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ON ITERATIVE COMBINATION OF BERNSTEIN–DURRMEYER POLYNOMIALS

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The Bernstein–Durrmeyer polynomials

$$M_n(f;t) = (n+1)\sum_{k=0}^{n} p_{n,k}(t)\int_0^1 p_{n,k}(u)f(u)\,\mathrm{d}u,$$

where $p_{n,k}(t) = \binom{n}{k} t^k(1-t)^{n-k}$, $0 \le t \le 1$, defined on $L_B[0,1]$, the space of bounded and integrable functions on $[0,1]$ were introduced by Durrmeyer and extensively studied by Derriennic and several other researchers . It turns out that the order of approximation by these operators is, at best $O(n^{-1})$, however smooth the function may be. In order to improve this rate of approximation we consider an iterative combination $T_{n,k}(f;t)$ of the operators $M_n(f;t)$. This technique of improving the rate of convergence was given by Micchelli who first used it to improve the order of approximation by Bernstein polynomials $B_n(f;t)$. The object of this paper is to study direct theorems in ordinary as well as in simultaneous approximation by the operators $T_{n,k}(f;t)$. We prove that the order of approximation by these operators is $O(n^{-k})$ for sufficiently smooth functions.

1. INTRODUCTION

For $f \in L_B[0,1]$ the operators $M_n(f;t)$ can be expressed as

$$M_n(f;t) = \int_0^1 W_n(u,t)f(u)\,\mathrm{d}u, \tag{1.1}$$

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where $W_n(u,t) = (n+1)\sum_{k=0}^{n} p_{n,k}(t)p_{n,k}(u)$ is the kernel of the operators.

For $m \in \mathbb{N}_0$ (the set of non-negative integers), the m-th order moment for the operators M_n is defined as

$$\mu_{n,m}(t) = M_n\big((u-t)^m;t\big). \tag{1.2}$$

The iterative combination $T_{n,k} : L_B[0,1] \to C^\infty[0,1]$ of the operators $M_n(f;t)$ is defined as

$$T_{n,k}(f;t) = \big(I-(I-M_n)^k\big)(f;t) = \sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}M_n^r(f;t), \quad k \in \mathbb{N}, \tag{1.3}$$

where $M_n^0 = I$, and $M_n^r = M_n(M_n^{r-1})$ for $r \in \mathbb{N}$.

In Section 2 of this paper we give some definitions and auxiliary results which will be needed to prove the main results. In Section 3 first we establish a Voronovskaja type asymptotic formula and then find the degree of approximation for functions of a given smoothness in ordinary approximation. Subsequently in Section 4 first we show that the operators $T_{n,k}$ possess simultaneous approximation property i.e. the property that the derivatives of the operators $T_{n,k}$ converge to the corresponding order derivatives of $f(x)$ and then extend the results of Section 3 to the case of simultaneous approximation.

2. PRELIMINARIES

In the sequel we shall require the following results:

Lemma 1 [2]. *For the function $\mu_{n,m}(t)$, we have $\mu_{n,0}(t) = 1$, $\mu_{n,1}(t) = \dfrac{1-2t}{n+2}$, and there holds the recurrence relation*

$$(n+m+2)\mu_{n,m+1}(t) = t(1-t)\big(\mu'_{n,m}(t) + 2m\mu_{n,m-1}(t)\big) + (m+1)(1-2t)\mu_{n,m}(t), \quad \text{for } m \geq 1.$$

Consequently, we have

(i) $\mu_{n,m}(t)$ are polynomials in t of degree m;

(ii) for every $t \in [0,1]$, $\mu_{n,m}(t) = O\Big(n^{-[(m+1)/2]}\Big)$, where $[\beta]$ is the integer part of β.

The m-th order moment for the operator M_n^p is defined as

$$\mu_{n,m}^{[p]}(t) = M_n^p\big((u-t)^m;t\big),$$

$p \in \mathbb{N}$ (the set of natural numbers). We denote $\mu_{n,m}^{[1]}(t)$ by $\mu_{n,m}(t)$.

Lemma 2 [7]. *For the function $p_{n,k}(t)$, there holds the result*

$$t^r(1-t)^r D^r p_{n,k}(t) = \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i(k-nt)^j q_{i,j,r}(t)p_{n,k}(t), \tag{2.1}$$

where D^r stands for $\dfrac{\mathrm{d}^r}{\mathrm{d}t^r}$ and $q_{i,j,r}(t)$ are certain polynomials in t independent of n and k.

Lemma 3. *There holds the recurrence relation*

$$\mu_{n,m}^{[p+1]}(t) = \sum_{j=0}^{m}\sum_{i=0}^{m-j}\binom{m}{j}\frac{1}{i!}\,D^i\big(\mu_{n,m-j}^{[p]}(t)\big)\mu_{n,i+j}(t). \tag{2.2}$$

Proof. We can write

$$\begin{aligned}\mu_{n,m}^{[p+1]}(t) &= M_n^{p+1}\big((u-t)^m;t\big)\\ &= M_n\Big(M_n^p\big((u-t)^m;x\big);t\Big) = M_n\Big(M_n^p\big((u-x+x-t)^m;x\big);t\Big)\\ &= \sum_{j=0}^{m}\binom{m}{j}M_n\Big((x-t)^j M_n^p\big((u-x)^{m-j};x\big);t\Big).\end{aligned} \tag{2.3}$$

Since $M_n^p\big((u-x)^{m-j};x\big)$ is a polynomial in x of degree $m-j$, by TAYLOR's expansion, we can write as

$$M_n^p\big((u-x)^{m-j};x\big) = \sum_{i=0}^{m-j}\frac{(x-t)^i}{i!}\,D^i\big(\mu_{n,m-j}^{[p]}(t)\big). \tag{2.4}$$

From (2.3) and (2.4) we get the required result.

Lemma 4. *For $k,\ell\in\mathbb{N}$, there holds $T_{n,k}\big((u-t)^\ell;t\big) = O(n^{-k})$.*

Proof. We apply induction on k. For $k=1$, the result follows from Lemma 1. Assume that it is true for a certain k, then by the definition of $T_{n,k}$ we get

$$\begin{aligned}T_{n,k+1}\big((u-t)^\ell;t\big) &= \sum_{r=1}^{k+1}(-1)^{r+1}\binom{k+1}{r}M_n^r\big((u-t)^\ell;t\big)\\ &= \sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}M_n^r\big((u-t)^\ell;t\big)\\ &\qquad\qquad + \sum_{r=1}^{k+1}(-1)^{r+1}\binom{k}{r-1}M_n^r\big((u-t)^\ell;t\big)\\ &= I_1+I_2,\ \text{say.}\end{aligned}$$

We can write I_1 as

$$I_1 = T_{n,k}\big((u-t)^\ell;t\big). \tag{2.5}$$

Next, by Lemma 3

$$\begin{aligned} I_2 &= \sum_{r=0}^{k}(-1)^r\binom{k}{r}\mu_{n,\ell}^{[r+1]}(t) \\ &= \mu_{n,\ell}(t) - \sum_{j=1}^{\ell}\sum_{i=0}^{\ell-j}\binom{\ell}{j}\frac{1}{i!}\Big(D^iT_{n,k}\big((u-t)^{\ell-j};t\big)\Big)\mu_{n,i+j}(t) \\ &\qquad - \sum_{i=0}^{\ell}\frac{1}{i!}\Big(D^iT_{n,k}\big((u-t)^{\ell};t\big)\Big)\mu_{n,i}(t) \\ &= \mu_{n,\ell}(t) - \sum_{j=1}^{\ell}\sum_{i=0}^{\ell-j}\binom{\ell}{j}\frac{1}{i!}\Big(D^iT_{n,k}\big((u-t)^{\ell-j};t\big)\Big)\mu_{n,i+j}(t) \\ &\qquad - \sum_{i=1}^{\ell}\frac{1}{i!}\Big(D^iT_{n,k}\big((u-t)^{\ell};t\big)\Big)\mu_{n,i}(t) - T_{n,k}\big((u-t)^\ell;t\big), \end{aligned}$$

$$\begin{aligned} I_2 &= -\sum_{j=1}^{\ell-1}\sum_{i=0}^{\ell-j}\binom{\ell}{j}\frac{1}{i!}\Big(D^iT_{n,k}\big((u-t)^{\ell-j};t\big)\Big)\mu_{n,i+j}(t) \\ &\qquad - \sum_{i=1}^{\ell}\frac{1}{i!}\Big(D^iT_{n,k}\big((u-t)^{\ell};t\big)\Big)\mu_{n,i}(t) - T_{n,k}\big((u-t)^\ell;t\big) \end{aligned} \tag{2.6}$$

From Lemma 1, (2.5) and (2.6) we get $T_{n,k+1}\big((u-t)^\ell;t\big) = O\big(n^{-(k+1)}\big)$. Thus, the result is proved for all $k \in \mathbb{N}$.

Lemma 5. *For $p \in \mathbb{N}$, $m \in \mathbb{N}_0$ and $t \in [0,1]$, we have*

$$\mu_{n,m}^{[p]}(t) = O\big(n^{-[(m+1)/2]}\big). \tag{2.7.}$$

Proof. For $p=1$, the result follows from Lemma 1. Suppose (2.7) is true for a certain p. Then $\mu_{n,m-j}^{[p]}(t) = O\big(n^{-[(m-j+1)/2]}\big)$, $0 \le j \le m$. Also $\mu_{n,m-j}^{[p]}(t)$ is a polynomial in t of degree $m-j$, therefore, we have

$$D^i\big(\mu_{n,m-j}^{[p]}(t)\big) = O\big(n^{-[(m-j+1)/2]}\big)\ \forall\ 0 \le i \le m-j.$$

Now, applying Lemma 3,

$$\mu_{n,m}^{[p+1]}(t) = \sum_{j=0}^{m}\sum_{i=0}^{m-j} O\big(n^{-[(m-j+1)/2]}\big)\cdot O\big(n^{-[(i+j+1)/2]}\big) = O\big(n^{-[(m+1)/2]}\big).$$

Hence, the lemma is proved by induction on p.

3. ORDINARY APPROXIMATION

Theorem 1. (VORONOVSKAJA type asymptotic formula). *Let* $f \in L_B[0,1]$ *admitting a derivative of order* $2k$ *at a point* $t \in [0,1]$. *Then*

$$\lim_{n\to\infty} n^k\big(T_{n,k}(f;t) - f(t)\big) = \sum_{v=1}^{2k} \frac{f^{(v)}(t)}{v!}\, Q(v,k,t) \tag{3.1}$$

and

$$\lim_{n\to\infty} n^k\big(T_{n,k+1}(f;t) - f(t)\big) = 0, \tag{3.2}$$

where $Q(v,k,t)$ *are certain polynomials in* t *of degree* v. *Further, the limits in* (3.1) *and* (3.2) *hold uniformly in* $[0,1]$ *if* $f^{(2k)}(t)$ *is continuous in* $[0,1]$.

Proof. Since $f^{(2k)}(t)$ exists, we can write an expansion of f as:

$$f(u) = \sum_{v=0}^{2k} \frac{f^{(v)}(t)}{v!}\,(u-t)^v + \varepsilon(u,t)(u-t)^{2k}, \tag{3.3}$$

where $\varepsilon(u,t) \to 0$ as $u \to t$ and is bounded and integrable in $[0,1]$. The proof is as follows:

Let $\varepsilon(u,t) = \dfrac{f(u) - \sum_{i=0}^{2k} \frac{f^{(i)}(t)}{i!}\,(u-t)^i}{(u-t)^{2k}}$. Then,

$$\begin{aligned}
\lim_{u\to t}\varepsilon(u,t) &= \lim_{u\to t} \frac{f(u) - \left(f(t) + (u-t)f'(t) + \cdots + \dfrac{(u-t)^{2k}}{(2k)!}\, f^{(2k)}(t)\right)}{(u-t)^{2k}}\\
&= \lim_{u\to t} \frac{f^{(2k-1)}(u) - \big(f^{(2k-1)}(t) + (u-t)f^{(2k)}(t)\big)}{2k!(u-t)}\\
&\qquad\qquad \text{(applying L'Hospital's rule successively } (2k-1) \text{ times)}\\
&= \frac{1}{2k!}\lim_{u\to t} \frac{f^{(2k-1)}(u) - f^{(2k-1)}(t)}{u-t} - \frac{f^{(2k)}(t)}{2k!}\\
&= 0.
\end{aligned}$$

Operating by $T_{n,k}$ on both sides of (3.3) we get

$$\begin{aligned}
n^k\big(T_{n,k}(f;t) - f(t)\big) &= n^k \sum_{v=1}^{2k} \frac{f^{(v)}(t)}{v!}\, T_{n,k}\big((u-t)^v;t\big) + n^k T_{n,k}\big(\varepsilon(u,t)(u-t)^{2k};t\big).\\
&= I_1 + I_2, \quad \text{say.}
\end{aligned}$$

Making use of Lemma 4, we obtain

$$I_1 = \sum_{v=1}^{2k} \frac{f^{(v)}(t)}{v!} Q(v,k,t) + o(1),$$

where $Q(v,k,t)$ is the coefficient of n^{-k} in $T_{n,k}\big((u-t)^v;t\big)$.

Since $\varepsilon(u,t) \to 0$ as $u \to t$, for a given $\varepsilon' > 0$ we can find a $\delta > 0$ such that $|\varepsilon(u,t)| < \varepsilon'$ whenever $0 < |u-t| < \delta$ and for $|u-t| \geq \delta$, $|\varepsilon(u,t)| \leq K$ for some $K > 0$. Suppose $\chi(u)$ is the characteristic function of the interval $(t-\delta, t+\delta)$, then

$$\begin{aligned} |I_2| &= n^k \sum_{r=1}^{k} \binom{k}{r} M_n^r\big(|\varepsilon(u,t)|(u-t)^{2k}\chi(u);t\big) \\ &\qquad + n^k \sum_{r=1}^{k} \binom{k}{r} M_n^r\Big(|\varepsilon(u,t)|(u-t)^{2k}\big(1-\chi(u)\big);t\Big) \\ &= I_3 + I_4, \text{ say.} \end{aligned}$$

In view of Lemma 5,

$$I_3 = \varepsilon' O(1).$$

Now, applying Lemma 5, we have for any integer $s > k$,

$$\begin{aligned} I_4 &\leq n^k \sum_{r=1}^{k} \binom{k}{r} M_n^r\big(K(u-t)^{2s}/\delta^{2s-2k};t\big) = O(n^{k-s}) \text{ for any integer } s > k. \\ &= o(1). \end{aligned}$$

Due to arbitrariness of ε' it follows that $|I_2| = o(1)$.

Combining the estimates of I_1 and I_2, we obtain (3.1). Similarly, the assertion (3.2) follows from the fact $T_{n,k+1}\big((u-t)^\ell;t\big) = O(n^{-k-1})$ for all $\ell \in \mathbb{N}$.

The uniformity assertion follows du to the uniform continuity of $f^{(2k)}$ on $[0,1]$ which enables δ to become independent of t and the uniformness of the term $o(1)$ in the estimate of I_1.

In our next result we obtain an estimate of the degree of approximation of a function with specified smoothness.

Theorem 2. *Let $1 \leq p \leq 2k$ be an integer and $f^{(p)} \in C[0,1]$. Then, for sufficiently large n there holds*

$$\big\|T_{n,k}(f;t) - f(t)\big\| \leq \max\big\{C_1 n^{-p/2}\omega(f^{(p)};n^{-1/2}), C_2 n^{-k}\big\}, \tag{3.4}$$

where $C_1 = C_1(k,p)$, $C_2 = C_2(k,p,f)$, $\|\cdot\|$ is sup-norm on $[0,1]$ and $\omega(f^{(p)};\delta)$ is the modulus of continuity of $f^{(p)}$ on $[0,1]$.

Proof. By Taylor's expansion, we can write

$$f(u) - f(t) = \sum_{i=1}^{p} \frac{f^{(i)}(t)}{i!}(u-t)^i + \frac{f^{(p)}(\xi) - f^{(p)}(t)}{p!}(u-t)^p, \tag{3.5}$$

where ξ lies between u and t.

Operating by $T_{n,k}$ on both sides of (3.5) and breaking the right hand side into two parts I_1 and I_2 say, corresponding to two terms on the right hand side of (3.5), we get

$$T_{n,k}(f;t) - f(t) = I_1 + I_2, \text{ say.}$$

In view of Lemma 4,

$$I_1 = \sum_{i=1}^{p} \frac{f^{(i)}(t)}{i!} T_{n,k}\big((u-t)^i;t\big) = O(n^{-k}), \text{ uniformly for every } t \in [0,1].$$

Since $f^{(p)} \in C[0,1]$, we have

$$\big|f^{(p)}(\xi) - f^{(p)}(t)\big| \le \omega\big(f^{(p)}; |\xi - t|\big) \le \big(1 + |u-t|/\delta\big)\omega(f^{(p)};\delta), \text{for any } \delta > 0.$$

Hence, using Schwarz inequality and Lemma 5,

$$|I_2| \le \frac{\omega(f^{(p)};\delta)}{p!} \sum_{r=1}^{k} \binom{k}{r} M_n^r\Big(\big|(u-t)\big|^p \big(1 + |u-t|/\delta\big);t\Big).$$

Choosing $\delta = n^{-1/2}$, we get

$$|I_2| \le \omega(f^{(p)}; n^{-1/2}) O(n^{-p/2}), \text{ uniformly in } [0,1].$$

Combining the estimates of I_1 and I_2, the theorem follows.

4. SIMULTANEOUS APPROXIMATION

In this section we discuss simultaneous approximation property of the operators $T_{n,k}$. First we prove that $T_{n,k}^{(p)}$ is an approximation process for $f^{(p)}$, $p = 1, 2, 3, \ldots$.

Theorem 3. *Let* $f \in L_B[0,1]$ *admitting a derivative of order* p *at a fixed point* $t \in (0,1)$. *Then*

$$\lim_{n\to\infty} T_{n,k}^{(p)}(f;t) = f^{(p)}(t). \tag{4.1}$$

Further, if $f^{(p)}$ *exists and is continuous on* $(a-\eta, b+\eta) \subset (0,1)$, $\eta > 0$, *then* (4.1) *holds uniformly in* $t \in [a,b]$.

Proof. We can expand $f(u)$ as

$$f(u) = \sum_{i=0}^{p} \frac{f^{(i)}(t)}{i!} (u-t)^i + \varepsilon(u,t)(u-t)^p,$$

where $\varepsilon(u,t) \to 0$ as $u \to t$ and is bounded and integrable on $[0,1]$.

In order to prove (4.1), it is sufficient to show that $\lim_{n\to\infty} D^p\big(M_n^r(f;t)\big) = f^{(p)}(t)$. Therefore, from the above expansion of f and the definition of M_n^r

$$\begin{aligned}D^pM_n^r(f;t) &= \sum_{i=0}^{p}\frac{f^{(i)}(t)}{i!}\int_0^1 W_n^{(p)}(s,t)M_n^{r-1}\big((u-t)^i;s\big)\,\mathrm{d}s\\ &\qquad + \int_0^1 W_n^{(p)}(s,t)M_n^{r-1}\big(\varepsilon(u,t)(u-t)^p;s\big)\,\mathrm{d}s\\ &= I_1+I_2,\ \text{say.}\end{aligned}$$

Now

$$\begin{aligned}I_1 &= \sum_{i=0}^{p}\frac{f^{(i)}(t)}{i!}\sum_{j=0}^{i}\binom{i}{j}(-t)^{i-j}\int_0^1 W_n^{(p)}(s,t)M_n^{r-1}(u^j;s)\,\mathrm{d}s\\ &= \sum_{i=0}^{p}\frac{f^{(i)}(t)}{i!}\sum_{j=0}^{i}\binom{i}{j}(-t)^{i-j}D^pM_n^r(u^j;t).\end{aligned}$$

Since $M_n^r(u^j;t)$ is a polynomial in t of degree j and the coefficient of t^j is equal to $\prod_{i=1}^{j}\big((n-i+1)/(n+i+1)\big)^r$, which tends to 1 as $n\to\infty$, it follows that $I_1\to f^{(p)}(t)$ as $n\to\infty$. Since $\varepsilon(u,t)\to 0$ as $u\to t$, for a given $\varepsilon'>0$ we can find a $\delta>0$ such that $|\varepsilon(u,t)|<\varepsilon'$ whenever $0<|u-t|<\delta$, $\varepsilon(u,t)$ is bounded by some $K>0$, say. Suppose $\chi(u)$ is the characteristic function of the interval $(t-\delta,t+\delta)$, then in view of Lemma 2

$$\begin{aligned}I_2 &= (n+1)\sum_{k=0}^{n}\int_0^1\Big(D^p\big(p_{n,k}(t)\big)\Big)p_{n,k}(s)M_n^{r-1}\big(\varepsilon(u,t)(u-t)^p;s\big)\,\mathrm{d}s\\ &= (n+1)\sum_{k=0}^{n}\sum_{\substack{2i+j\le p\\ i,j\ge 0}}\frac{n^i(k-nt)^j}{t^p(1-t)^p}\,q_{i,j,p}(t)p_{n,k}(t)\,\times\\ &\qquad\times\Bigg(\int_0^1 p_{n,k}(s)M_n^{r-1}\big(\varepsilon(u,t)(u-t)^p\chi(u);s\big)\,\mathrm{d}s\\ &\qquad\qquad+\int_0^1 p_{n,k}(s)M_n^{r-1}\big(\varepsilon(u,t)(u-t)^p(1-\chi(u));s\big)\,\mathrm{d}s\Bigg)\\ &= I_3+I_4,\ \text{say.}\end{aligned}$$

Let $C_1 = \sup_{\substack{2i+j\le p\\ i,j\ge 0}}\big|q_{i,j,p}(t)/(t^p(1-t)^p)\big|$, applying SCHWARZ inequality three times we get

$$|I_3| \le \varepsilon' C_1 \sum_{\substack{2i+j\le p\\ i,j\ge 0}} n^i \Big(\sum_{k=0}^{n} p_{n,k}(t)(k-nt)^{2j} \Big)^{1/2} \times$$
$$\times \Big((n+1) \sum_{k=0}^{n} p_{n,k}(t) \int_0^1 p_{n,k}(s) M_n^{r-1}\big((u-t)^{2p}; s\big)\,\mathrm{d}s \Big)^{1/2}.$$

Now, it is known [**3**] that for $0 \le t \le 1$ and $m \in \mathbb{N}_0$,

$$\sum_{k=0}^{n} p_{n,k}(t)(k-nt)^{2j} = O(n^j). \tag{4.2}$$

Therefore, using Lemma 5 we get

$$I_3 = \varepsilon' O(1). \tag{4.3}$$

Again,

$$|I_4| \le \sum_{k=0}^{n} (n+1) \binom{k}{r} \sum_{\substack{2i+j\le p\\ i,j\ge 0}} C_1 n^i p_{n,k}(t) \big|(k-nt)\big|^j \times$$
$$\times \int_0^1 p_{n,k}(s) M_n^{r-1}\Big(|\varepsilon(u,t)|\, |(u-t)|^p \big(1-\chi(u)\big); s\Big)\,\mathrm{d}s.$$

Using Schwarz inequality, (4.2) and Lemma 5, for any integer $s > p$ we obtain

$$\begin{aligned} |I_4| &\le C_1 O(n^{p/2}) K \delta^{-s+p} \times \\ &\quad \times \Big((n+1) \sum_{k=0}^{n} p_{n,k}(t) \int_0^1 p_{n,k}(s) M_n^{r-1}\Big((u-t)^{2s}\big(1-\chi(u)\big); s\Big) \mathrm{d}s \Big)^{1/2} \\ &\le K'(n^{(p-s)/2}). \end{aligned}$$

Therefore we have

$$I_4 = o(1). \tag{4.4}$$

As $\varepsilon' > 0$ is arbitrary, from (4.3) and (4.4) we see that $I_2 = o(1)$. Hence (4.1) follows from the estimates of I_1 and I_2. The second assertion follows due to the fact that $\delta(\varepsilon')$ can be chosen independent of $t \in [a,b]$ and all the other estimates hold uniformly in $[a,b]$.

In our next theorem we study an asymptotic result for $T_{n,k}$ in simultaneous approximation.

Theorem 4. *Let $f \in L_B[0,1]$. If $f^{(2k+p)}(t)$ exists at the point $t \in (0,1)$, then we have*

$$\lim_{n\to\infty} n^k \big(T_{n,k}^{(p)}(f;t) - f^{(p)}(t)\big) = \sum_{j=p}^{2k+p} Q_1(j,k,p,t) f^{(j)}(t), \tag{4.5}$$

where $Q_1(j,k,p,t)$ are certain polynomials in t. Further, if $f^{(2k+p)}$ is continuous in $(a-\eta, b+\eta) \subset (0,1)$, $\eta > 0$, then (4.5) holds uniformly in $[a,b]$.

Proof. By our hypothesis we can write

$$\begin{aligned} T_{n,k}^{(p)}(f;t) &= \sum_{r=1}^{k} (-1)^{r+1} \binom{k}{r} \int_0^1 W_n^{(p)}(s,t) M_n^{r-1} \Bigg(\sum_{i=0}^{2k+p} \frac{f^{(i)}(t)}{i!} (u-t)^i \\ &\qquad\qquad + \varepsilon(u,t)(u-t)^{2k+p}; s \Bigg) \mathrm{d}s \\ &= I_1 + I_2, \text{ say,} \end{aligned}$$

where $\varepsilon(u,t) \to 0$ as $u \to t$ and is bounded and integrable on $[0,1]$.

On an application of Lemma 1 and Theorem 1 we obtain

$$\begin{aligned} I_1 &= \sum_{i=p}^{2k+p} \frac{f^{(i)}(t)}{i!} \sum_{\ell=0}^{i} \binom{i}{\ell} (-t)^{i-\ell} T_{n,k}^{(p)}(u^\ell; t) \\ &= \sum_{i=p}^{2k+p} \frac{f^{(i)}(t)}{i!} \sum_{\ell=0}^{i} \binom{i}{\ell} (-t)^{i-\ell} \bigg(D^p t^\ell + n^{-k} \sum_{j=1}^{2k} D^p \Big(\frac{Q(j,k,t)}{j!} D^j t^\ell \Big) + o(n^{-k}) \bigg) \\ &= f^{(p)}(t) + \sum_{i=p}^{2k+p} n^{-k} \sum_{\ell=0}^{i} \binom{i}{\ell} (-t)^{i-\ell} \frac{f^{(i)}(t)}{i!} \bigg(\sum_{j=1}^{2k} D^p \Big(\frac{Q(j,k,t)}{j!} D^j t^\ell \Big) \bigg) + o(n^{-k}) \\ &= f^{(p)}(t) + n^{-k} \sum_{j=p}^{2k+p} Q_1(j,k,p,t) f^{(j)}(t) + o(n^{-k}), \end{aligned}$$

where we used the identities $\displaystyle\sum_{\ell=0}^{i} (-1)^\ell \binom{i}{\ell}\binom{\ell}{p} = \begin{cases} 0, & i > p \\ (-1)^p, & i = p. \end{cases}$

To estimate $\displaystyle I_2 = \sum_{r=1}^{k} (-1)^{r+1} \binom{k}{r} \int_0^1 W_n^{(p)}(s,t) M_n^{r-1}\big(\varepsilon(u,t)(u-t)^{2k+p}; s\big)\,\mathrm{d}s$, proceeding as in the estimate of I_4 in Theorem 3, it follows that $n^k I_2 \to 0$ as $n \to \infty$. Hence, combining the estimates of I_1 and I_2, (4.5) is established. The uniformity assertion follows as in Theorem 3.

Theorem 5. *Let $p, q \in \mathbb{N}$, $p \le q \le 2k + p$ and $f \in L_B[0, 1]$. If $f^{(q)}$ exists and is continuous on $(a - \eta, b + \eta) \subset (0, 1)$, for some $\eta > 0$ then*

$$\big\|T^{(p)}_{n,k}(f; t) - f^{(p)}(t)\big\| \le \max\big\{C_1 n^{-(q-p)/2}\omega(f^{(q)}; n^{-1/2}), C_2 n^{-k}\big\}, \tag{4.6}$$

where $C_1 = C_1(k, p)$, $C_2 = C_2(k, p, f)$, $\|\cdot\|$ is the sup-norm on $[a, b]$ and the modulus of continuity of $f^{(q)}$ on $(a - \eta, b + \eta)$ is $\omega(f^{(q)}; n^{-1/2})$.

Proof. By our hypothesis, we may write for all $u \in [0, 1]$ and $t \in [a, b]$

$$f(u) = \sum_{i=0}^{q} \frac{f^{(i)}(t)}{i!}(u-t)^i + \frac{f^{(q)}(\xi) - f^{(q)}(t)}{q!}(u-t)^q\chi(u) + F(u, t)\big(1-\chi(u)\big), \tag{4.7}$$

where $\chi(u)$ is the characteristic function of $(a-\eta, b+\eta)$, ξ lies between u and t and $F(u, t)$ is defined as $F(u, t) = f(u) - \sum_{i=0}^{q} \frac{f^{(i)}(t)}{i!}(u - t)^i$, $\forall u \in [0, 1]$ and $t \in [a, b]$.

The function $F(u, t)$ is bounded by $M|u - t|^q$ for $t \in [a, b]$ and M is some positive number. Now operating by $T^{(p)}_{n,k}$ on both sides of (4.7) and breaking the right hand side into three parts I_1, I_2 and I_3 say, corresponding to the three terms on the right hand side of (4.7), we get

$$T^{(p)}_{n,k}(f; t) - f^{(p)}(t) = I_1 + I_2 + I_3, \text{ say.}$$

Now,

$$I_1 = \sum_{i=1}^{q}\sum_{j=0}^{i}(-t)^{i-j}\binom{i}{j}\frac{f^{(i)}(t)}{i!}T^{(p)}_{n,k}(u^j; t).$$

Proceeding as in the estimate of I_1 of Theorem 4

$$\begin{aligned} I_1 &= \sum_{i=1}^{q}\sum_{j=0}^{i}(-t)^{i-j}\binom{i}{j}\frac{f^{(i)}(t)}{i!}D^p\bigg(t^j + n^{-k}\bigg(\sum_{r=1}^{2k} D^p\Big(\frac{Q(r, k, t)}{r!}D^r t^j\Big) + o(n^{-k})\bigg)\bigg) \\ &= O(n^{-k}), \text{ uniformly in } t \in [a, b]. \end{aligned}$$

Next, applying Lemma 2,

$$\begin{aligned} |I_2| \le \frac{\omega(f^{(q)}; \delta)}{q!}\sum_{r=1}^{k}\binom{k}{r}(n+1)\sum_{v=0}^{n}\sum_{\substack{2i+j\le p\\ i,j\ge 0}} n^i p_{n,v}(t)\frac{|q_{i,j,p}(t)|}{t^p(1-t)^p}|v - nt|^j \times \\ \times \int_0^1 p_{n,v}(s) M_n^{r-1}\Big(|u-t|^q\big(1 + |u-t|/\delta\big); s\Big)\,\mathrm{d}s. \end{aligned}$$

Let $C' = \sup\limits_{\substack{2i+j\le p\\ i,j\ge 0,\ t\in[a,b]}} \big|q_{i,j,p}(t)/\big(t^p(1-t)^p\big)\big|$. Using Schwarz inequality, (4.2) and Lemma 5 we obtain $|I_2| \le C'\omega(f^{(q)}; \delta)\big(O(n^{-(q-p)/2}) + O(n^{-(q+1-p)/2})\big)$, uniformly in $t \in [a, b]$.

Choosing $\delta = n^{-1/2}$, it follows that $I_2 = \omega(f^{(q)}; n^{-1/2})O(n^{-(q-p)/2})$ uniformly in $t \in [a, b]$. Lastly, to estimate $I_3 = T_{n,k}^{(p)}\big(F(u,t)(1-\chi(u)); t\big)$, proceeding in a manner similar to the estimate of I_4 in Theorem 3, it follows that $I_3 = O(n^{(p-s)/2})$, where s is an integer greater than $2k + p + 2$. Thus $I_3 = o(n^{-(k+1)})$, uniformly in $t \in [a, b]$.

Combining the estimates of I_1, I_2 and I_3, (4.6) is established. This completes the proof.

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