Applicable Analysis and Discrete Mathematics - Accepted manuscript
Received: April 17, 2009
Revised: July 8, 2009
Please cite this article as: Letıcia R. Bueno, Luerbio Faria, Celina M. H. de Figueiredo, Guilherme D. da Fonseca: Hamiltonian Paths in Odd Graphs, Appl. Anal. Discrete Math. (to appear)

Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.bg.ac.yu

Appl. Anal. Discrete Math. **x** (xxxx), xxx–xxx. doi:10.2298/AADMxxxxxxxx

Hamiltonian Paths in Odd Graphs

Letícia R. Bueno
Luerbio Faria
Celina M. H. de Figueiredo
Guilherme D. da Fonseca

Lovász conjectured that every connected vertex-transitive graph has a Hamiltonian path. The odd graphs O_k form a well-studied family of connected, k-regular, vertex-transitive graphs. It was previously known that O_k has Hamiltonian paths for $k \leq 14$. A direct computation of Hamiltonian paths in O_k is not feasible for large values of k, because O_k has $\binom{2k-1}{k-1}$ vertices and $k\binom{2k-1}{k-1}/2$ edges. We show that O_k has Hamiltonian paths for $15 \leq k \leq 18$. Instead of directly running any heuristics, we use existing results on the middle levels problem, therefore further relating these two fundamental problems, namely finding a Hamiltonian path in the odd graph and finding a Hamiltonian cycle in the corresponding middle levels graph. We show that further improved results for the middle levels problem can be used to find Hamiltonian paths in O_k for larger values of k.

1. Introduction

A spanning cycle in a graph is a *Hamiltonian cycle* and a graph which contains such cycle is said to be *Hamiltonian*. A *Hamiltonian path* is a path that contains every vertex of the graph precisely once. Lovász [11] conjectured that every connected vertex-transitive graph has a Hamiltonian path. The odd graphs O_k, $k \geq 2$, form a well-studied family of connected, k-regular, vertex-transitive graphs (see [2] for an introduction). Therefore, the study of Hamiltonian paths in odd graphs may provide more evidence to support Lovász conjecture, or offer a counterexample for it.

A combinatorial conjecture formulated by Kneser [10] states that whenever the k-subsets of an n-set, where $n = 2k + r$ with positive r, are divided into $r + 1$ classes, then two disjoint subsets end up in the same class. Lovász [12] gave a proof based on graph theory showing that the Kneser graph $K(n, k)$, whose

A preliminary version of this work was presented at SGT in Rio – Workshop on Spectral Graph Theory with applications on Computer Science, Combinatorial optimization and Chemistry.

2000 Mathematics Subject Classification: 05C10, 05C45, 05C70.
Keywords and Phrases: Hamilton path, odd graphs, Kneser graphs, middle levels problem.

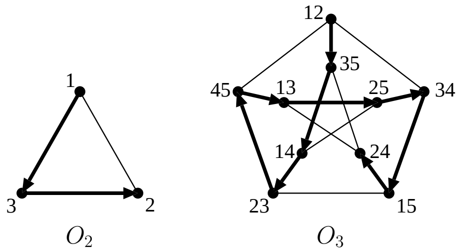

FIGURE 1. The odd graphs O_2 and O_3, with highlighted Hamiltonian paths and vertex labels corresponding to $(k-1)$-subsets of $\{1,\ldots,2k-1\}$.

vertices represent the k-subsets and each edge connects two disjoint subsets, is not $(r+1)$-colorable. Note that $K(n,k)$ is by definition connected vertex-transitive, it has $\binom{n}{k}$ vertices and is regular of degree $\binom{n-k}{k}$. Chen showed that Kneser graphs are Hamiltonian (have a Hamiltonian cycle) for $n \geq 3k$ [3], and later for $n \geq 2.72\,k+1$ [4]. Shields and Savage [17] showed that Kneser graphs are Hamiltonian for $n \leq 27$, with the exception of the Petersen graph.

The *odd graph* O_k is defined as $K(2k-1,k-1)$, for $k \geq 2$. Being k-regular, the odd graphs are the Kneser graphs $K(n,\cdot)$ of smallest degree for any fixed odd n. The graph O_k has $\binom{2k-1}{k-1}$ vertices and $k\binom{2k-1}{k-1}/2$ edges. Therefore, a direct computation of Hamiltonian paths in O_k is not feasible for large values of k.

The odd graphs are connected vertex-transitive graphs. Therefore, if Lovász conjecture [11] is true, then the odd graphs have a Hamiltonian path. The graph O_2 is a triangle and O_3 is the Petersen graph, both of which have Hamiltonian paths (Figure 1). Balaban [1] showed that O_4 and O_5 have Hamiltonian paths. Meredith and Lloyd [14] showed that O_6 and O_7 have Hamiltonian paths. Mather [13] showed that O_8 has a Hamiltonian path. Shields and Savage [17] used a carefully designed heuristic to find Hamiltonian paths in O_k for $k \leq 14$. In fact, the previous references show that, for $4 \leq k \leq 14$, O_k has not only a Hamiltonian path, but also a Hamiltonian cycle. It is well-known that the Petersen graph O_3 has a Hamiltonian path, but no Hamiltonian cycle.

In this paper, we show that O_k has a Hamiltonian path for $15 \leq k \leq 18$. Instead of directly running any heuristics, we use existing results on the middle levels problem [16, 18], therefore further relating these two fundamental problems, namely finding a Hamiltonian path in the odd graph and finding a Hamiltonian cycle in the corresponding middle levels graph.

Given a pair of positive integers n, k with $1 \leq k \leq n$, the *bipartite Kneser graph* $H(n,k)$ has as vertices the k-subsets and the $(n-k)$-subsets of $\{1,\ldots,n\}$, and two vertices are adjacent when one subset is contained in the other. The *middle levels graph* B_k is defined as $H(2k-1,k-1)$, for $k \geq 2$. The middle levels graph is the subgraph of the $(2k-1)$-hypercube induced by the middle two levels. The middle levels graphs B_k are also connected vertex-transitive graphs. The famous *middle levels problem* asks whether B_k has a Hamiltonian cycle.

While the middle levels problem remains unsolved, it is known that B_k has Hamiltonian cycles for some values of k. Dejter, Cordova and Quintana [5] presented explicit Hamiltonian cycles in B_k, for $k = 10$; additionally Dejter, Cedeño and Jáuregui [6] presented explicit Hamiltonian cycles in B_k, for $2 \leq k \leq 9$. Shields and Savage [16] showed that B_k has Hamiltonian cycles for $2 \leq k \leq 16$. Shields, Shields and Savage [18] recently showed that B_k also has Hamiltonian cycles for $k = 17$ and 18. These results [5, 6, 16, 18] were based on properties of smaller graphs obtained by identifying vertices that are equivalent according to some equivalence relation. The results from [16, 18] were based on computationally determining a special Hamiltonian path P in the reduced graph (defined in Section 2) and converting P into a Hamiltonian cycle in B_k. In the present paper, we show how to convert P into a Hamiltonian path in O_k, consequently showing that O_k has a Hamiltonian path for $2 \leq k \leq 18$.

An alternative approach consists of showing that B_k and O_k, for any $k \leq 2$, contain substructures that are related to Hamiltonian cycles. Horák, Kaiser, Rosenfeld, and Ryjáček [7] showed that the prism over the middle levels graph is Hamiltonian, and consequently B_k has a closed spanning 2-trail. Savage and Winkler [15] showed that if B_k has a Hamiltonian cycle for $k \leq h$, then B_k has a cycle containing a fraction $1 - \varepsilon$ of the graph vertices for all k, where ε is a function of h. For example, since B_k has a Hamiltonian cycle for $2 \leq k \leq 18$, then B_k has a cycle containing at least 86.7% of the graph vertices, for any $k \geq 2$. Johnson [8] showed that both B_k and O_k have cycles containing $(1-o(1))|V|$ vertices, where $|V|$ is the number of vertices in the graph. Johnson and Kierstead [9] used matchings of B_k to show that O_k has not only a 2-factor (spanning collection of cycles), but also a 2-factorisation.

In Section 2, we give two equivalent definitions for the reduced graph, one based on the middle levels graph B_k, and one based on the odd graph O_k. In Section 3, we show how to obtain a Hamiltonian path in O_k through a special Hamiltonian path in the reduced graph. Concluding remarks and open problems are presented in Section 4.

2. The Reduced Graph

In this section, we give two equivalent definitions for reduced graph. First, we define the reduced graph as the result of a quotient operation $\sim$ applied to the middle levels graph B_k. Then, we prove that the same operation applied to the odd graph O_k results in the same reduced graph. We start with some definitions.

Let $\mathbb{Z}_n$ denote the set $\{1, \ldots, n\}$ with arithmetic modulo n. Throughout this paper, we consider the vertices of O_k and B_k to be subsets of $\mathbb{Z}_n$ and $n = 2k - 1$. We define two special $(k-1)$-subsets of $\mathbb{Z}_n$, which are $r_1 = \{1, \ldots, k-1\}$ and $r_2 = \{2, 4, 6, \ldots, n-1\}$.

Given a set $v \subseteq \mathbb{Z}_n$, let $v + \delta$ denote the set $\{a + \delta : a \in v\}$ and $\overline{v}$ denote the complement of v with respect to $\mathbb{Z}_n$. We say that $u, v \subset \mathbb{Z}_n$ satisfy $u \sim v$ if either

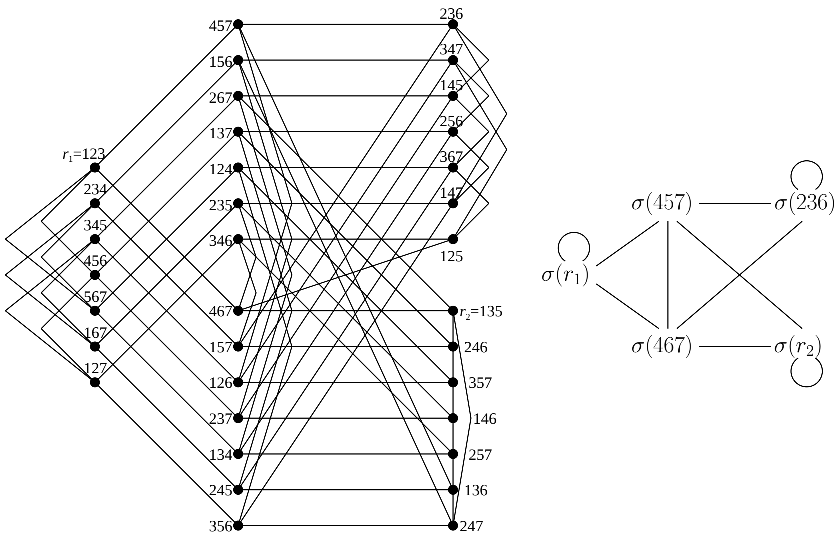

FIGURE 2. The odd graph O_4 and the reduced graph $\widetilde{O_4}$.

(i) $u = v + \delta$ or (ii) $\overline{u} = v + \delta$ for some $\delta \in \mathbb{Z}_n$. It is easy to verify that $\sim$ is an equivalence relation. We refer to the equivalence class of v defined by $\sim$ as $\sigma(v)$.

Given a graph G, with $V(G) \subseteq 2^{\mathbb{Z}_n}$, we define the *quotient graph* $\widetilde{G}$ as the graph obtained from G by identifying vertices that are equivalent according to $\sim$. More precisely, the vertices of $\widetilde{G}$ are the equivalence classes $\sigma(v)$ for $v \in V(G)$, and if $uv \in E(G)$ then $\sigma(u)\sigma(v) \in E(\widetilde{G})$. Note that if $uv \in E(G)$ satisfies $u \sim v$, then the vertex $\sigma(u) \in V(\widetilde{G})$ has a loop. For the special case when graph G is the odd graph O_k, the graphs O_4 and $\widetilde{O_4}$ are illustrated in Figure 2.

For the special case when graph G is the middle levels graph B_k, the quotient graph $\widetilde{B_k}$ is called the *reduced graph*. The following lemma is proved as Lemma 1 of [16].

Lemma 2.1. *Each equivalence class $\sigma(v)$ of $\widetilde{B_k}$ consists of exactly $n = 2k - 1$ $(k-1)$-subsets and n k-subsets.*

As a consequence, the reduced graph $\widetilde{B_k}$ has $2n$ times fewer vertices than B_k. For example, B_{18} has 9,075,135,300 vertices, while $\widetilde{B_{18}}$ has 129,644,790 vertices, which is 70 times smaller, but still quite large.

Shields and Savage [16] showed that the existence of a Hamiltonian path in the reduced graph $\widetilde{B_k}$, starting at the vertex $\sigma(r_1)$ and ending at the vertex $\sigma(r_2)$ implies that B_k is Hamiltonian. We refer to a Hamiltonian path starting at $\sigma(r_1)$ and ending at $\sigma(r_2)$ as a *useful path*. Using heuristics, Shields and Savage [16] determined useful paths in $\widetilde{B_k}$ for $1 \leq k \leq 16$. Recently, Shields, Shields and Savage [18] extended this result for $1 \leq k \leq 18$.

We prove the following lemma, which relates $\widetilde{O_k}$ and $\widetilde{B_k}$.

Lemma 2.2. *The quotient graphs $\widetilde{O_k}$ and $\widetilde{B_k}$ are equal.*

Proof. The vertices of O_k are $(k-1)$-subsets $\mathbb{Z}_n$, with $n = 2k-1$, and the vertices of B_k are the $(k-1)$-subsets and k-subsets of $\mathbb{Z}_n$. Since the complement of a $(k-1)$-subset is a k-subset, we have $V(\widetilde{O_k}) = V(\widetilde{B_k})$.

There is an edge $uv \in O_k$ when $u \cap v = \emptyset$ and there is an edge $u\overline{v} \in B_k$ when $u \subset \overline{v}$. Since $|u| \neq |\overline{v}|$, both statements are equivalent, and we have $E(\widetilde{O_k}) = E(\widetilde{B_k})$. □

3. Hamiltonian Paths in the Odd Graph

In this section, we show how to obtain a Hamiltonian path in O_k through a useful path in $\widetilde{O_k}$. First, we examine the structure of the subgraphs of O_k induced by $\sigma(r_1)$ and $\sigma(r_2)$.

Lemma 3.1. *The subgraph of O_k induced by $\sigma(r_1)$ is the cycle $r_1,\ r_1+k,\ r_1+2k,\ \ldots,\ r_1+(n-1)k$.*

Proof. The cycle has n vertices because k and $n = 2k-1$ are relatively prime. The fact that the only vertices in $\sigma(r_1)$ that are disjoint from r_1 are r_1+k and $r_1+(n-1)k = r_1-k$ follows from the definition that $r_1 = \{1, \ldots, k-1\}$. □

Lemma 3.2. *The subgraph of O_k induced by $\sigma(r_2)$ is the cycle $r_2,\ r_2+1,\ r_2+2,\ \ldots,\ r_2+(n-1)$.*

Proof. Note that r_2+j is adjacent to r_2+i+j if and only if r_2 is adjacent to r_2+j. Therefore, it suffices to prove that r_2 is only adjacent to r_2+1 and r_2+n. Remember the definition that $r_2 = \{2, 4, 6, \ldots, n-1\}$ and that O_k has edges between disjoint sets. The graph O_k has an edge between r_2 and r_2+1, because r_2 contains only even numbers and r_2+1 contains only odd numbers. There is no edge between r_2 and r_2+i for an even number $i \leq n-1$ because $n-1 \in r_2 \cap (r_2+i)$. There is no edge between r_2 and r_2+i for an odd number i with $3 \leq i \leq n-3$ because $2 \in r_2 \cap (r_2+i)$. □

Next, we study the relation between adjacencies in $\widetilde{O_k}$ and adjacencies in O_k.

Lemma 3.3. *If there is an edge $\sigma(u)\sigma(v)$ in $\widetilde{O_k}$, then there is a perfect matching between the vertices of $\sigma(u)$ and the vertices of $\sigma(v)$ in O_k.*

Proof. If $\sigma(u)\sigma(v) \in E(\widetilde{O_k})$, then there are two vertices $u' \in \sigma(u)$ and $v' \in \sigma(v)$ that are adjacent in O_k. We can form a perfect matching by pairing $u'+i$ with $v'+i$ for $1 \leq i \leq n$. □

The following lemma is an immediate consequence of Lemma 3.3, and shows how to build disjoint paths in O_k using a path in $\widetilde{O_k}$.

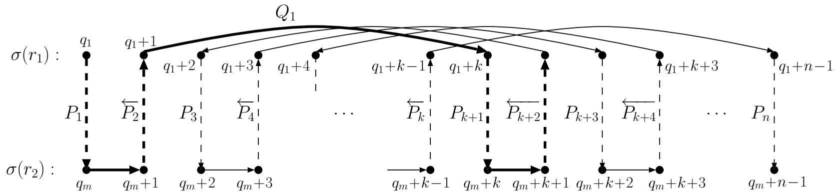

FIGURE 3. Construction of the Hamiltonian path in O_k, with Q_1 highlighted.

Lemma 3.4. *If there is a path $P = (p_1, \ldots, p_m)$ in $\widetilde{O_k}$, then O_k has n disjoint paths $(q_1+(i-1), \ldots, q_m+(i-1))$, for $1 \le i \le n$, such that $q_j \in p_j$, for $1 \le j \le m$.*

In order to build a Hamiltonian path in O_k, we traverse all the n disjoint paths given by Lemma 3.4. When traversing the paths, we alternate between forward and backward order. We carefully pick edges from the cycles defined in Lemmas 3.1 and 3.2 in order to connect n paths into a single Hamiltonian path.

Theorem 3.5. *If there is a useful path $P = (p_1, \ldots, p_m)$ in $\widetilde{O_k}$, then there is a Hamiltonian path in O_k.*

Proof. Given a path Q, we denote by $\overleftarrow{Q}$ the path Q traversed from the last to the first vertex. Given two paths Q_1, Q_2 with no vertices in common and the last vertex of Q_1 being adjacent to the first vertex of Q_2, we denote by $Q_1 \circ Q_2$ the path obtained by traversing the vertices of Q_1 followed by the vertices of Q_2.

By definition of useful path, $P = (p_1, \ldots, p_m)$ is Hamiltonian in $\widetilde{O_k}$, $m = |V(O_k)|/n$, $p_1 = \sigma(r_1)$, and $p_m = \sigma(r_2)$. By Lemma 3.4 there are n disjoint paths P_i of the following form, for $1 \le i \le n$:

$P_i = (q_1+(i-1), \ldots, q_m+(i-1))$ with $q_1+(i-1) \in \sigma(r_1)$ and $q_m+(i-1) \in \sigma(r_2)$.

By Lemma 3.1 and since $n = 2k-1$, q_1+i is adjacent to $q_1+i+k-1$. Therefore, $\overleftarrow{P_{i+1}} \circ P_{i+k}$ is a valid path. By Lemma 3.2, q_m+i is adjacent to q_m+i+1. Therefore, $P_i \circ \overleftarrow{P_{i+1}}$ is a valid path. Consequently, the following is a valid path:

$$Q_i = P_i \circ \overleftarrow{P_{i+1}} \circ P_{i+k} \circ \overleftarrow{P_{i+k+1}}, \text{ where}$$

$P_i = (q_1+(i-1), \ldots, q_m+(i-1))$, $\overleftarrow{P_{i+1}} = (q_m+i, \ldots, q_1+i)$, $P_{i+k} = (q_1+i+(k-1), \ldots, q_m+i+(k-1))$, and $\overleftarrow{P_{i+k+1}} = (q_m+i+k, \ldots, q_1+i+k)$.

The idea to conclude the proof is to build a Hamiltonian path $Q_1 \circ Q_3 \circ Q_5 \circ \ldots$. This construction is illustrated in Figure 3. Next, we discuss two separate cases, even and odd k, which have different conditions to define the end of the path.

If k is odd, then we show that

$$R_{odd} = Q_1 \circ Q_3 \circ \ldots \circ Q_{k-2}$$

is a valid path. By Lemma 3.1, the last vertex of Q_i, q_1+i+k, is adjacent to q_1+i+1, the first vertex of Q_{i+2}, since $P_{i+2} = (q_1+i+1, \ldots, q_m+i+1)$, and q_1+i+1 is adjacent to q_1+i+k. Also, R_{odd} contains either P_i or $\overleftarrow{P_i}$, for

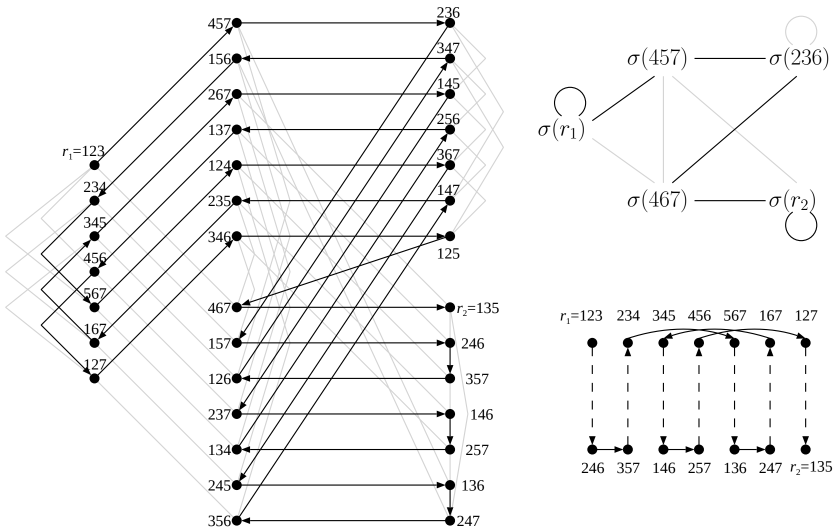

FIGURE 4. On the top-right corner: Useful Hamiltonian path in $\widetilde{O_4}$. Unused edges are gray. On the bottom-right corner: Vertices in $\sigma(r_1)$ and $\sigma(r_2)$, and dashed paths $P_1, P_2, \ldots, P_7$ used in the construction of the Hamiltonian path in O_4. On the left: Hamiltonian path in O_4 between r_1 and r_2.

$i \in \{1, \ldots, 2k-1\} \setminus \{k\}$. To include the missing path P_k, we define the Hamiltonian path in O_k as

$$H_{odd} = R_{odd} \circ P_k.$$

If k is even, then the same argument shows that

$$R_{even} = Q_1 \circ Q_3 \circ \ldots \circ Q_{k-3}$$

is a valid path. In the even case, R_{even} contains either P_i or $\overleftarrow{P_i}$, for $i \in \{1, \ldots, 2k-1\} \setminus \{k-1, k, 2k-1\}$. To include the missing paths, we define the Hamiltonian path in O_k as

$$H_{even} = R_{even} \circ P_{k-1} \circ \overleftarrow{P_k} \circ P_{2k-1}.$$

Note Figure 3 actually illustrates the construction of H_{even}. The full construction of a Hamiltonian path in O_4 is illustrated in Figure 4. □

4. Conclusion and Open Problems

In this paper, we show a relationship between the reduced graphs $\widetilde{B_k}$ [16] and the odd graphs O_k. We determine a Hamiltonian path in the odd graph O_k by using a useful path in the reduced graph $\widetilde{O_k}$. This way, we determine Hamiltonian paths in O_k for k up to 18. It is natural to ask whether a Hamiltonian cycle in O_k can be constructed in a similar manner.

All Hamiltonian paths known for the reduced graph were determined by computer using heuristics. Finding an useful path in the reduced graph $\widetilde{B_{18}}$ [18] took more than 20 days of processing on a AMD Athlon 3500+. Further studies in the structure of the reduced graph may help finding useful paths faster, and possibly tell whether all reduced graphs have a useful path. It is important to note that if the reduced graph does not have a useful path, then the corresponding odd graph may still have a Hamiltonian path.

Two different kinds of approximations for Hamiltonian cycles in the middle levels graphs are known. Savage and Winkler [15] showed that B_k has a cycle containing at least 86.7% of the graph vertices, for $k \geq 19$. Horák, Kaiser, Rosenfeld, and Ryjáček [7] showed the middle levels graph has a closed spanning 2-trail. It remains open to prove similar results for the odd graphs.

Since vertex-transitive graphs defined by a single parameter, such as the odd graphs and the middle levels graphs, are not known to have Hamiltonian paths, Lovász conjecture [11] remains challengingly open.

Acknowledgments

We would like to thank Peter Horák for introducing us to the problem of Hamiltonian cycles in Kneser and bipartite Kneser graphs. We thank the referee for the careful reading and valuable suggestions, which helped improve the presentation of this paper.

References

[1] A. T. Balaban. Chemical graphs, part XIII; combinatorial patterns. *Rev. Roumain Math. Pures Appl.*, 17:3–16, 1972.
[2] N. Biggs. *Algebraic Graph Theory.* Cambridge University Press, 2008.
[3] Y.-C. Chen. Kneser graphs are Hamiltonian for $n \geq 3k$. *J. Combin. Theory Ser. B*, 80:69–79, 2000.
[4] Y.-C. Chen. Triangle-free hamiltonian Kneser graphs. *J. Combin. Theory Ser. B*, 89(1):1–16, 2003.
[5] I. J. Dejter, J. Cordova, and J. A. Quintana. Two hamilton cycles in bipartite reflective kneser graphs. *Discrete Math.*, 72(1):63–70, 1988.
[6] I. J. Dejter, W. Cedeño, and V. Jáuregui. Frucht diagrams, boolean graphs and hamilton cycles. *Sci. Ser. A Math. Sci. (N.S.)*, 5(1):21–37, 1992/93.
[7] P. Horák, T. Kaiser, M. Rosenfeld, and Z. Ryjáček. The prism over the middle-levels graph is Hamiltonian. *Order*, 22(1):73–81, 2005.
[8] J. R. Johnson. Long cycles in the middle two layers of the discrete cube. *J. Combin. Theory Ser. A*, 105(2):255–271, 2004.
[9] J. R. Johnson and H. A. Kierstead. Explicit 2-factorisations of the odd graph. *Order*, 21(1):19–27, 2004.
[10] M. Kneser. Aufgabe 360. *Jahresbericht der DMV*, 58(2):27, 1955.
[11] L. Lovász. Problem 11. In *Combinatorial Structures and their Applications.* Gordon and Breach, 1970.
[12] L. Lovász. Kneser's conjecture, chromatic number, and homotopy. *J. Combin. Theory Ser. A*, 25:319–324, 1978.
[13] M. Mather. The Rugby footballers of Croam. *J. Combin. Theory Ser. B*, 20(1):62–63, 1976.

[14] G. H. J. Meredith and E. K. Lloyd. The Hamiltonian graphs O_4 to O_7. In *Combinatorics (Proc. Conf. Combinatorial Math., Math. Inst., Oxford, 1972)*, pages 229–236. Inst. Math. Appl., Southend, 1972.
[15] C. D. Savage and P. Winkler. Monotone gray codes and the middle levels problem. *J. Combin. Theory Ser. A*, 70(2):230–248, 1995.
[16] I. Shields and C. D. Savage. A Hamilton path heuristic with applications to the middle two levels problem. In *Proceedings of the Thirtieth Southeastern International Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, FL, 1999). Congr. Numer.*, volume 140, pages 161–178, 1999.
[17] I. Shields and C. D. Savage. A note on Hamilton cycles in Kneser graphs. *Bull. Inst. Combin. Appl.*, 40:13–22, 2004.
[18] I. Shields, B. J. Shields, and C. D. Savage. An update on the middle levels problem. *Discrete Math. (in press)*, 2007, doi:10.1016/j.disc.2007.11.010.

Letícia R. Bueno[1], Celina M. H. de Figueiredo, Guilherme D. da Fonseca[2]
Programa de Engenharia de Sistemas e Computação - COPPE
Universidade Federal do Rio de Janeiro
Caixa Postal: 68511
Rio de Janeiro, RJ 21941-972
Brazil
E–mail: letbueno@cos.ufrj.br, celina@cos.ufrj.br, fonseca@cos.ufrj.br

Luerbio Faria
Departamento de Matemática, Faculdade de Formação de Professores
Universidade do Estado do Rio de Janeiro
São Gonçalo, RJ 24435-000
Brazil
E–mail: luerbio@cos.ufrj.br

[1]Research supported by CNPq grant.
[2]Research supported by CNPq grant PDJ-151194/2007-6.