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ON A CLASS OF MAXIMAL REFLEXIVE θ-GRAPHS GENERATED BY SMITH GRAPHS

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A simple graph is said to be reflexive if its second largest eigenvalue does not exceed 2. The property $\lambda_2 \leq 2$ is a hereditary one, i.e. any induced subgraph of a reflexive graph preserves this property and that is why reflexive graphs are usually represented by maximal graphs within a given class. Bicyclic graphs whose two cycles have a common path are called θ-graphs. We consider classes of maximal reflexive θ-graphs arising from a Smith tree and a cycle attached to it in a specified way.

1. INTRODUCTION

Let $P_G(\lambda) = \det(\lambda I - A)$ be the *characteristic polynomial* of the $(0,1)$- adjacency matrix of a simple graph G (an undirected graph without loops or multiple edges). The roots of $P_G(\lambda)$ are the *eigenvalues* of G. The family of these roots forms the *spectrum* of G. The eigenvalues of a simple graph are real, and we assume their non-increasing order: $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$. The relation between the spectrum of a graph and the spectra of its induced subgraphs is established by the *interlacing theorem*:

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of a graph G and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$ eigenvalues of its induced subgraph H. Then the inequalities $\lambda_{n-m+i} \leq \mu_i \leq \lambda_i$ $(i = 1, \ldots, m)$ hold.

Thus, for example, if $m = n-1$, $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2$. Also, $\lambda_1 > \mu_1$ if G is connected.

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Graphs with the property $\lambda_2 \leq 2$ are called *reflexive graphs* and if $\lambda_2 \leq 2 \leq \lambda_1$ they are also called *hyperbolic graphs,* ([**7**], [**8**]).

The terminology concerning graph spectra follows [**2**], while for general graph theoretic concepts one can see [**4**].

Since the graphic property $\lambda_2 \leq 2$ is hereditary (every induced subgraph maintains the property), the result is expressed through the set of maximal graphs within a given class.

Bicyclic graphs whose two cycles have a common path are called *θ-graphs*.

Smith graphs are connected graphs with the property $\lambda_1 = 2$. SMITH graphs are widely present in the sets of maximal reflexive graphs investigated so far. Many families of such graphs can be described completely or almost completely by SMITH graphs.

So far various classes of reflexive graphs have been studied such as: reflexive trees ([**5**], [6]), bicyclic reflexive graphs with a bridge between the cycles [**13**], treelike reflexive graphs with three or more cycles ([**9**], [**11**], [**12**], [**14**], [**15**]), some classes of bicyclic reflexive graphs [**9**], [**10**], and there are some preliminary results on θ-graphs [**10**], [**14**], [**16**].

In this paper we construct a class of maximal reflexive θ-graphs using Smith graphs.

Some general and auxiliary results to be used in our investigations are presented in the next section. At some stages the work has been supported by using the expert system GRAPH ([**1**], [**3**]).

2. PRELIMINARIES

The following theorem gives useful interrelations between the characteristic polynomial of a graph and its induced subgraphs.

Lemma 1. (SCHWENK [**17**]). *Given a graph G, let $C(v)$ $(C(uv))$ denote the set of all cycles containing a vertex v and an edge uv of G, respectively. Then*

(i) $P_G(\lambda) = \lambda P_{G-v}(\lambda) - \sum\limits_{u \in \mathrm{Adj}(v)} P_{G-v-u}(\lambda) - 2 \sum\limits_{C \in \mathcal{C}(v)} P_{G-V(C)}(\lambda),$

(ii) $P_G(\lambda) = P_{G-uv}(\lambda) - P_{G-v-u}(\lambda) - 2 \sum\limits_{C \in \mathcal{C}(uv)} P_{G-V(C)}(\lambda),$

where $\mathrm{Adj}(v)$ *denotes the set of neighbors of v, while $G - V(C)$ is the graph obtained from G by removing the vertices belonging to the cycle C.*

These relations have the following consequences (see, e.g. [**2**], p. 59).

Corollary 1. *Let G be a graph obtained by joining a vertex v_1 of a graph G_1 to a vertex v_2 of a graph G_2 by an edge. Let G_1' $\left(G_2'\right)$ be the subgraph of G_1 (G_2) obtained by deleting the vertex v_1 (v_2) from G_1 (resp. G_2). Then*

$$P_G(\lambda) = P_{G_1}(\lambda) P_{G_2}(\lambda) - P_{G_1'}(\lambda) P_{G_2'}(\lambda).$$

Corollary 2. *Let G be a graph with a pendant edge v_1v_2, v_1 being of degree* 1. *Then*

$$P_G(\lambda) = \lambda P_{G_1}(\lambda) - P_{G_2}(\lambda),$$

where $G_1\,(G_2)$ is the graph obtained from G (resp. G_1) by deleting the vertex v_1 (resp. v_2).

3. SMITH GRAPHS

The set of connected graphs for which $\lambda_1 = 2$ is depicted in Fig. 1. These graphs are known as SMITH graphs. The set contains cycles of all possible lengths, a family W_n of trees of arbitrary diameter and four small trees, one of which is actually W_0 but sometimes it is convenient to be treated separately. Proper induced subgraphs of SMITH graphs all have $\lambda_1 < 2$ (they are also known as COXETER-DYNKIN graphs).

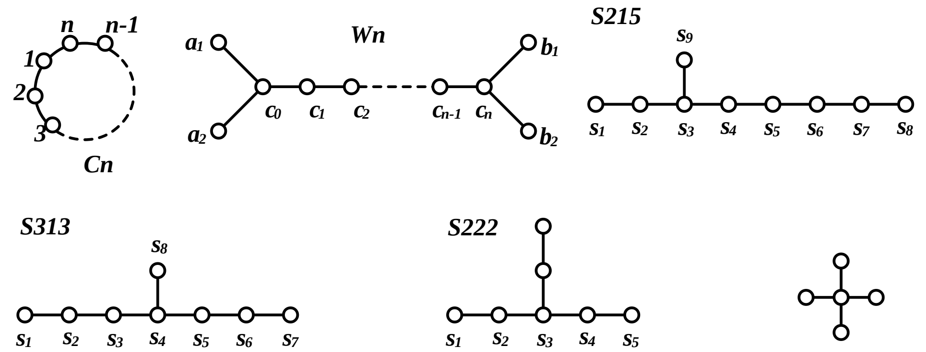

Figure 1.

Theorem S. (SMITH [**18**], see also [**2**, p.79]) $\lambda_1\,(G) \le 2$ (*resp.* $\lambda_1\,(G) < 2$) *if and only if each component of graph G is a subgraph* (*resp. proper subgraph*) *of one of the graphs of Fig.* 1, *all of which have index equal to* 2.

Any connected graph is either an induced subgraph or an induced supergraph of some SMITH graphs.

Lemma 2. (RADOSAVLJEVIĆ and SIMIĆ, [**13**]). *Let G be a graph obtained by extending any of Smith graphs by a vertex of arbitrary positive degree. Then $P_G\,(2) < 0$ (i.e. $\lambda_2\,(G) < 2 < \lambda_1\,(G)$).*

Theorem RS. (RADOSAVLJEVIĆ and SIMIĆ, [**13**]). *Let G be a graph with cut-vertex u.*

(i) *If at least two components of $G - u$ are induced supergraphs of Smith graphs, and if at least one of them is a proper supergraph, then $\lambda_2\,(G) > 2$.*

(ii) *If at least two components of* $G - u$ *are Smith graphs, and the rest are induced subgraphs of Smith graphs, then* $\lambda_2(G) = 2$.

(iii) *If at most one component of* $G - u$ *is a Smith graph, and the rest are proper induced subgraphs of Smith graphs, then* $\lambda_2(G) < 2$.

If $G - u$ (u being a cut-vertex) has one proper supergraph and the remaining components are proper (induced) subgraphs of Smith graphs, Theorem RS is not applicable and these cases are interesting for further investigations.

4. A CLASS OF θ-GRAPHS

If two cycles of a bicyclic graph have a common path, we shall say that they form a θ-graph (Figure 2) and the same name will be used for any bicyclic graph with such cyclic structure.

Research on maximal reflexive θ-graphs is the first step in the area of new classes of reflexive graphs which are not trees or treelike graphs. In previous investigations on maximal reflexive treelike graphs we have noticed a constant presence of Smith trees.

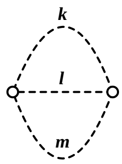

Figure 2.

Therefore, it is obvious that Smith graphs have an important role in forming of maximal reflexive graphs. This is the reason why we are making first steps in this area by constructing θ-graphs from Smith graphs.

Consider the Smith tree S depicted in Figure 3. $P_S(2) = 0$. Let us introduce the notation: $U_S = P_{S-u}(2)$, $V_S = P_{S-v}(2)$, $C = P_{S-p}(2)$, where p is the unique path connecting vertices u and v (within the Smith tree).

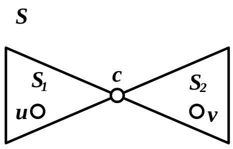

Figure 3.

Consider θ-graph in Figure 4. This graph is formed from a Smith tree (S) and a cycle (length n). Smith tree is connected with the cycle by two paths of length 2 (one starting at u and ending at u_1, and another one starting at v and ending at v_1). Lengths of paths connecting vertices u_1 and v_1 within the cycle are and n_1 and n_2, ($n_1 + n_2 = n, n_1, n_2 \geq 4$).

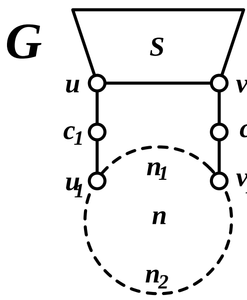

Figure 4.

Proposition 1. *Let* G *be the graph in Figure* 4. *Then* $P_G(2) = n\,(U_S + V_S - 2C)$.

Proof. Let us remove the vertex c_1 from graph G, and then vertices u_1 and u. We get graphs H_1, H_2 and H_3 of Figure 5, respectively. Applying Theorem RS we get $P_{H_1}(2) = 0$. Application of Lemma 1 to the graph H_2 at the vertex c_2 gives the following result:

$$P_{H_2}(2) = 2P_{H_2-c_2}(2) - P_{H_2-c_2-v_1}(2) - P_{H_2-c_2-v}(2) = -nV_S.$$

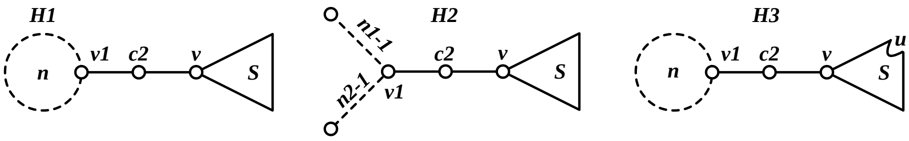

Figure 5.

Applying Lemma 1 to the graph H_3 at the vertex c_2 we get:

$$P_{H_3}(2) = 2P_{H_3-c_2}(2) - P_{H_3-c_2-v_1}(2) - P_{H_3-c_2-v}(2) = -nU_S.$$

Finally, we use these results to get $P_G(2)$.

$$\begin{aligned} P_G(2) &= 2P_{H_1}(2) - P_{H_2}(2) - P_{H_3}(2) - 2n_1C - 2n_2C \\ &= nU_S + nV_S - 2(n_1+n_2)C = nU_S + nV_S - 2nC. \end{aligned}$$

We see that $P_G(2) = n\,(U_S + V_S - 2C)$ and this completes the proof.

The next step is to go through all SMITH trees and find all cases in which $U_S + V_S - 2C = 0$ holds, because in Proposition 1 we proved that then 2 belongs to the spectrum of the corresponding θ-graph.

5. ANALYSIS OF SMITH TREES

5.1 SMITH TREE $S215$

We can find now all pairs of vertices (u, v) of $S215$ (Fig. 1), for which $U_S + V_S - 2C = 0$ holds. They are:

$$(u, v) \in \{(s_1, s_7), (s_7, s_1), (s_2, s_5), (s_5, s_2), (s_6, s_9), (s_9, s_6)\}.$$

Those θ-graphs corresponding to these pairs are shown in Fig. 6. They are all maximal reflexive graphs in their class, and $n_1 = n_2 = 4$. In all three cases $\lambda_2 = \lambda_3 = 2$ holds.

In these proofs the expert system GRAPH is used in the final stages to check wether the graph is maximal (wether it could be extended at some vertices) and determine the limits of the lengths n_1 and n_2 of the given cycle (for larger n_1 and n_2 2 would still belong to the spectrum, but it would no longer be λ_2, but λ_3 or λ_4, etc.).

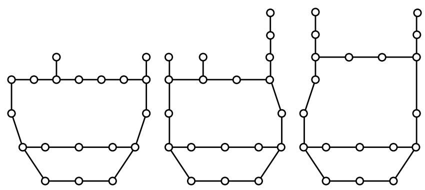

Figure 6.

5.2 SMITH TREE S313

All pairs of vertices (u,v) of $S313$ (Fig. 1) for which $U_S + V_S - 2C = 0$ holds are:

$$(u,v) \in \{(s_1,s_7),(s_2,s_6),(s_2,s_8),(s_8,s_2),(s_3,s_5)\}.$$

Consider the pair (s_1,s_7). The corresponding maximal reflexive θ-graphs are shown in Figure 7.

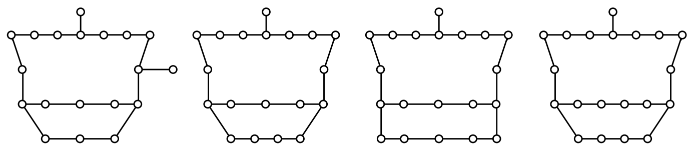

Figure 7.

Application of Theorem RS gives the explanation why the extension of the starting graph is possible only at vertices c_1 and c_2.

For the remaining pairs (u,v), the corresponding maximal θ-graphs are shown in Fig. 8.

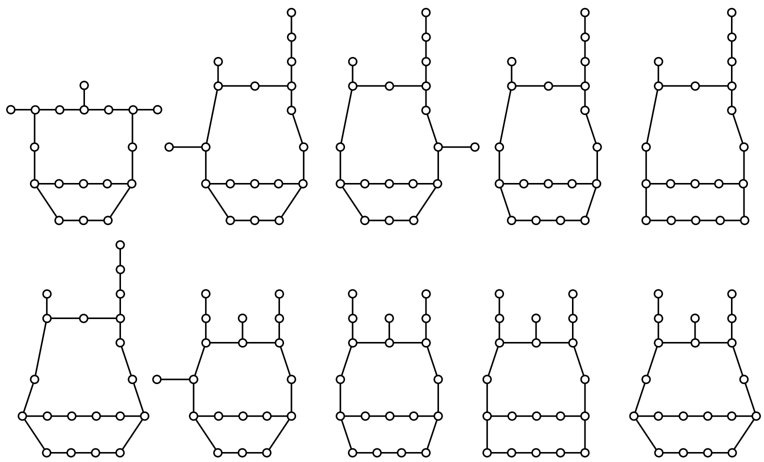

Figure 8

5.3 SMITH TREE $S222$

All pairs of vertices (u,v) of $S222$ (Fig. 1) for which $U_S + V_S - 2C = 0$ holds are:

$(u,v) \in \{(s_1,s_5),(s_2,s_4)\}$. Corresponding maximal reflexive θ-graphs are shown in Fig. 9.

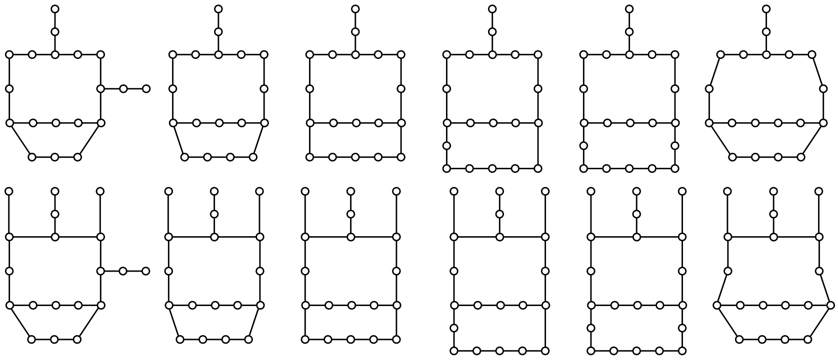

Figure 9.

5.4 SMITH TREE W_n

All pairs of vertices (u, v) of the Smith graph W_n (Fig. 1) for which $U_S + V_S - 2C = 0$ holds are: $(u, v) = (a_1, a_2)\,, (u, v) = (c_k, c_{k+l})\,, k \in \{0, 1, \ldots, n-1\}\,, k + l \in \{1, \ldots, n\}$ and $(u, v) = (a_1, b_1)\,.$

Maximal reflexive θ-graphs corresponding to the pair (a_1, a_2) are shown in Fig 10.

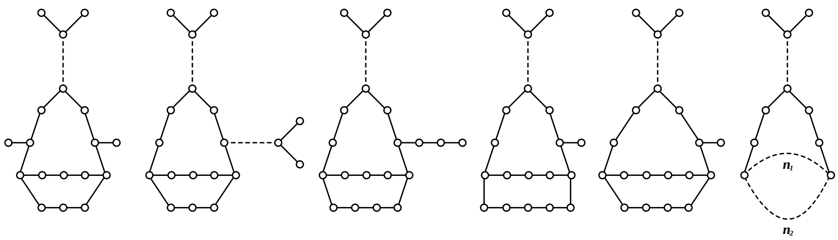

Figure 10.

The values n_1 and n_2 for the last graph in Figure 10 are:

$$(n_1, n_2) \in \{(4,7)\,, (4,8)\,, (4,9)\,, (4,10)\,, (4,11)\,, (4,12)\,, (5,6)\,, (5,7)\,, (6,6)\}.$$

To the pairs $(u, v) = (c_k, c_{k+\ell},\ k \in \{0, 1, 2, \ldots, n-1\}\,, k + \ell \in \{1, 2, \ldots, n\}$ there corresponds θ-graph in Figure 11(a)

For $\ell \geq 3$ it holds $\lambda_2 > 2$ $(\lambda_3 = 2, \ldots)$. For $\ell = 2$ we get maximal reflexive θ-graph shown in Figure 11(b).

For $\ell = 1$ we get maximal reflexive θ-graph shown in Figure 12.

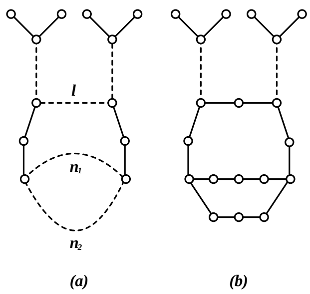

Figure 11.

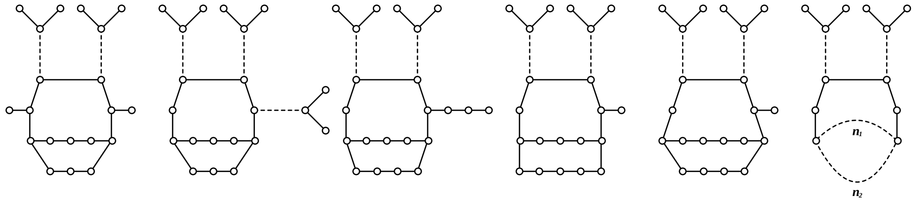

Figure 12.

$$(n_1, n_2) \in \{(4,7), (4,8), (4,9), (4,10), (4,11), (4,12), (5,6), (5,7), (6,6)\}.$$

From the pair $(u, v) = (a_1, b_1)$ we get θ-graphs in Figure 13.

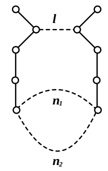

Figure 13.

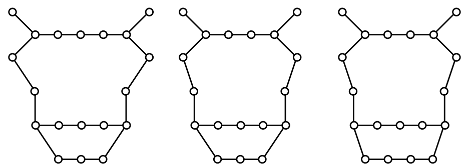

Figure 14.

For $\ell > 4$ corresponding θ -graphs are not reflexive, $\lambda_2 > 2$ ($\lambda_3 = 2, \ldots$). For $\ell = 4$ and $\ell = 3$ the corresponding maximal reflexive θ-graphs are shown in Figure 14.

For $\ell = 2$ the corresponding maximal reflexive θ-graphs are shown in Figure 15.

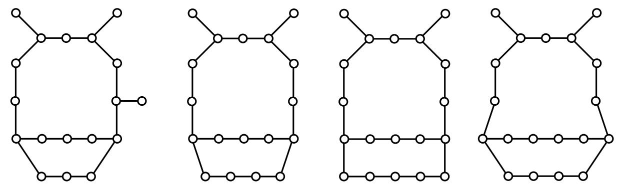

Figure 15.

For $\ell = 1$ the corresponding maximal reflexive θ-graphs are shown in Figure 16.

$$(n_1, n_2) \in \{(4,6), (4,7), (4,8), (5,5), (5,6)\}.$$

Based on previously shown results we have proven the following theorem.

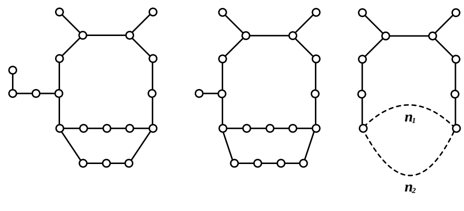

Figure 16.

Theorem 1. *Consider the graph with the cyclic structure of graph G in Figure* 4. *Then, graph is maximal reflexive θ-graph if and only if it is one of the* 72 *graphs in Figures* $6-10$, 11(b), 12 *and* $14-16$.

6. CONCLUSION

This is only one of many cases where the presence of SMITH graphs is noticed when investigating reflexive graphs. Currently we are working on determining all maximal reflexive θ-graphs for various values of parameters k, l, m (Figure 3). Various forms of presence of SMITH graphs are noticed in most of the resulting maximal reflexive graphs and this is one of the areas for us to focus on in the future work.

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