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ON THE ALEKSANDROV PROBLEM FOR ISOMETRIC MAPPINGS

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In this paper some relations between linearity and isometry are investigated for mappings which preserve some distance. Several open problems are discussed.

1. INTRODUCTION

Let X, Y be two metric spaces, d_1, d_2 the distances on X and Y, respectively. A mapping $f : X \to Y$, of X onto Y, is defined to be an **isometry** if

$$d_2\big(f(x), f(y)\big) = d_1(x, y)$$

for all elements x, y of X.

S. Mazur and S. Ulam [**14**] have proved that every isometry of a normed real vector space onto a normed real vector space is a linear mapping up to translation. Consider then the following condition (**distance one preserving property**) for the mapping $f : X \to Y$.

(DOPP) *Given* $x, y \in X$ *with* $d_1(x, y) = 1$. *Then* $d_2\big(f(x), f(y)\big) = 1$.

A. D. Aleksandrov [**1**] posed the following problem:

Under what conditions is a mapping of a metric space into itself preserving unit distance an isometry?

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The basic "**problem of conservative distances**" is whether the existence of a single conservative distance for f implies that f is an isometry of X into Y (cf. [**6**, 17]).

F. S. Beckman and D. A. Quarles [**2**] proved that if $f : E^n \to E^n$ for $2 \leq n < \infty$ satisfies condition (DOPP), then f is an isometry, where E^n is a finite-dimensional real Euclidean space. Independently from Beckman and Quarles, R. L. Bishop [**5**], P. Zvengrowski [**23**], D. Greenwell and P. D. Johnson [**7**] have obtained different proofs of the same result. For non-Euclidean spaces the Beckman-Quarles result has been obtained by the Russian school, notably by A. Guc [**8**], A. V. Kuz'minyh [**13**].

This property does not hold for E^1, the Euclidean line. A simple counterexample is the following:

Let $f : E^1 \to E^1$ be defined by

$$f(x) = \begin{cases} x+1 & \text{if } x \text{ is an integer point,} \\ x & \text{otherwise.} \end{cases}$$

Nevertheless, one may ask about a solution with additional assumptions (for instance continuity or differentiability of f). The answer is still negative:

Example 1.1. Define $f : E^1 \to E^1$ by

$$f(x) = x + \frac{1}{7}\sin(2\pi x).$$

The function f is an analytic diffeomorphism satisfying the (DOPP), but is not an isometry.

Also this property does not hold for E^∞, a Hilbert space. A counterexample can be made in the following way: Let $\{y_i\}$ be a countable everywhere dense set of points. Define $g : E^\infty \to \{y_i\}$ such that $d(x, g(x)) < 1/2$. Define $h : \{y_i\} \to \{a_i\}$ such that $h(y_i) = a_i$, where a_i is the point in E^∞ with coordinates $(a_{i1}, a_{i2}, \ldots)$ such that $a_{ij} = \delta_{ij}/\sqrt{2}$, where δ_{ij} is the Kronecker delta. Then

$$f = gh : E^\infty \to E^\infty$$

satisfies condition (DOPP). If $d(x, y) = 1$, then $g(x) \neq g(y)$ and hence $f(x) \neq f(y)$, but f is not an isometry.

It is not yet known what does it happen in E^∞ even with the additional condition of continuity of the mapping.

Conjecture 1.2. *A continuous mapping $f : E^\infty \to E^\infty$ satisfying condition* (DOPP) *must be an isometry.*

In this paper, we will survey recent developments on the Aleksandrov problem and the Mazur-Ulam theorem for mappings which preserve some distances.

2. RESULTS AND OPEN PROBLEMS

B. Mielnik and Th. M. Rassias [**15**] have proved the following

Theorem 2.1. *Every homeomorphism* $f : E^n \to E^n$ $(2 < n \leq \infty)$ *with a non-trivial conservative distance* $\ell > 0$ *is an isometry.*

The case of mapping $f : E^n \to E^m$ $(2 \leq n < m < +\infty)$

In the following we outline a method to show how to construct examples to prove that for each positive integer n there exists a positive integer m and a unit distance preserving mapping $f : E^n \to E^m$ that is not an isometry. The following example illustrates the case of a mapping $f : E^2 \to E^8$. For this consider partitioning the plane into squares of unit diagonal as follows:

Each square contains the bottom edge, the left edge and the bottom left corner but none of the other corners. Now label the nine vertices of the unit 8-simplex in E^8 and map each square labeled i to the i-th vertex. This mapping satisfies condition (DOPP) but is not an isometry.

Remark. Using hexagons instead of squares one can construct such a mapping from $E^2 \to E^6$. This idea extends easily to higher dimensions.

Th. M. Rassias [**16**] has proved the following

Theorem 2.2. *For any integer* $n \geq 1$, *there exists an integer* n_m *such that for* $N \geq n_m$ *it follows that there exists a mapping* $f : E^n \to E^N$ *which is distance one preserving but is not an isometry.*

It is not yet known whether there is a distance 1-preserving mapping $f : E^2 \to E^3$ **which is not an isometry. It is also an open problem whether there is a continuous mapping** $f : E^n \to E^m$ **for** $m > n$ **which satisfies the (DOPP) but is not an isometry.**

Combining continuity and distance preserving properties for the mapping we can formulate the following

Conjecture 2.3. *If* M *is a locally Euclidean manifold of finite dimension greater or equal to two, then there is a distance* a *such that for any* $b < a$, *every mapping* $f : M \to M$ *preserving distance* b *is an isometry.*

In E^n three classical metrics induce the same topology:

$$d_m(x, y) = \max\{|x_i - y_i| : i = 1, 2, \ldots, n\},$$
$$d_\Sigma(x, y) = \sum_{i=1}^{n} |x_i - y_i|,$$

and the Euclidean metric d_E, where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$.

In the following we consider the isometry problem with respect to these metrics (see [**6**]).

PROBLEM. Does the condition (DOPP) suffice for a mapping $f : E^n \to E^k$ with respect to these metrics to be an isometry if $2 \le n < k < +\infty$?

It is obvious that for $n = 1$ all three metrics are the same.

Consider the space E^2 with the metric d_m. In this case the mapping may satisfy (DOPP) and not be an isometry. For this consider the following

EXAMPLE 2.4. Let $f : E^2 \to E^2$ be defined by

$$f(x, y) = ([x], [y])$$

(in Cartesian coordinates, $[x]$ denotes the integer part of x). This mapping, which corresponds every point to the left-bottom corner of a suitable square with sides of length equal to one, with range equal to $\mathbb{Z}^2$ ($\mathbb{Z}$ denotes the set of integers) is not an isometry but it preserves distance one.

Let us consider now the metric d_Σ.

EXAMPLE 2.5. Consider the mapping g defined by

$$g = \left(\sqrt{2} \cdot R_{\pi/4}\right) \circ f \circ \left(\frac{1}{\sqrt{2}} \cdot R^{-1}_{\pi/4}\right),$$

where f is as in Example 2.4 and $R_{\pi/4}$ is the rotation:

$$(x, y) \mapsto \left(\frac{x + y}{\sqrt{2}}, \ \frac{y - x}{\sqrt{2}}\right).$$

The rotation maps unit balls in metric d_m to balls of radius $\sqrt{2}$ with respect to metric d_Σ. The mapping g satisfies (DOPP) but is not an isometry.

REMARK. In the general case for E^n, $n > 2$, a rotation as in E^2 does not do the job. This happens because the balls in metrics d_m and d_Σ are of the same shape only for $n = 1, 2$. In E^2 one has squares in both cases, but in E^3 one has cubes for d_m and octahedrons for d_Σ.

EXAMPLE 2.6. For (E^n, d_m), $n > 2$, a mapping satisfying (DOPP) need not be an isometry. For this it is enough to consider the mapping $f : E^n \to E^n$ defined by $f(x_1, \ldots, x_n) = ([x_1], \ldots, [x_n])$.

For d_Σ the following problem is still open:

PROBLEM. Must the mapping $f : (E^n, d_\Sigma) \to (E^n, d_\Sigma)$ satisfying (DOPP) be an isometry for $n \ge 3$?

TH. M. RASSIAS and P. ŠEMRL [**18**] introduced the following condition: Let X and Y be two real normed vector spaces. A mapping $f : X \to Y$ satisfies the **strong distance one preserving property** (SDOPP) if and only if for all $x, y \in X$ with $\|x - y\| = 1$ it follows that $\|f(x) - f(y)\| = 1$ and conversely.

The following two theorems were proved in [**18**]:

Theorem 2.7. *Let X and Y be real normed vector spaces such that one of them has dimension greater than one. Suppose that $f : X \to Y$ is a surjective mapping satisfying* (SDOPP)*. Then f is an injective mapping satisfying*

$$|\,\|f(x) - f(y)\| - \|x - y\|\,| < 1$$

for all $x, y \in X$. Moreover, f preserves distance n in both directions for any positive integer n.

The assumption that one of the spaces has dimension greater than one cannot be omitted in the theorem.

In the theorem (SDOPP) cannot be replaced by (DOPP).

The inequality

$$|\,\|f(x) - f(y)\| - \|x - y\|\,| < 1 \quad \text{for all} \quad x, y \in X$$

in the theorem is sharp.

Theorem 2.8. ([**18**]) *Let X and Y be real normed vector spaces such that one of them has dimension greater than one. Suppose that $f : X \to Y$ is a Lipschitz mapping with $k = 1$:*

$$\|f(x) - f(y)\| \leq \|x - y\| \quad \textit{for all} \quad x, y \in X.$$

Assume also that f is a surjective mapping satisfying (SDOPP)*. Then f is an isometry. Thus f is a linear isometry up to translation.*

Corollary 2.9. *Let X and Y be real normed vector spaces such that one of them has dimension greater than one. Assume also that one of the spaces is strictly convex. Suppose that $f : X \to Y$ is a surjective mapping satisfying* (SDOPP)*. Then f is a linear isometry up to translation.*

Corollary 2.10. *Let X and Y be real normed vector spaces with* $\dim X > 1$*, such that one of them is strictly convex. Suppose that $f : X \to Y$ is a homeomorphism satisfying* (DOPP)*. Then f is a linear isometry up to translation.*

Open problems

1. Let X and Y be Banach spaces such that Y is strictly convex, $\dim Y > 2$, and $f : X \to Y$ be a mapping. Suppose that f preserves the two distances a and λa for some non-integer $\lambda > 2$. It is an open problem whether f must be an isometric mapping.

2. Examine whether a mapping $f : S^n \to S^n$ for $1 < n \leq \infty$, which preserves two distances, both different from $\pi/2$ and π, can be an isometry (S^n denotes the n-sphere in $\mathbb{R}^{n+1}$).

If $f : S^n \to S^n$ maps every point of S^n onto itself, except the north and south poles, and maps these two points onto each other, then f is not an isometry. This mapping f does preserve the two distances $\pi/2$ and π. The mapping is not continuous.

Let f be a mapping of a metric space X into itself. A nonnegative number r is called a **nonexpanding** (or **contractive**) distance of f if and only if for any $x, y \in X$, $d(x, y) = r$ implies $d\big(f(x), f(y)\big) \leq r$. A nonnegative number r is called a **nonshrinking** (or **extensive**) distance of f if and only if for all $x, y \in X$, $d(x, y) = r$ implies $d\big(f(x), f(y)\big) \geq r$. The distance r is called **preserved** (or **conservative**) by f if and only if for all $x, y \in X$ with $\|x - y\| = r$, it follows that $\|f(x) - f(y)\| = r$.

Th. M. Rassias and S. Xiang [**19**] proved the following two theorems:

Theorem 2.11. *Let X and Y be real Hilbert spaces with the dimension of X greater than one. Suppose that $f : X \to Y$ satisfies* (DOPP) *and the distances a, b are contractive by f, where a and b are positive numbers with $|a - b| < 1$. Then the distance $\sqrt{2a^2 + 2b^2 - 1}$ is contractive by f. Especially, if the distance $\sqrt{2a^2 + 2b^2 - 1}$ is extensive by f, then the distances a, b and $\sqrt{2a^2 + 2b^2 - 1}$ are preserved by f.*

Theorem 2.12. *Let X and Y be real Hilbert spaces with the dimension of X greater than one. Suppose that $f : X \to Y$ satisfies* (DOPP). *Assume that the distance $n\sqrt{4^m k^2 - \frac{4^m - 1}{3}}$ is extensive by f for some positive integers n, k and m. Then f must be a linear isometry up to translation.*

Recently, S.-M. Jung and K.-S. Lee [**10**] proved a general inequality for distances between points: Let X be a real (or complex) inner product space, let n be an integer not less than 2, and let p_{ik}, $i \in \{1, \ldots, n\}$ and $k \in \{1, 2\}$, be any distinct $2n$ points of X.

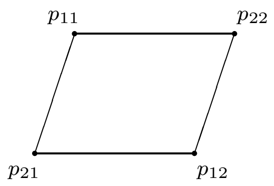
(a) It holds that

$$\sum_{\substack{1 \leq i < j \leq n \\ k, \ell \in \{1, 2\}}} \|p_{ik} - p_{j\ell}\|^2 \geq (n - 1) \sum_{i \in \{1, \ldots, n\}} \|p_{i1} - p_{i2}\|^2.$$

(b) The equality sign holds true in the above inequality if and only if for all $i, j \in \{1, \ldots, n\}$ with $i < j$, the pair of four points $\{p_{i1}, p_{i2}, p_{j1}, p_{j2}\}$ comprises the vertices of an appropriate (possibly degenerate) parallelogram such that p_{i1} and p_{j1} are the opposite vertices to p_{i2} and p_{j2}, respectively.

(Inequality (a) for $n = 2$ was proved in Lemma 1 of [**9**] and the case for $n = 3$ was treated in Theorem 2 of [**9**].)

We will label the vertices of any (possibly degenerate) parallelogram by p_{11}, p_{12}, p_{21}, and p_{22} as we see in the left-hand side of Fig. 1. We label the vertices of any (possibly degenerate) octahedron by p_{11}, p_{12}, p_{21}, p_{22}, p_{31}, and p_{32} as we see in the right-hand side of Fig. 1.



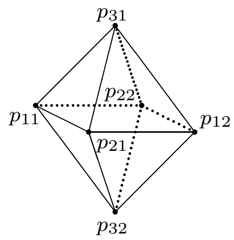


Fig. 1

We can continue this construction for the general case. Assume that we have constructed an n-dimensional polyhedron with $2n$ vertices, p_{11}, $p_{12}, \ldots$, p_{n1}, p_{n2}. Now, we add two more points, denoted by $p_{(n+1)1}$ and $p_{(n+1)2}$, to construct an $(n+1)$-dimensional polyhedron in the following manner: Each of the new points, $p_{(n+1)1}$ and $p_{(n+1)2}$, is connected to the existing $2n$ vertices, p_{11}, p_{12}, $\ldots$, p_{n1}, p_{n2}.

For a given n-dimensional polyhedron constructed as above, we will denote its $2n$ vertices by p_{11}, p_{12}, $\ldots$, p_{n1}, p_{n2} as the above construction. We define

$$\alpha_{ij} = \|p_{i1} - p_{j1}\|, \quad \beta_{ij} = \|p_{i2} - p_{j2}\|, \quad \gamma_{ij} = \|p_{i1} - p_{j2}\|$$

for all $i, j \in \{1, \ldots, n\}$. In the following theorem, we will assume that for any $i, j \in \{1, \ldots, n\}$ with $i < j$, each pair of four points, p_{i1}, p_{i2}, p_{j1}, p_{j2}, comprises the vertices of a corresponding parallelogram.

With these notations Jung and Lee [**10**] obtained the following

Theorem 2.13. *Let X and Y be either real inner product spaces or complex inner product spaces with $\dim X \geq n$ and $\dim Y \geq n$, where $n \geq 2$. Assume that the distances α_{ij}, β_{ij}, γ_{ij} are contractive by a mapping $f : X \to Y$ for all $i, j \in \{1, \ldots, n\}$ with $i < j$ and that the distances γ_{ii} are extensive by f for each $i \in \{1, \ldots, n\}$. Then f preserves the distances α_{ij}, β_{ij}, γ_{ij} for all $i, j \in \{1, \ldots, n\}$ with $i \leq j$.*

Sketch of the proof. First, we denote by p'_{ik} the image of p_{ik} under f. Since $\gamma_{ii} = \|p_{i1} - p_{i2}\|$ are extensive by f and α_{ij}, β_{ij}, γ_{ij} are contractive by f for all $1 \leq i < j \leq n$, we have

$$\begin{aligned}
(n-1) \sum_{i \in \{1,\ldots,n\}} \|p'_{i1} - p'_{i2}\|^2 &\geq (n-1) \sum_{i \in \{1,\ldots,n\}} \|p_{i1} - p_{i2}\|^2 \\
&= \sum_{\substack{1 \leq i < j \leq n \\ k, \ell \in \{1,2\}}} \|p_{ik} - p_{j\ell}\|^2 \\
&\geq \sum_{\substack{1 \leq i < j \leq n \\ k, \ell \in \{1,2\}}} \|p'_{ik} - p'_{j\ell}\|^2 \\
&\geq (n-1) \sum_{i \in \{1,\ldots,n\}} \|p'_{i1} - p'_{i2}\|^2,
\end{aligned}$$

where the last inequality follows from inequality (a). Hence, we get

$$\textstyle\sum_{i\in\{1,\ldots,n\}} \|p'_{i1} - p'_{i2}\|^2 = \sum_{i\in\{1,\ldots,n\}} \|p_{i1} - p_{i2}\|^2,$$

$$\sum_{\substack{1\le i<j\le n \\ k,\ell\in\{1,2\}}} \|p_{ik} - p_{j\ell}\|^2 = \sum_{\substack{1\le i<j\le n \\ k,\ell\in\{1,2\}}} \|p'_{ik} - p'_{j\ell}\|^2.$$

Since $\|p'_{i1} - p'_{i2}\| \ge \|p_{i1} - p_{i2}\|$ and $\|p_{ik} - p_{j\ell}\| \ge \|p'_{ik} - p'_{j\ell}\|$ for all $1 \le i < j \le n$ and $k, \ell \in \{1, 2\}$, we may conclude that

$$\|p'_{i1} - p'_{i2}\| = \|p_{i1} - p_{i2}\| = \gamma_{ii}$$

and

$$\|p'_{ik} - p'_{j\ell}\| = \|p_{ik} - p_{j\ell}\| = \begin{cases} \alpha_{ij} & (\text{for } k = \ell = 1) \\ \beta_{ij} & (\text{for } k = \ell = 2) \\ \gamma_{ij} & (\text{for } k = 1 \text{ and } \ell = 2) \\ \gamma_{ij} & (\text{for } k = 2 \text{ and } \ell = 1) \end{cases}$$

for any $1 \le i < j \le n$.

As we see in Theorem 4 and Corollary 5 of [9], if we set $n = 3$, $\alpha_{ij} = \beta_{ij} = \gamma_{ij} = \rho$ for $1 \le i < j \le 3$, and $\gamma_{ii} = \sqrt{2}\rho$ for $i \in \{1, 2, 3\}$, then we obtain the following

Corollary 2.14. *Let X and Y be real Hilbert spaces with* $\dim X \ge 3$ *and* $\dim Y \ge 3$. *For a given $\rho > 0$, assume that the distance ρ is contractive and the distance $\sqrt{2}\rho$ is extensive by a mapping $f : X \to Y$. Then, f is a linear isometry up to translation.*

We now consider an octahedron determined by the six vertices

$$p_{11} = \left(\frac{\sqrt{3}}{2}\rho, 0, 0, 0, \ldots, 0\right), \qquad p_{12} = \left(-\frac{\sqrt{3}}{2}\rho, 0, 0, 0, \ldots, 0\right),$$
$$p_{21} = \left(0, \frac{1}{2}\rho, 0, 0, \ldots, 0\right), \qquad p_{22} = \left(0, -\frac{1}{2}\rho, 0, 0, \ldots, 0\right),$$
$$p_{31} = \left(0, 0, \frac{1}{2}\rho, 0, \ldots, 0\right), \qquad p_{32} = \left(0, 0, -\frac{1}{2}\rho, 0, \ldots, 0\right),$$

where ρ is a given positive number. Applying Theorem 2.13 for $n = 3$ to the above octahedron and using Theorem 2.1 of S. Xiang [**22**], we can prove the following

Corollary 2.15. *Let X and Y be real Hilbert spaces with* $\dim X \ge 3$ *and* $\dim Y \ge 3$. *For a given $\rho > 0$, assume that the distance ρ is preserved, $\frac{1}{\sqrt{2}}\rho$ is contractive, and that the distance $\sqrt{3}\rho$ is extensive by a mapping $f : X \to Y$. Then, f is a linear isometry up to translation.*

Now, let X and Y denote n-dimensional Euclidean spaces, where $n \ge 3$, for which there exists a unit vector $w \in X$ and a subspace X_s of X such that

$X = X_s \oplus Sp(w)$ and X_s is orthogonal to $Sp(w)$, where $Sp(w)$ is the subspace of X which is spanned by w.

We define

$$r_0 = \theta, \quad r_1 = \theta + \rho, \quad r_2 = \theta + \rho + \rho_1, \quad r_3 = \theta + \left(1 + \frac{1}{n}\right)\rho + \rho_1,$$

where θ is a real number, ρ is a positive real number and

$$\rho_1 = \sqrt{\frac{2(n+1)}{n}}\,\rho.$$

By using these r_k's we define

$$E_k = \{x + \lambda w : x \in X_s\,;\, \lambda > r_k\}$$

for $k \in \{0, 1, 2, 3\}$.

Using these notations, S.-M. Jung and Th. M. Rassias [**11**] have proved the classical theorem of Beckman and Quarles for a restricted domain (see also [**12**]):

Theorem 2.16. *If a mapping $f : E_0 \to Y$ preserves the distance ρ, then the restriction $f|_{E_3}$ is an isometry. In particular, for any $x, y \in E_2$ with $x_s \neq y_s$, it holds that $\|f(x) - f(y)\| = \|x - y\|$, where x_s and y_s denote the X_s-components of x and y, respectively.*

Sketch of the proof. Lemma 13 of [**11**] implies that the distance $\frac{2(n+1)}{n}\rho$ preserved (extensive) by $f|_{E_2}$, while Lemma 14 of [**11**] shows the contractive property of the distance $\frac{2}{n}\rho$ under $f|_{E_2}$. Thus, in view of Theorem 9 of [**11**], we can conclude that the restriction $f|_{E_3}$ is an isometry. The second part of this theorem also follows from the second part of Theorem 9 of [**11**]. (We may remark that the proofs of Theorem 9 and Lemmas 13 and 14 are strongly based on the papers [**3, 4**] of W. Benz.)

B. Mielnik and Th. M. Rassias [**15**] have proved the following

Theorem 2.17. *Let f be a homeomorphism of the unit sphere X in a real Hilbert space H $(3 \leq \dim H \leq \infty)$ which preserves the angular distance $\pi/2$. Then f is an isometry.*

The proof of the above theorem is based on a very fundamental theorem that was proposed by Eugene Wigner [**21**].

This theorem asserts that mappings from a Hilbert space to itself which preserve the absolute values of inner products are in a certain sense equivalent to isometries (for a precise statement and proof of Wigner's theorem see [**20**]).

Absolute values of inner products are related to probabilities of transitions between states of a quantum system and the time evolution of such a system is supposed to preserve these probabilities.

Wigner used his theorem to define two linear mappings from a Hilbert space to itself which have played very fundamental roles in the development of quantum theory. These mappings are known to physicists as **time reversal** and **charge conjugation operators**.

It is an *open problem* to examine if the above theorem holds when f satisfies a condition weaker than that of a homeomorphism.

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