

ON SO'S CONJECTURE FOR INTEGRAL CIRCULANT GRAPHS

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Each integral circulant graph $ICG(n, \mathcal{D})$ is characterised by its order n and a set \mathcal{D} of positive divisors of n in such a way that it has vertex set $\mathbb{Z}/n\mathbb{Z}$ and edge set $\{(a, b) : a, b \in \mathbb{Z}/n\mathbb{Z}, \gcd(a - b, n) \in \mathcal{D}\}$. According to a conjecture of SO two integral circulant graphs are isomorphic if and only if they are isospectral, i.e. they have the same eigenvalues (counted with multiplicities). We prove a weaker form of this conjecture, namely, that two integral circulant graphs with multiplicative divisor sets are isomorphic if and only if their spectral vectors coincide.

1. INTRODUCTION AND RESULTS

The vivid question “Can one hear the shape of a drum?”, posed by KAC [11] in 1966, has become a synonym for the considerably older and much more general problem to decide whether a Riemannian manifold is determined by its spectrum. In [7] FISHER formulated the discrete analogue of KAC’S question and thus transferred it to the examination of spectra of linear graphs by use of their adjacency matrices. Originating from a problem in chemistry [8], it has been asked in general since the mid-twentieth century which graphs are determined by their spectrum, and some answers were given for different types of graphs (cf. [6] for a survey). Our paper addresses this problem with regard to the class of integral circulant graphs and partially proves a conjecture of SO [19].

Integral circulant graphs are those having a circulant adjacency matrix with integral spectrum, i.e. all eigenvalues of the adjacency matrix are integers. By the works of SO [19] and KLOTZ and T. SANDER [12] each integral circulant graph

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$\text{ICG}(n, \mathcal{D})$ is characterised by its order n and a set $\mathcal{D} \subseteq D(n) := \{d > 0 : d \mid n\}$ of positive divisors of n in such a way that it has vertex set $\mathbb{Z}/n\mathbb{Z}$ and edge set $\{(a, b) : a, b \in \mathbb{Z}/n\mathbb{Z}, \gcd(a - b, n) \in \mathcal{D}\}$. These graphs comprise algebraic, arithmetic and combinatorial features at the same time, and quite a lot of interesting results have been obtained in recent years (see [16] for references). In particular, the examination of the spectra of integral circulant graphs attracted a lot of attention (cf. [1], [2], [9], [17], [18]). It should be noted that one usually assumes $n \notin \mathcal{D}$, since $\text{ICG}(n, \mathcal{D})$ has loops otherwise. Moreover, it is known that $\text{ICG}(n, \mathcal{D})$ is connected only if the elements of \mathcal{D} are coprime (cf. [5, Prop. 1]). However, our results also hold if these conditions are violated.

According to a Conjecture of SO (cf. [19, Conj. 7.3] and [10, page 2]) two integral circulant graphs are isomorphic if and only if they are isospectral (or, synonymously, cospectral, i.e. they have the same spectrum). More precisely,

Conjecture of So. *Let n be a positive integer, and let $\mathcal{D}, \mathcal{E} \subseteq D(n)$ be two arbitrary divisor sets of n . Then the integral circulant graphs $\text{ICG}(n, \mathcal{D})$ and $\text{ICG}(n, \mathcal{E})$ are isomorphic if and only if the sets of eigenvalues (counted with multiplicities) of the two graphs are equal.*

As already observed by SO the conjecture holds for all prime powers n (cf. [19, Sect. 7]), assuming the graphs to have no loops. This can be shown quite easily, for instance by verifying that two loopless integral circulant graphs of the same prime power order with different divisor sets have different largest eigenvalues (see Corollary 2.2 below). This is no longer true for graphs of order different from prime powers as shown by the two connected graphs $\text{ICG}(6, \{1, 3\})$ and $\text{ICG}(6, \{2, 3\})$, both having largest eigenvalue 3. Apart from the aforementioned result, the conjecture has been verified only for very special cases, e.g. n being a product of two primes, or n being squarefree and divisor sets having at most two prime elements (cf. [10]).

In [19, Sect. 3] one can find an example of two circulant graphs, i.e. Cayley graphs on cyclic groups, which are non-isomorphic but have the same spectrum. This shows that SO's Conjecture in general requires the integrality condition. It is also easy to find non-isomorphic integral circulant graphs of the same order with different divisor sets having the same spectrum if we neglect the multiplicities of the eigenvalues. For instance, $\text{ICG}(48, \{2, 3, 4, 24\})$ and $\text{ICG}(48, \{2, 3, 8, 16, 24\})$ have the same set $\{21, 5, 3, -1, -3, -13\}$ of eigenvalues, but these occur with different multiplicities and therefore the two graphs are not isomorphic.

In the sequel we shall prove a weaker form of the Conjecture of SO. In order to formulate our additional requirements we need some further terminology, namely *multiplicative* divisor sets and *spectral vectors*.

The idea to study the subclass of integral circulant graphs $\text{ICG}(n, \mathcal{D})$ with multiplicative divisor sets \mathcal{D} was introduced by LE and the first author in [13]. They applied this new concept successfully in [14] to improve results on extremal energies of integral circulant graphs for arbitrary n . In [15] the first author used it again to consider questions concerning integral circulant Ramanujan graphs. For a

positive integer d and a number p in the set \mathbb{P} of all primes, we denote by $e_p(d)$ the order of p in d . A non-empty finite set \mathcal{D} of positive integers is called *multiplicative* if $\mathcal{D} = \prod_{p \in \mathbb{P}} \mathcal{D}_p$, where $\mathcal{D}_p := \{p^{e_p(a)} : a \in \mathcal{D}\}$ for each prime p , and the product of sets D_1, \dots, D_t of positive integers is defined as $\prod_{i=1}^t D_i := \{d_1 \cdots d_t : d_i \in D_i\}$. Observe that $\mathcal{D}_p \neq \{1\}$ only for those finitely many primes dividing at least one of the $d \in \mathcal{D}$. Hence $\prod_{p \in \mathbb{P}} \mathcal{D}_p$ can be regarded as a finite product for any finite set \mathcal{D} .

The *spectral vector* of an integral circulant graph $\text{ICG}(n, \mathcal{D})$ with an arbitrary positive integer n and arbitrary divisor set $\mathcal{D} \subseteq D(n)$ is defined as

$$\vec{\lambda}(n, \mathcal{D}) := (\lambda_1(n, \mathcal{D}), \lambda_2(n, \mathcal{D}), \dots, \lambda_n(n, \mathcal{D})),$$

where

$$(1) \quad \lambda_\ell(n, \mathcal{D}) = \sum_{d \in \mathcal{D}} c\left(\ell, \frac{n}{d}\right) \quad (1 \leq \ell \leq n)$$

are the eigenvalues of $\text{ICG}(n, \mathcal{D})$ with multiplicities [12, Theor. 16]. Here

$$c(\ell, n) := \sum_{\substack{j \bmod n \\ (j, n)=1}} e\left(\frac{\ell j}{n}\right)$$

denotes the well-known Ramanujan sum (cf. [3, chapt. 8.3-8.4]), and we use the notation $e(x) := e^{2\pi i x}$ for real x . Note that

$$(2) \quad \vec{\lambda}(n, \mathcal{D}_1 \cup \mathcal{D}_2) = \vec{\lambda}(n, \mathcal{D}_1) + \vec{\lambda}(n, \mathcal{D}_2)$$

for disjoint $\mathcal{D}_1, \mathcal{D}_2 \subseteq D(n)$.

Weak Conjecture of So. *Let n be a positive integer, and let $\mathcal{D}, \mathcal{E} \subseteq D(n)$ be two multiplicative divisor sets of n . Then $\text{ICG}(n, \mathcal{D})$ and $\text{ICG}(n, \mathcal{E})$ are isomorphic if and only if $\vec{\lambda}(n, \mathcal{D}) = \vec{\lambda}(n, \mathcal{E})$.*

Our first result provides an explicit formula for the eigenvalues of arbitrary integral circulant graphs with multiplicative divisor sets.

Theorem 1.1. *Let n be a positive integer with prime factorisation $n = p_1^{k_1} \cdots p_r^{k_r}$, and let $\mathcal{D} = \mathcal{D}_{p_1} \cdots \mathcal{D}_{p_r} \subseteq D(n)$ be a multiplicative divisor set of n , where $\mathcal{D}_{p_j} = \{p_j^{k_{j,1}}, p_j^{k_{j,2}}, \dots, p_j^{k_{j,s_j}}\}$, say, with $0 \leq k_{j,1} < k_{j,2} < \dots < k_{j,s_j} \leq k_j$ for $1 \leq j \leq r$. Given some ℓ , $1 \leq \ell \leq n$, we define $p_j^{t_j} := \gcd(\ell, p_j^{k_j})$ for $1 \leq j \leq r$. Then*

$$\lambda_\ell(n, \mathcal{D}) = \prod_{j=1}^r \left(-\chi(p_j^{k_j}, \mathcal{D}_{p_j}, t_j) p_j^{t_j} + \sum_{\substack{i=1 \\ k_{j,i} \geq k_j - t_j}}^{s_j} \varphi(p_j^{k_j - k_{j,i}}) \right),$$

where φ denotes Euler's totient function and

$$(3) \quad \chi(p^k, \mathcal{D}, t) := \begin{cases} 1 & \text{if } k_i = k - t - 1 \text{ for some } 1 \leq i \leq s, \\ 0 & \text{otherwise,} \end{cases}$$

for $0 \leq t \leq k$ and $\mathcal{D} = \{p^{k_1}, p^{k_2}, \dots, p^{k_s}\}$, say, with $0 \leq k_1 < k_2 < \dots < k_s \leq k$.

The following theorem proves the Weak Conjecture of So.

Theorem 1.2. *Let $\mathcal{D}, \mathcal{E} \subseteq D(n)$ be two multiplicative divisor sets of the positive integer n . Then the integral circulant graphs $\text{ICG}(n, \mathcal{D})$ and $\text{ICG}(n, \mathcal{E})$ are isomorphic if and only if the corresponding spectral vectors satisfy $\vec{\lambda}(n, \mathcal{D}) = \vec{\lambda}(n, \mathcal{E})$.*

In order to give an impression of the significance of Theorem 1.2 as compared to the original Conjecture of So we determine the proportion of multiplicative divisor sets among all divisor sets for any given positive integer n . To this end, we define $\delta(n)$ to be the number of divisor sets of n and $\mu(n)$ to be the number of multiplicative divisor sets of n .

Proposition 1.1. *Let $n > 1$ be an integer with prime factorisation $n = p_1^{k_1} \dots p_r^{k_r}$. Then $\delta(n) = 2^{(k_1+1)\dots(k_r+1)}$ and $\mu(n) = 2^{(k_1+1)+\dots+(k_r+1)}$.*

Proof. Clearly, the set of all divisor sets is the power set $\mathfrak{P}(D(n))$ of the set $D(n)$ of all divisors of n . Hence $|D(n)| = d(n)$ for the well-known multiplicative divisor function $d(n)$. A standard result in elementary number theory says that

$$d(n) = \prod_{i=1}^r (k_i + 1), \text{ and therefore}$$

$$\delta(n) = |\mathfrak{P}(D(n))| = 2^{d(n)} = 2^{(k_1+1)\dots(k_r+1)}.$$

In comparison, each multiplicative divisor set \mathcal{D} of n is uniquely represented by $\mathcal{D} = \mathcal{D}_{p_1} \dots \mathcal{D}_{p_r}$, where each $\mathcal{D}_{p_j} \subseteq D(p_j^{k_j})$. Hence

$$\mu(n) = \prod_{j=1}^r |\mathfrak{P}(D(p_j^{k_j}))| = \prod_{j=1}^r 2^{|D(p_j^{k_j})|} = \prod_{j=1}^r 2^{k_j+1} = 2^{(k_1+1)+\dots+(k_r+1)}. \quad \square$$

For any prime power p^k all divisor sets $\mathcal{D} \subseteq D(p^k)$ are trivially multiplicative. More precisely, Proposition 1.1 shows that $\delta(p^k) = \mu(p^k) = 2^{k+1}$. In general, Proposition 1.1 reveals that for integers n with at least two distinct prime factors the proportion of multiplicative divisor sets among all divisor sets of n decreases rapidly with growing number of distinct prime factors and with growing exponents in the prime factorisation of n .

However, the study of integral circulant graphs with multiplicative divisor sets turned out to be fruitful with regard to problems concerning extremal energies of ICGs [14] or the identification of Ramanujan graphs in the class of ICGs [15]. Integral circulant graphs with multiplicative divisor sets seem to have a certain degree of representativity for ICGs in general. Apart from that Theorem 1.2 is the only non-trivial case where the Conjecture of So could be verified so far, namely for graphs $\text{ICG}(n, \mathcal{D})$ with arbitrary n and arbitrarily large divisor sets \mathcal{D} .

Since $p \nmid m$, the set of integers mj with $j \bmod p^{k-t-k_i}$, $p \nmid j$, runs through a reduced residue system mod p^{k-t-k_i} . Hence

$$(5) \quad \sum_{\substack{j \bmod p^{k-t-k_i} \\ p \nmid j}} e\left(\frac{mj}{p^{k-t-k_i}}\right) = \sum_{\substack{j \bmod p^{k-t-k_i} \\ p \nmid j}} e\left(\frac{j}{p^{k-t-k_i}}\right),$$

which shows that $\lambda_\ell(p^k, \mathcal{D})$ does not depend on m , and this already proves (i).

Some elementary computations involving geometric sums reveal that for any positive integer u

$$\sum_{\substack{j \bmod p^u \\ p \nmid j}} e\left(\frac{j}{p^u}\right) = \sum_{j=1}^{p^u} e\left(\frac{1}{p^u}\right)^j - \sum_{j=1}^{p^u-1} e\left(\frac{pj}{p^u}\right) = - \sum_{j=1}^{p^u-1} e\left(\frac{1}{p^{u-1}}\right)^j = \begin{cases} -1 & \text{for } u = 1, \\ 0 & \text{for } u \geq 2. \end{cases}$$

Inserting this identity into (5) and the result into (4), (ii) follows. \square

Before we turn our attention to the crucial observation concerning the position and value of the first non-zero entry in the spectral vector $\vec{\lambda}(p^k, \mathcal{D})$ of $\text{ICG}(p^k, \mathcal{D})$ (cf. Proposition 2.4), let us deduce several easy consequences of Proposition 2.2.

Corollary 2.1. *Let p^k be a prime power and $\mathcal{D} \subseteq D(p^k)$.*

(i) *For $0 \leq u \leq v \leq k$, we have $|\lambda_{p^u}(p^k, \mathcal{D})| \leq |\lambda_{p^v}(p^k, \mathcal{D})|$.*

(ii) *$\lambda_{p^k}(p^k, \mathcal{D})$ is positive and the largest eigenvalue of $\text{ICG}(p^k, \mathcal{D})$.*

Proof. Assertion (ii) is an immediate consequence of (i), Proposition 2.2 (i) and the fact that $\chi(p^k, \mathcal{D}, k) = 0$. Therefore it remains to verify (i).

It suffices to prove that $|\lambda_{p^u}(p^k, \mathcal{D})| \leq |\lambda_{p^{u+1}}(p^k, \mathcal{D})|$ for $0 \leq u < k$. In order to apply Proposition 2.2 (ii) let $\mathcal{D} = \{p^{k_1}, \dots, p^{k_s}\}$, say, with $0 \leq k_1 < k_2 < \dots < k_s \leq k$ and set $\mathcal{K} := \{k_1, \dots, k_s\}$. Then Proposition 2.2 (ii) implies for $0 \leq t \leq k$ that

$$\begin{aligned} \lambda_{p^t}(p^k, \mathcal{D}) &= -\chi(p^k, \mathcal{D}, t)p^t + \sum_{\substack{i=1 \\ k_i \geq k-t}}^s \varphi(p^{k-k_i}) \\ &= \begin{cases} \sum_{\substack{i=1 \\ k_i \geq k-t}}^s \varphi(p^{k-k_i}) & \text{if } k-t-1 \notin \mathcal{K}, \\ -p^t + \sum_{i=j+1}^s \varphi(p^{k-k_i}) & \text{if } k_j := k-t-1 \in \mathcal{K}, \end{cases} \end{aligned}$$

hence

$$(6) \quad |\lambda_{p^t}(p^k, \mathcal{D})| = \begin{cases} \sum_{\substack{i=1 \\ k_i \geq k-t}}^s \varphi(p^{k-k_i}) & \text{if } k-t-1 \notin \mathcal{K}, \\ p^{k-k_j-1} - \sum_{i=j+1}^s \varphi(p^{k-k_i}) & \text{if } k_j := k-t-1 \in \mathcal{K}. \end{cases}$$

Case 1: $k - u - 1 \notin \mathcal{K}$ and $k - (u + 1) - 1 \notin \mathcal{K}$.

It follows from (6) that

$$|\lambda_{p^u}(p^k, \mathcal{D})| = \sum_{\substack{i=1 \\ k_i \geq k-u}}^s \varphi(p^{k-k_i}) \leq \sum_{\substack{i=1 \\ k_i \geq k-(u+1)}}^s \varphi(p^{k-k_i}) = |\lambda_{p^{u+1}}(p^k, \mathcal{D})|.$$

Case 2: $k_j := k - u - 1 \in \mathcal{K}$ and $k - (u + 1) - 1 \notin \mathcal{K}$.

Again by (6), we obtain that

$$\begin{aligned} |\lambda_{p^u}(p^k, \mathcal{D})| &= p^{k-k_j-1} - \sum_{i=j+1}^s \varphi(p^{k-k_i}) \leq p^{k-k_j-1} \leq \varphi(p^{k-k_j}) \\ &\leq \sum_{\substack{i=1 \\ k_i \geq k_j}}^s \varphi(p^{k-k_i}) = \sum_{\substack{i=1 \\ k_i \geq k-(u+1)}}^s \varphi(p^{k-k_i}) = |\lambda_{p^{u+1}}(p^k, \mathcal{D})|. \end{aligned}$$

Case 3: $k - u - 1 \notin \mathcal{K}$ and $k_{j-1} := k - (u + 1) - 1 \in \mathcal{K}$.

Since $k - u - 1 \notin \mathcal{K}$, we have $k_j \geq k_{j-1} + 2$ in this case. With this (6) yields that

$$\begin{aligned} |\lambda_{p^u}(p^k, \mathcal{D})| &= \sum_{\substack{i=1 \\ k_i \geq k-u}}^s \varphi(p^{k-k_i}) = \sum_{\substack{i=1 \\ k_i \geq k_{j-1}+2}}^s \varphi(p^{k-k_i}) = \sum_{i=j}^s \varphi(p^{k-k_i}) \\ &\leq p^{k-k_j+1} - \sum_{i=j}^s \varphi(p^{k-k_i}) \leq p^{k-k_{j-1}-1} - \sum_{i=j}^s \varphi(p^{k-k_i}) \\ &= |\lambda_{p^{u+1}}(p^k, \mathcal{D})|. \end{aligned}$$

Case 4: $k_j := k - u - 1 \in \mathcal{K}$ and $k_{j-1} := k - (u + 1) - 1 \in \mathcal{K}$.

In this situation (6) implies

$$\begin{aligned} |\lambda_{p^u}(p^k, \mathcal{D})| &= p^{k-k_j-1} - \sum_{i=j+1}^s \varphi(p^{k-k_i}) = p^{k-k_j-1} + \varphi(p^{k-k_j}) - \sum_{i=j}^s \varphi(p^{k-k_i}) \\ &= p^{k-k_j} - \sum_{i=j}^s \varphi(p^{k-k_i}) = p^{k-k_{j-1}-1} - \sum_{i=j}^s \varphi(p^{k-k_i}) = |\lambda_{p^{u+1}}(p^k, \mathcal{D})|, \end{aligned}$$

which completes the proof. \square

It is well known that integral circulant graphs are regular, and incidentally their largest eigenvalue $\Lambda(p^k, \mathcal{D}) = \lambda_{p^k}(p^k, \mathcal{D})$ (see Corollary 2.1 (ii)) equals their degree of regularity (cf. [4]). The proof of the following result shows that loopless integral circulant graphs of any prime power order p^k are uniquely determined not only by their spectrum, but even by $\Lambda(p^k, \mathcal{D})$ alone.

Corollary 2.2. *Let p^k be an arbitrary prime power, and let $\mathcal{D}, \mathcal{E} \subseteq D(p^{k-1})$ be arbitrary divisor sets. If $\Lambda(p^k, \mathcal{D}) = \Lambda(p^k, \mathcal{E})$, then $\mathcal{D} = \mathcal{E}$.*

Proof. Let $\mathcal{D} = \{p^{k_1}, \dots, p^{k_s}\}$ and $\mathcal{E} = \{p^{\ell_1}, \dots, p^{\ell_t}\}$ with $0 \leq k_1 < \dots < k_s < k$ and $0 \leq \ell_1 < \dots < \ell_t < k$, say. By our assumption it follows from Proposition 2.2 and Corollary 2.1 that

$$(7) \quad \sum_{i=1}^s \varphi(p^{k-k_i}) = \lambda_{p^k}(p^k, \mathcal{D}) = \lambda_{p^k}(p^k, \mathcal{E}) = \sum_{i=1}^t \varphi(p^{k-\ell_i}).$$

Since $k_s < k$ and $\ell_t < k$, it follows from (7) that

$$(p-1) \sum_{i=1}^s p^{k-k_i-1} = (p-1) \sum_{j=1}^t p^{k-\ell_j-1}.$$

Dividing by $(p-1)$ we have equality between two integers in p -adic representations. Hence $\{k_1, \dots, k_s\} = \{\ell_1, \dots, \ell_t\}$, i.e. $\mathcal{D} = \mathcal{E}$. \square

The next statement implies that the Conjecture of SO holds for loopless graphs of arbitrary prime power order.

Corollary 2.3. *Two loopless integral circulant graphs $\text{ICG}(p^k, \mathcal{D})$ and $\text{ICG}(p^k, \mathcal{E})$ of prime power order p^k are isomorphic if and only if $\mathcal{D} = \mathcal{E}$.*

Proof. As pointed out in the introduction, an integral circulant graph of order p^k has loops if and only if p^k lies in the divisor set. By assumption, $p^k \notin \mathcal{D}$ and $p^k \notin \mathcal{E}$. Hence Corollary 2.2 implies our claim. \square

Given a prime power p^k and a divisor set $\mathcal{D} \subseteq D(p^k)$, we call $\lambda_1(p^k, \mathcal{D})$, i.e. the first entry of the spectral vector $\vec{\lambda}(p^k, \mathcal{D})$ of $\text{ICG}(p^k, \mathcal{D})$, its *dominating eigenvalue*. As another consequence of Proposition 2.2 we obtain the justification for the chosen naming.

Corollary 2.4. *Let p^k be a prime power and $\mathcal{D} \subseteq D(p^k)$ any divisor set.*

(i) *The dominating eigenvalue of $\text{ICG}(p^k, \mathcal{D})$ satisfies*

$$\lambda_1(p^k, \mathcal{D}) = \begin{cases} +1 & \text{if } p^k \in \mathcal{D} \text{ and } p^{k-1} \notin \mathcal{D}, \\ -1 & \text{if } p^k \notin \mathcal{D} \text{ and } p^{k-1} \in \mathcal{D}, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) *The multiplicity of $\lambda_1(p^k, \mathcal{D})$ is at least $p^k - p^{k-1}$.*

Proof. The formula in (i) follows from Proposition 2.2 (ii) with $t = 0$. By Proposition 2.2 (i) we have $\lambda_\ell(p^k, \mathcal{D}) = \lambda_1(p^k, \mathcal{D})$ for all $1 \leq \ell \leq p^k$ satisfying $p \nmid \ell$. The number of these ℓ equals $\varphi(p^k) = p^k - p^{k-1}$, which proves (ii). \square

For later purposes we have to extend Corollary 2.2 to integral circulant graphs which might have loops. Unfortunately, this relaxation complicates matters quite a bit, since now integral circulant graphs of prime power order cannot be distinguished anymore by their largest eigenvalue alone.

Proposition 2.3. *Let p^k be an arbitrary prime power.*

- (i) *If $\Lambda(p^k, \mathcal{D}) = \Lambda(p^k, \mathcal{E})$ for some $\mathcal{D}, \mathcal{E} \subseteq D(p^k)$, then either*
- (α) *$\mathcal{D} = \mathcal{E}$ or*
 - (β) *$p = 2$, $\mathcal{D} \subseteq D(2^{k-1})$ arbitrary and, with $2^{k_{\max}} := \max \mathcal{D}$,
 $\mathcal{E} = \mathcal{D} \setminus \{2^{k_{\max}}\} \cup \{2^{k_{\max}+1}, 2^{k_{\max}+2}, \dots, 2^{k-1}, 2^k\}$, or vice versa.*
- (ii) *Let $p = 2$, and let \mathcal{D} and \mathcal{E} be as in (β). Then $\lambda_{2^t}(2^k, \mathcal{D}) \neq \lambda_{2^t}(2^k, \mathcal{E})$ for $t := k - k_{\max} - 1$, in particular $\vec{\lambda}(2^k, \mathcal{D}) \neq \vec{\lambda}(2^k, \mathcal{E})$.*

Proof. In order to verify (i) let $\mathcal{D} = \{p^{k_1}, \dots, p^{k_s}\}$ and $\mathcal{E} = \{p^{\ell_1}, \dots, p^{\ell_t}\}$ with $0 \leq k_1 < \dots < k_s \leq k$ and $0 \leq \ell_1 < \dots < \ell_t \leq k$, say. As in the proof of Corollary 2.2 it follows from our assumption in (i), Proposition 2.2 and Corollary 2.1 that

$$(8) \quad \sum_{i=1}^s \varphi(p^{k-k_i}) = \lambda_{p^k}(p^k, \mathcal{D}) = \lambda_{p^k}(p^k, \mathcal{E}) = \sum_{i=1}^t \varphi(p^{k-\ell_i}).$$

For $k_s < k$ and $\ell_t < k$ assertion (i) was already shown in Corollary 2.2. If $k_s = \ell_t = k$, we subtract the equal summands for $i = s$ and $j = t$, respectively, in (8), and the argument in the proof of Corollary 2.2 yields $\{k_1, \dots, k_{s-1}\} = \{\ell_1, \dots, \ell_{t-1}\}$, thus $\mathcal{D} = \mathcal{E}$.

We are left with the case where exactly one of the integers k_s and ℓ_t equals k . Without loss of generality we may assume that $k_s < k$ and $\ell_t = k$, hence $\mathcal{D} \neq \mathcal{E}$. We obtain from (8) that

$$(p-1) \sum_{i=1}^s p^{k-k_i-1} = 1 + (p-1) \sum_{j=1}^{t-1} p^{k-\ell_j-1}.$$

It follows immediately that $p = 2$, hence

$$(9) \quad \sum_{i=1}^s 2^{k-k_i-1} = 1 + \sum_{j=1}^{t-1} 2^{k-\ell_j-1}.$$

Let $m \in \{1, 2, \dots, \min\{s, t\}\}$ be minimal with $k_m \neq \ell_m$. Clearly, m exists due to the fact that $\mathcal{D} \neq \mathcal{E}$. Consequently,

$$\mathcal{X} := \{2^{k_1}, 2^{k_2}, \dots, 2^{k_{m-1}}\} = \{2^{\ell_1}, 2^{\ell_2}, \dots, 2^{\ell_{m-1}}\},$$

where \mathcal{X} is possibly empty. Then (9) yields

$$(10) \quad \sum_{i=m}^s 2^{k-k_i-1} = 1 + \sum_{j=m}^{t-1} 2^{k-\ell_j-1}.$$

Trivially, $k - k_m - 1 \geq k - \ell_m - 1$, and obviously

$$2^{k-k_m-1} \leq \sum_{i=m}^s 2^{k-k_i-1} = 1 + \sum_{j=m}^{t-1} 2^{k-\ell_j-1} \leq 2^{k-\ell_m}.$$

We obtain $k - \ell_m - 1 \leq k - k_m - 1 \leq k - \ell_m$. Since $k_m \neq \ell_m$, we conclude that $k_m = \ell_m - 1$. Inserting this into (10) we get

$$(11) \quad 2^{k-\ell_m} \leq 2^{k-\ell_m} + \underbrace{\sum_{i=m+1}^s 2^{k-k_i-1}}_{(\diamond)} = 1 + \underbrace{2^{k-\ell_m-1} + \sum_{i=m+1}^{t-1} 2^{k-\ell_i-1}}_{(*)} \\ \leq 1 + \sum_{j=0}^{k-\ell_m-1} 2^j = 2^{k-\ell_m}.$$

Therefore, equality holds in (11), thus $(*)$ contains all powers of 2 up to the exponent $k - \ell_m - 1$ and (\diamond) vanishes. This implies that $\ell_{m+i} = \ell_m + i$ for $i = 1, 2, \dots, t - m - 1$ and $t - m = k - \ell_m$. Consequently,

$$\mathcal{D} = \mathcal{X} \cup \{2^{k_m}\} = \mathcal{X} \cup \{2^{\ell_m-1}\} \quad \text{and} \quad \mathcal{E} = \mathcal{X} \cup \{2^{\ell_m}, 2^{\ell_m+1}, \dots, 2^{k-1}, 2^k\},$$

where $\ell_m > \ell_{m-1} + 1$. This proves (i).

It remains to show (ii). As a consequence of (2) we have for $\mathcal{X} := \mathcal{D} \setminus \{2^{k_{\max}}\}$ and $\mathcal{Y} := \{2^{k_{\max}+1}, 2^{k_{\max}+2}, \dots, 2^{k-1}, 2^k\}$ that

$$(12) \quad \vec{\lambda}(2^k, \mathcal{E}) - \vec{\lambda}(2^k, \mathcal{D}) = (\vec{\lambda}(2^k, \mathcal{X}) + \vec{\lambda}(2^k, \mathcal{Y})) - (\vec{\lambda}(2^k, \mathcal{X}) + \vec{\lambda}(2^k, \{2^{k_{\max}}\})) \\ = \vec{\lambda}(2^k, \mathcal{Y}) - \vec{\lambda}(2^k, \{2^{k_{\max}}\}).$$

By use of Proposition 2.2 (ii) we obtain for $t = k - k_{\max} - 1$ that $\chi(2^k, \mathcal{Y}, t) = 0$ and $\chi(2^k, \{2^{k_{\max}}\}, t) = 1$, hence

$$\lambda_{2^t}(2^k, \mathcal{Y}) = \sum_{i=k_{\max}+1}^k \varphi(2^{k-i}) = 2^{k-k_{\max}-1}$$

and $\lambda_{2^t}(2^k, \{2^{k_{\max}}\}) = -2^t$. By (12) it follows that $\lambda_{2^t}(2^k, \mathcal{E}) - \lambda_{2^t}(2^k, \mathcal{D}) = 2^{t+1}$, which confirms (ii). \square

It will be crucial for our proof of the Weak SO Conjecture to determine the position and the value of the first non-zero entry in the spectral vector of integral circulant graphs of prime power order.

Proposition 2.4. *Let p^k be a prime power and $\mathcal{D} \subseteq D(p^k)$. For*

$$(13) \quad m_0 = m_0(p^k, \mathcal{D}) := \min \{0 \leq m \leq k : \lambda_{p^m}(p^k, \mathcal{D}) \neq 0\}$$

we have

(i) $m_0 = k - m^* - 1$, where

$$m^* = m^*(p^k, \mathcal{D}) := \begin{cases} -1 & \text{if } \mathcal{D} = D(p^k), \\ \max \{0 \leq j < k : p^j \notin \mathcal{D}\} & \text{if } p^k \in \mathcal{D} \neq D(p^k), \\ \max \{0 \leq j < k : p^j \in \mathcal{D}\} & \text{if } p^k \notin \mathcal{D}; \end{cases}$$

(ii) $\lambda_{p^{m_0}}(p^k, \mathcal{D}) = \varepsilon(p^k, \mathcal{D})p^{m_0}$, where

$$\varepsilon(p^k, \mathcal{D}) := \begin{cases} +1 & \text{if } p^k \in \mathcal{D}, \\ -1 & \text{if } p^k \notin \mathcal{D}. \end{cases}$$

Proof. According to Proposition 2.2 (ii), we have $\lambda_{p^m}(p^k, \mathcal{D}) = 0$ if and only if

$$(14) \quad \sum_{\substack{i=1 \\ k_i \geq k-m}}^s \varphi(p^{k-k_i}) = \chi(p^k, \mathcal{D}, m)p^m$$

with $\mathcal{D} = \{p^{k_1}, \dots, p^{k_s}\}$, say. Clearly

$$\sum_{\substack{i=1 \\ k_i \geq k-m}}^s \varphi(p^{k-k_i}) \leq \sum_{j=0}^m \varphi(p^j) = p^m,$$

where equality holds if and only if $\{p^{k-m}, p^{k-m+1}, \dots, p^k\} \subseteq \mathcal{D}$. Hence

$$\sum_{\substack{i=1 \\ k_i \geq k-m}}^s \varphi(p^{k-k_i}) = \begin{cases} p^m & \text{if } \{p^{k-m}, p^{k-m+1}, \dots, p^k\} \subseteq \mathcal{D}, \\ 0 & \text{if } \{p^{k-m}, p^{k-m+1}, \dots, p^k\} \cap \mathcal{D} = \emptyset. \end{cases}$$

Therefore, (14) holds if and only if $\{p^{k-m}, p^{k-m+1}, \dots, p^k\} \subseteq \mathcal{D}$ and $\chi(p^k, \mathcal{D}, m) = 1$, i.e. $\{p^{k-m-1}, p^{k-m}, \dots, p^k\} \subseteq \mathcal{D}$, or $\{p^{k-m}, p^{k-m+1}, \dots, p^k\} \cap \mathcal{D} = \emptyset$ and $\chi(p^k, \mathcal{D}, m) = 0$, i.e. $\{p^{k-m-1}, p^{k-m}, \dots, p^k\} \cap \mathcal{D} = \emptyset$. In other words, (14) holds for $m \leq M$ if and only if $m^* \leq k - M - 2$. Setting $M = m_0 - 1$ implies $m_0 = M + 1 = k - m^* - 1$, thus (i) is proven.

By Proposition 2.2 (ii) and the fact that $m^* = k - m_0 - 1$ by (i), we obtain

$$\lambda_{p^{m_0}}(p^k, \mathcal{D}) = \begin{cases} 0 + (\varphi(p^{m_0}) + \varphi(p^{m_0-1}) + \dots + \varphi(p) + \varphi(1)) = p^{m_0} & \text{if } p^k \in \mathcal{D}, \\ -p^{m_0} + 0 = -p^{m_0} & \text{if } p^k \notin \mathcal{D}, \end{cases}$$

and this verifies (ii).

3. SPECTRAL VECTORS FOR ICGS OF ARBITRARY ORDER

Proof of Theorem 1.1. LE and the first author proved in [13, Prop. 4.1] that

$$\lambda_\ell(n, \mathcal{D}) = \prod_{j=1}^r \lambda_\ell(p_j^{k_j}, \mathcal{D}_{p_j}) \text{ for } 1 \leq \ell \leq n, \text{ where } \ell \text{ in } \lambda_\ell(p_j^{k_j}, \mathcal{D}_{p_j}) \text{ is to be understood}$$

as the residue of $\ell \bmod p_j^{k_j}$. As a consequence of this and Proposition 2.2 we obtain that

$$\begin{aligned}
(15) \quad \lambda_\ell(n, \mathcal{D}) &= \prod_{j=1}^r \lambda_{\gcd(\ell, p_j^{k_j})}(p_j^{k_j}, \mathcal{D}_{p_j}) = \prod_{j=1}^r \lambda_{p_j^{t_j}}(p_j^{k_j}, \mathcal{D}_{p_j}) \\
&= \prod_{j=1}^r \left(-\chi(p_j^{k_j}, \mathcal{D}_{p_j}, t_j) p_j^{t_j} + \sum_{\substack{i=1 \\ k_{j,i} \geq k_j - t_j}}^{s_j} \varphi(p_j^{k_j - k_{j,i}}) \right).
\end{aligned}$$

Proof of Theorem 1.2. If $\text{ICG}(n, \mathcal{D})$ and $\text{ICG}(n, \mathcal{E})$ are isomorphic, we trivially have $\vec{\lambda}(n, \mathcal{D}) = \vec{\lambda}(n, \mathcal{E})$. We have to prove the converse and assume that $\vec{\lambda}(n, \mathcal{D}) = \vec{\lambda}(n, \mathcal{E})$.

Let $n = p_1^{k_1} \cdots p_r^{k_r}$, say, be the prime power factorisation of n , and let $\mathcal{D} = \mathcal{D}_{p_1} \cdots \mathcal{D}_{p_r}$ and $\mathcal{E} = \mathcal{E}_{p_1} \cdots \mathcal{E}_{p_r}$ be the factorisations of the multiplicative divisor sets \mathcal{D} and \mathcal{E} . It follows from (15) and $\vec{\lambda}(n, \mathcal{D}) = \vec{\lambda}(n, \mathcal{E})$ that for all $0 \leq \ell_j \leq k_j$, $1 \leq j \leq r$,

$$(16) \quad \prod_{j=1}^r \lambda_{p_j^{\ell_j}}(p_j^{k_j}, \mathcal{D}_{p_j}) = \lambda_{p_1^{\ell_1} \cdots p_r^{\ell_r}}(n, \mathcal{D}) = \lambda_{p_1^{\ell_1} \cdots p_r^{\ell_r}}(n, \mathcal{E}) = \prod_{j=1}^r \lambda_{p_j^{\ell_j}}(p_j^{k_j}, \mathcal{E}_{p_j}).$$

We set (cf. (13) for the definition of m_0)

$$d_j := m_0(p_j^{k_j}, \mathcal{D}_{p_j}), \quad e_j := m_0(p_j^{k_j}, \mathcal{E}_{p_j})$$

for $1 \leq j \leq r$. Applying (16) for $\ell_j := d_j$, $1 \leq j \leq r$, we obtain by use of Proposition 2.4 (ii)

$$\pm \prod_{j=1}^r p_j^{d_j} = \prod_{j=1}^r \varepsilon(p_j^{k_j}, \mathcal{D}_{p_j}) p_j^{d_j} = \prod_{j=1}^r \lambda_{p_j^{d_j}}(p_j^{k_j}, \mathcal{E}_{p_j}),$$

hence $\lambda_{p_j^{d_j}}(p_j^{k_j}, \mathcal{E}_{p_j}) \neq 0$ for each j , i.e. $d_j \geq e_j$ for all j . By symmetry, we also have $e_j \geq d_j$ for all j , which means that $d_j = e_j$ for each j . Knowing this, we use identity (16) once more, setting $\ell_j := d_j = e_j$ for $1 \leq j \leq r$, $j \neq i$ for some i . Now Proposition 2.4 (ii) implies

$$\lambda_{p_i^{\ell_i}}(p_i^{k_i}, \mathcal{D}_{p_i}) \prod_{\substack{j=1 \\ j \neq i}}^r p_j^{d_j} = \pm \lambda_{p_i^{\ell_i}}(p_i^{k_i}, \mathcal{E}_{p_i}) \prod_{\substack{j=1 \\ j \neq i}}^r p_j^{e_j},$$

thus $\lambda_{p_i^{\ell_i}}(p_i^{k_i}, \mathcal{D}_{p_i}) = \pm \lambda_{p_i^{\ell_i}}(p_i^{k_i}, \mathcal{E}_{p_i})$ for all ℓ_i and all $1 \leq i \leq r$. In particular, $\lambda_{p_i^{k_i}}(p_i^{k_i}, \mathcal{D}_{p_i}) = \pm \lambda_{p_i^{k_i}}(p_i^{k_i}, \mathcal{E}_{p_i})$ for all i . By Corollary 2.1 (ii) we know that $\lambda_{p_i^{k_i}}(p_i^{k_i}, \mathcal{D}_{p_i})$ and $\lambda_{p_i^{k_i}}(p_i^{k_i}, \mathcal{E}_{p_i})$ are both positive and the largest eigenvalues of the corresponding spectra, thus

$$(17) \quad \Lambda(p_i^{k_i}, \mathcal{D}_{p_i}) = \lambda_{p_i^{k_i}}(p_i^{k_i}, \mathcal{D}_{p_i}) = \lambda_{p_i^{k_i}}(p_i^{k_i}, \mathcal{E}_{p_i}) = \Lambda(p_i^{k_i}, \mathcal{E}_{p_i}) \quad \text{for } 1 \leq i \leq r.$$

It follows from Proposition 2.3 (i) that the divisor sets \mathcal{D}_{p_i} and \mathcal{E}_{p_i} are equal for all i with $p_i \neq 2$. Hence, if $2 \nmid n$ or $\mathcal{D}_2 = \mathcal{E}_2$, the multiplicativity of the divisor sets immediately yields $\mathcal{D} = \mathcal{E}$, which proves the theorem.

We are left with the case $2 \mid n$ and $\mathcal{D}_2 \neq \mathcal{E}_2$, and may assume that $p_1 = 2$. According to Proposition 2.3 (i) we have without loss of generality that

$$\mathcal{D}_2 \subseteq D(2^{k_1-1}) \quad \text{and} \quad \mathcal{E}_2 = \mathcal{D}_2 \setminus \{2^{k_{\max}}\} \cup \{2^{k_{\max}+1}, 2^{k_{\max}+2}, \dots, 2^{k_1-1}, 2^{k_1}\},$$

where $2^{k_{\max}} := \max \mathcal{D}_2$. Now Proposition 2.3 (ii) tells us that $\lambda_{2^t}(2^{k_1}, \mathcal{D}_2) \neq \lambda_{2^t}(2^{k_1}, \mathcal{E}_2)$ for $t := k_1 - k_{\max} - 1$. We know from (16)

$$\begin{aligned} \lambda_{2^t}(2^{k_1}, \mathcal{D}_2) \prod_{j=2}^r \lambda_{p_j^{k_j}}(p_j^{k_j}, \mathcal{D}_{p_j}) &= \lambda_{2^t \cdot p_2^{k_2} \dots p_r^{k_r}}(n, \mathcal{D}) = \lambda_{2^t \cdot p_2^{k_2} \dots p_r^{k_r}}(n, \mathcal{E}) \\ &= \lambda_{2^t}(2^{k_1}, \mathcal{E}_2) \prod_{j=2}^r \lambda_{p_j^{k_j}}(p_j^{k_j}, \mathcal{E}_{p_j}), \end{aligned}$$

but taking account of (17) we obtain $\lambda_{2^t}(2^{k_1}, \mathcal{D}_2) = \lambda_{2^t}(2^{k_1}, \mathcal{E}_2)$. This contradiction completes the proof.

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