambda) + m\lambda\,R_n(-a,\lambda)$

if n *is odd.*

Corollary 4. *Let* $P_G^c(x+y\sqrt{m}\,) = Q_n(x,y) + \sqrt{m}\,R_n(x,y)$. *Then we have*:

- $R_{n+1}^{(k)}(-a,\lambda) = (\lambda \pm 2b)\,Q_n(-a,\lambda) - aR_n(-a,\lambda)$ *if* n *is odd;*
- $R_{n+1}^{(k)}(-a,\lambda) = \lambda\,Q_n(-a,\lambda) + \dfrac{mb^2}{a}\,R_n(-a,\lambda)$

if n *is even.*

3. THE CONJUGATE ANGLE MATRIX

Let $\mu_1^c > \mu_2^c > \cdots > \mu_m^c$ be the distinct conjugate eigenvalues of a graph G of order n and let $\mathcal{E}_{A^c}(\mu_i^c)$ denote the eigenspace of the conjugate eigenvalue μ_i^c. Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be the standard orthonormal basis of $\mathbb{R}^n$ and let $\mathbf{P}_i^c$ denote the orthogonal projection of the space $\mathbb{R}^n$ onto $\mathcal{E}_{A^c}(\mu_i^c)$.

Definition 2. *The numbers* $||\mathbf{P}_i^c\mathbf{e}_j|| = \cos\beta_{ij}^c$ $(i=1,2,\ldots,m;\ j=1,2,\ldots,n)$, *where* β_{ij}^c *is the angle between* $\mathcal{E}_{A^c}(\mu_i^c)$ *and* $\mathbf{e}_j$, *are called the conjugate angles of* G. *The* $m\times n$ *matrix* $\mathcal{A}^c = [\mathbf{P}_i^c\mathbf{e}_j]$ *is called the conjugate angle matrix of* G. *The* (i,j)*-entry of* $\mathcal{A}^c$ *is* $||\mathbf{P}_i^c\mathbf{e}_j||$.

The conjugate angle matrix $\mathcal{A}^c$ is an algebraic (graph) invariant, provided that its columns are ordered lexicographically.

Using the same arguments as in [**2**], we have (i) $\sum_{j=1}^{n} \gamma_{ij}^2 = \dim \mathcal{E}_{A^c}(\mu_i^c)$, where $\gamma_{ij} = \cos \beta_{ij}^c$; (ii) $\sum_{i=1}^{m} \gamma_{ij}^2 = 1$. Besides, using the spectral decomposition of A^c, the following relation is obtained

$$c_{ii}^{(k)} = \sum_{j=1}^{m} (\mu_j^c)^k \gamma_{ji}^2 \qquad (k = 0, 1, 2, \dots)\,. \tag{19}$$

We note from (19) that $\sum_{j=1}^{m} \mu_j^c \, \gamma_{ji}^2 = 0$ and $\sum_{j=1}^{m} (\mu_j^c)^2 \, \gamma_{ji}^2 = d_i \, c^2 + ((n-1) - d_i)\, \overline{c}^{\,2}$, where d_i denotes the degree of the vertex i. Moreover, from (11) and (19) we find the next result.

Proposition 11. *Let G be a graph of order n. Then for any vertex deleted subgraph G_i we have*

$$P_{G_i}^c(\lambda) = P_G^c(\lambda) \sum_{j=1}^{m} \frac{\gamma_{ji}^2}{\lambda - \mu_j^c}\,.$$

Proposition 12 (Lepović [**7**]). *If G and H are two conjugate cospectral graphs then their complementary graphs $\overline{G}$ and $\overline{H}$ are also conjugate cospectral.*

Corollary 5. *If $\gamma_{ki} = \gamma_{kj}$ for $k = 1, 2, \dots, m$ then $\sigma^c(G_i) = \sigma^c(G_j)$ and $\sigma^c(\overline{G}_i) = \sigma^c(\overline{G}_j)$.*

Finally, using the conjugate formal product and conjugate formal generating functions, we shall determine the conjugate characteristic polynomial of the graph $(K_n)_S$, where K_n is the complete graph on n vertices and $S \subseteq V(K_n)$. First, we note from (2) and Proposition 6 that

$$P_{G_S}^c(\lambda) = P_G^c(\lambda) \Big[\lambda - \frac{1}{\lambda}\, \mathfrak{F}_{[S]}^c\Big(\frac{1}{\lambda}\Big)\Big]\,, \tag{20}$$

where $\mathfrak{F}_{[S]}^c(t) = c^2\, \mathfrak{F}_S^c(t) + \overline{c}^{\,2}\, \mathfrak{F}_T^c(t) + 2c\overline{c}\, \mathfrak{F}_{S,T}^c(t)$.

Proposition 13. *For $S \subseteq V(K_n)$ let $s = |S|$ and $r = sc^2 + (n-s)\,\overline{c}^{\,2}$. Then $P_{(K_n)_S}^c(\lambda) = (\lambda + c)^{n-2}\, \Delta(\lambda)$, where*

$$\Delta(\lambda) = \lambda^3 - (n-2)\, c\, \lambda^2 - \big(r + (n-1)\, c^2\big)\, \lambda - \big(r - s(n-s)(c - \overline{c}\,)^2\big) c\,.$$

Proof. Since K_n is a regular graph of degree $n-1$, we obtain that $W_k^c = n(n-1)^k\, c^k$. Let $\alpha_k = c_{11}^{(k)}$ and $\beta_k = c_{12}^{(k)}$ $(k = 0, 1, 2, \dots)$. It is clear that $c_{ii}^{(k)} = \alpha_k$ $(i = 1, 2, \dots, n)$ and $c_{ij}^{(k)} = \beta_k$ $(i \neq j)$. Consequently, we get $n(n-1)^k\, c^k = n\alpha_k + (n^2 - n)\beta_k$. Since $\alpha_k = (n-1)\, c\, \beta_{k-1}$, the expressions for

$$\alpha_k = \Big[\frac{(n-1)^k + (-1)^k (n-1)}{n}\Big] c^k \quad \text{and} \quad \beta_k = \Big[\frac{(n-1)^k + (-1)^{k-1}}{n}\Big] c^k$$

can be obtained by solving the linear recursions

$$\beta_k = (n-1)^{k-1}\, c^k - c\,\beta_{k-1} \quad \text{and} \quad \alpha_k = (n-1)\, c\,\beta_{k-1}$$

with $\alpha_0 = 1$ and $\beta_0 = 0$.

Since $c_k = \sum\limits_{i\in S}\sum\limits_{j\in S} c_{ij}^{(k)}$ and $|S| = s$, we have $c_k = s\alpha_k + (s^2 - s)\beta_k$. In view of this we get

$$c_k = \Big[\frac{s^2(n-1)^k + (-1)^{k-1}\, s^2 + (-1)^k\, sn}{n}\Big] c^k\,,$$

which results in

$$\mathfrak{F}_S^c(t) = \frac{s^2}{n(1-(n-1)\,ct)} - \frac{s^2}{n(1+ct)} + \frac{s}{1+ct}\,. \tag{21}$$

Similarly, we obtain

$$\mathfrak{F}_T^c(t) = \frac{(n-s)^2}{n(1-(n-1)\,ct)} - \frac{(n-s)^2}{n(1+ct)} + \frac{n-s}{1+ct}\,. \tag{22}$$

Now, denote by e_k the corresponding coefficients of the function $\mathfrak{F}_{S,T}^c(t)$. Since $e_k = s(n-s)\beta_k$, we arrive at

$$\mathfrak{F}_{S,T}^c(t) = \frac{s(n-s)}{n(1-(n-1)\,ct)} + \frac{s^2}{n(1+ct)} - \frac{s}{1+ct}\,.$$

From (21), (22) and the last relation, by an easy calculation we obtain that the corresponding function $\mathfrak{F}_{[S]}^c(t)$ reads

$$\mathfrak{F}_{[S]}^c(t) = \frac{\big(r - s(n-s)(c-\overline{c}\,)^2\big)\, ct + r}{(1-(n-1)ct)(1+ct)}\,.$$

Finally, using that $P_{K_n}^c(\lambda) = (\lambda + c)^{n-1}(\lambda - (n-1)c)$ and using (20), we obtain the statement. $\square$

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