Applicable Analysis and Discrete Mathematics: **1** (2007), 284–292.
Available electronically at **http://pefmath.etf.bg.ac.yu**
Presented at the conference: *Topics in Mathematical Analysis and Graph Theory*, Belgrade, September 1–4, 2006.

# NEW CONCEPTS AND RESULTS ON THE AVERAGE DEGREE OF A GRAPH

*Maximiliano Pinto Damas, Lilian Markenzon*
*Nair Maria Maia de Abreu*

The idea of equilibrium of a graph $G$, initially applied to maximal outerplanar graphs (mops), was conceived to observe how the vertex degree distribution affects the average degree of the graph, $d(G)$. In this work, we formally extend the concept to graphs in general. From $d(G)$, two new parameters are introduced - the *top* and the *gap* of $G$, sustaining the definitions of tuner set, balanced and non-balanced graphs. We show properties of the new concepts when applied to particular families of graphs as trees and unicyclic graphs. We also establish bounds to the top of non-balanced graphs with integer average degree and we characterize their tuner sets.

## 1. INTRODUCTION

The idea of equilibrium of a graph $G$, initially applied to maximal outerplanar graphs (mops) in Rodrigues, Abreu and Markenzon [**3**], was conceived to observe how the vertex degree distribution affects the average degree of the graph, $d(G)$. In this work, we extend the concept to graphs in general.

We introduce two new parameters of a graph $G$ – the *top* and the *gap* – both given as a function of the number of vertices and the average degree of $G$. We formalize *balanced* and *non-balanced* graphs and we show how particular families behave when the new concepts are applied to them.

The definition of *tuner sets*, a subset of vertices of $G$ with degree lower than $d(G)$ that are able to compensate the presence of vertices with degree greater than $d(G)$ is presented. We show that a graph can have distinct tuner sets. An important

2000 Mathematics Subject Classification. 05C75, 05A18.
Key Words and Phrases. Average degree, tuner set, trees, unicyclic graphs.
Supported by CNPq-Brazil grants 140139/2003-6, 301068/2003-8 and 300563/1994-9.

result is the characterization of tuner sets for graphs that have the average degree as an integer. For these graphs, the following property holds: *the tuner set is unique and it is composed of all vertices with degree lower than d(G).*

Throughout this paper, $G$ is a simple and connected graph, $V$ its vertex set and $E$ its edge set; $|V| = n$ and $|E| = m$. In Section 2 basic concepts are reviewed and we define the *top*, $\mu(G)$, and the *gap*, $h(G)$, of $G$. In Section 3, properties concerning the new parameters are shown, particularly of trees and unicyclic graphs. We determine bounds to $\mu(G)$ and, for a given feasible value of $\mu(G)$, a graph $G$ is built. In Section 4, it is proved a characterization of graphs which have the average degree as an integer number.

## 2. BASIC NOTIONS

In this section, basic concepts already known are reviewed (see, for instance, Diestel [**1**] and Gross and Yellen [**2**]) and new definitions are presented. Two new parameters are defined, both related to the average degree of the graph: $\mu(G)$, the *top*, and $h(G)$, the *gap* of a graph $G$.

Let $d(v_i)$ be the *degree* of vertex $v_i$ and $\Delta(G) = \max\limits_{1\le i\le n} d(v_i)$ be the *maximal degree* of $G$. The *average degree of* $G$ is

$$d(G) = \frac{\sum\limits_{1\le i\le n} d(v_i)}{n}. \tag{1}$$

So, $0 \le d(G) \le n-1$, and it is not necessarily an integer. We define the *top* of graph $G$, $\mu(G)$, as $\lceil d(G) \rceil$.

For all $1 \le i \le n$, $0 \le d(v_i) \le n-1$, there is $j$, $0 \le j \le n-1$, such that $d(v_i) = j$. Let $Y \subseteq V(G)$. The *frequency of the degree* $v_i$ is the cardinality of the set $\{v \in Y : d(v) = j\}$, denoted $\omega_Y(j)$. When $Y = V(G)$, we simply use $\omega(j)$.

As the sum of the degrees of all vertices is twice the number of edges in $G$, we have

$$\sum_{i=1,\ldots,n} i\,\omega(i) = 2m. \tag{2}$$

The *gap of* $G$ is $h(G) = n(\mu(G) - d(G))$. If the gap of $G$ is zero, then $d(G) = \mu(G)$ and we say that $G$ is a *graph with zero gap.* Consequently, if $d(G) \in \mathbb{N}$, $h(G) = 0$.

For instance, if $G$ is a $k$-regular graph, $d(G) = k$ and $G$ is a graph with zero gap. However, non-regular graphs can also have this property. Figure 1 displays a non-regular graph with zero gap.

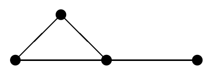

Figure 1. A non-regular graph $G$ with $d(G) = \mu(G) = 2$.

From (1) and (2), we get

$$h(G) = \mu(G)\, n - \sum_{i=1,\ldots,n} i\, \omega(i). \tag{3}$$

According to (3), $h(G) \in \mathbb{N}$.

Figure 2 displays a graph $G$ with $d(G) \approx 2.571$. Then, $\mu(G) = 3$ and, from (3), we have $h(G) = 3$.

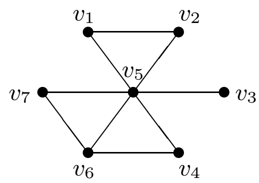


Figure 2. A graph $G$ with $d(G) \approx 2.571$ and $\mu(G) = h(G) = 3$

We call $B_G = \{v \in V(G) : d(v) = \mu(G)\}$ the *balanced vertex set*; $U_G = \{v \in V(G) : d(v) > \mu(G)\}$, the *upper vertex set* and $L_G = \{v \in V(G) : d(v) < \mu(G)\}$ the *lower vertex set* of $G$. Consequently, $|B_G| + |U_G| + |L_G| = n$ and if the subsets are different from the empty set, they determine a partition of $V(G)$. We simply use $B$, $U$ and $L$, when it is not necessary to specify which graph we refer to. If $U = \emptyset$, $G$ is said to be a *balanced graph.* If not, $G$ is a *non-balanced graph.* For example, the graph $G$ in Figure 2 is a non-balanced graph because $U = \{v_5\}$. It also has $B = \{v_6\}$ and $L = \{v_1, v_2, v_3, v_4, v_7\}$ as balanced and lower vertex sets, respectively.

## 3. PROPERTIES CONCERNING THE NEW PARAMETERS

In this section we present some properties of graphs concerning the new concepts and parameters presented in Section 2. Their proofs come straightforward from the definitions.

**Basic Properties**:

(1) For every graph with zero gap, $m = \dfrac{d(G)}{2} n = \dfrac{\mu(G)}{2} n$. Since $d(G) = \mu(G)$, if $n$ is odd, $\mu(G)$ must be even.

(2) Every regular graph is balanced and it is a graph with zero gap. Of course $U = L = \emptyset$.

(3) There are balanced and non-regular graphs for which $L \neq \emptyset$. As an example, the graph $G = C_n + \{e\}$ has $\mu(G) = 3$ and $L$ is constituted by all vertices of degree 2, being $C_n$ a cycle with length $n$.

(4) Let $G$ be a connected graph with $\mu(G) \leq n-2$. If $G$ has at least one universal vertex $v$, $d(v) = n - 1$, then $G$ is a non-balanced graph. This property gives enough condition for the existence of non-balanced graphs with zero gap. A necessary condition is given by Proposition (5).

(5) Every graph $G$ such that $\mu(G) = n - 1$ is a balanced graph.

The new concepts turn out to be particularly interesting when dealing with well known families such as trees and unicyclic graphs. The following three simple results are stated here without proof.

**Proposition 1.** *For every tree $T \neq K_2$, $T$ is not a graph with zero gap. Moreover, $\mu(T) = h(T) = 2$.*

**Proposition 2.** *For $n > 2$, $P_n$ is a non-regular balanced graph.*

**Proposition 3.** *Let $T$ be a tree with $n > 2$. $T$ is a non-balanced graph if and only if $T \neq P_n$.*

Let $G$ be a unicyclic graph. As $d(G) = 2$, $G$ is a graph with zero gap. Proposition 4 gives a more general property for members of this family.

**Proposition 4.** *Let $G$ be an unicyclic graph with $n$ vertices. Then, $G$ is a non-balanced graph with zero gap if and only if $G \neq C_n$.*

**Proof.** Suppose that $G$ is an unicyclic non-balanced graph with $n$ vertices. So, $m = n$, $d(G) = 2$ and $G$ is a graph with zero gap. Since $G$ is a non-balanced graph, $U \neq \emptyset$. Then, there is a vertex $u$ in $G$ such that $d(u) > 2$. Consequently, $G \neq C_n$. The reciprocal proof is equivalent. □

**Proposition 5.** *Let $G$ with $n$ vertices be a connected non-balanced graph. Then, $2 \leq \mu(G) \leq n - 2$. Moreover, the upper bound is achieved if and only if $n$ is even.*

**Proof.** From Propositions 1 and 3, if $G$ is a tree $T \neq P_n$, $T$ satisfies all the hypotheses above and $\mu(T) = 2$.

Let $G$ be a graph with $\mu(G) = n - 1$. From Basic Property (5), $U = \emptyset$ and $G$ is a balanced graph. Consequently, if $G$ is a non-balanced graph, $\mu(G) \leq n - 2$. If $n$ is odd, $\mu(G) = n - 2$ is odd too. This disagrees with Basic Property (1). □

We are particularly interested in the behavior of the top of graphs with zero gap. The next result characterizes non-balanced graphs for which the top attains the upper bound given by Proposition 5.

**Proposition 6.** *Let $n$ be even. $G$ with $n$ vertices is a connected non-balanced graph with zero gap and $\mu(G) = n - 2$ if and only if $G$ has $\dfrac{n(n-2)}{2}$ edges and it has at least one universal vertex.*

**Proof.** Let $n$ be even and $G$ be a graph with $n$ vertices. $G$ is a graph with zero gap and $\mu(G) = n - 2$ if and only if $d(G) = \mu(G) = n - 2$. Besides, $d(G) = \dfrac{2m}{n}$. So, $m = \dfrac{n(n-2)}{2}$.

The following sentences are equivalent:

1. $G$ is a non-balanced graph $\Leftrightarrow$
2. $U_G \neq \emptyset \Leftrightarrow$
3. There is a vertex $u$ such that $d(u) > n - 2 \Leftrightarrow$
4. $d(u) = n - 1$.

Therefore $u$ is an universal vertex. $\square$

Now, the following question can be raised: For every $q \in \mathbb{N}$, $2 \leq q \leq n-2$, is there a non-balanced graph with zero gap $G$ such that $\mu(G) = q$? For $n$ even, the answer is affirmative and the proof is presented in the next theorem. For $n$ odd, the subject will be handled later.

**Theorem 1.** *Let $n > 2$ be even. For every $q \in \mathbb{N}$, $2 \leq q \leq n-2$, there is a connected non-balanced graph with zero gap $G$ of order $n$ such that $\mu(G) = q$.*

**Proof.** The lower bound $\mu(G) = 2$ is achieved by unicyclic graphs, excluding $C_n$. The upper bound is proved in Proposition 6.

Let $n > 4$ and $q \in \mathbb{N}$, $2 < q < n-2$. Let us build a non-balanced graph $G$ with zero gap and $\mu(G) = q$. Let $T$ be a tree with $n$ vertices. In order to obtain the desired graph, we must add $t$ new edges to $T$ such that

1. $T$ is a spanning tree of $G$ ($V(T) = V(G)$);
2. $\mu(G) = d(G) = q$;
3. there is $v \in V(G)$ such that $d(v) > \mu(G)$.

We know that, for a graph with zero gap, $\mu(G) = d(G) = \dfrac{2m}{n}$. So, $q = \dfrac{2m}{n}$ and $m = \dfrac{nq}{2}$. As $t = m - (n-1)$, being $m$ the number of edges of $G$, we have

$$t = \left(\frac{q}{2} - 1\right) n + 1.$$

As $n$ is even and $q > 2$, $t \in \mathbb{N}$ and $t > 1$. $G$ will have $n - 1 + t$ edges; by construction it is a graph with zero gap.

If $\Delta(T) > q$, the $t$ edges can be inserted randomly, and $G$, the resultant graph, is non-balanced. If $\Delta(T) \leq q$, we need to make sure that at least one vertex $v$ of $G$ attains $d(v) > q$. Let $v$ be a vertex of $T$ with $d(v) = \Delta(T)$. Adding $q - \Delta(T) + 1$ edges $\{v, w\}$ such that $w \in V(T), w \neq v$ and $\{v, w\} \notin E(T)$ the degree of $v$ becomes exactly $q + 1$; this ensures that $G$ is a non-balanced graph. The remaining $t - q + \Delta(T) - 1$ edges can be inserted randomly. $\square$

**Corollary 1.** *Let $G$ and $G'$ be non-balanced graphs with zero gap of order $n$ with $m$ and $m'$ edges, respectively. If $n$ is even and $\mu(G') = \mu(G) + 1$ then $m' = m + \dfrac{n}{2}$.*

**Proof.** Let $n$ be even. From Theorem 1, in order to obtain $G$ and $G'$ such that $\mu(G) = q$ and $\mu(G') = q + 1$, we must add $t = \left(\frac{q}{2} - 1\right) n + 1$ and $t' = \left(\frac{q+1}{2} - 1\right) n + 1$, respectively, to a tree $T$. So, $m' - m = t' - t = \dfrac{n}{2}$ and $m' = m + \dfrac{n}{2}$. $\square$

Theorem 1 and Corollary 1 establish the theoretical foundation for the systematic generation of graphs with zero gap. The proof of Theorem 1 even provides the outline of an efficient algorithm to perform this task.

Actually, for a fixed even $n$, the algorithm presented below produces a whole sequence of graphs $G_2, G_3, \dots, G_{n-2}$, being $\mu(G_i) = i$ and such that $G_{i-1} \subset G_i, 3 \leq i \leq n-2$.

**Algorithm** *GEN.*

**Input:** $n$, even.

**Initial Step:** Generate a random tree $T \neq P_n$; compute $T + \{e\}$, obtaining $G_2$, an unicyclic graph. $G_2$ is a graph with zero gap and $\mu(G_2) = 2$.

The following steps are repeated $n-4$ times, generating $G_3, G_4, \dots G_{n-2}$.

**Step 1:** Let $G_i$ be the last graph generated (for the first time, $i = 2$). Find a vertex $v$ such that $d(v)$ is maximum in $G_i$. If $d(v) \leq i+1$ then add $m' = i + 2 - d(v)$ edges adjacent to vertex $v$; else $m'$ is equal to zero.

**Step2:** Add $(n/2 - m')$ edges to $G_i$ obtaining $G_{i+1}$.

For $n$ even, Algorithm *GEN* generates $n-3$ graphs. From Basic Property (1), when $n$ is odd, $\mu(G)$ must be even. So, for $n$ odd, Theorem 1 holds if and only if $q = 2k, k \in \mathbb{N}$. The proof for this affirmative is similar to the one of Theorem 1. Algorithm *GEN* can also be slightly modified to obey the convenient conditions. Therefore, it will be possible to obtain a sequence of $\frac{n-3}{2}$ graphs $G_2, G_4, \dots, G_{n-3}$ with $n$ vertices, zero gap and $\mu(G_i) = i$.

## 4. TUNER SETS FOR GRAPHS

In this section, we introduce the concept of *tuner set* for a graph $G$, a subset of vertices of $G$ with degree lower than $d(G)$ that are able to compensate the presence of vertices with degree greater than $d(G)$. We show that a graph can have distinct tuner sets. Finally, we characterize tuner sets for graphs with zero gap (see Theorem 2).

Let $L$ be the lower vertex set and $U$ be the upper vertex set of $G$ such that $U \neq \emptyset$. If there is $\Psi \subseteq L$ for which the following equality holds:

$$\mu(G) = \frac{\sum_{t \in \Psi} d(t) + \sum_{u \in U} d(u)}{|\Psi| + |U|}, \tag{4}$$

we say that $G$ has a *tuner set* $\Psi$ *determined by* $U$ or that $U$ *determines a tuner set* $\Psi$ *in* $G$. If $|\Psi| = k$, $k \leq |L|$. When $|L| = k$, $\Psi = L$ is unique and it is called the *full tuner set* of $G$ *determined by* $U$. When $k < |L|$, $\Psi$ is strictly contained in $L$ and $\Psi$ is called a *proper tuner set* of $G$. In general, proper tuner sets are not unique of $G$. Moreover, $G$ can have more than one proper tuner set of the same order $k$ or $G$

cannot have any tuner sets. For instance, there are not any tuner sets for regular graphs.

As an example, the graph $G$ showed in Figure 2 (Section 2) has $\mu(G) = 3$, $U = \{v_5\}$, $B = \{v_6\}$ and $L = \{v_1, v_2, v_3, v_4, v_7\}$. From (4), $\Psi_1 = \{v_1, v_3\}$ is a tuner set of $G$ determined by $U$. Observe that $\Psi_2 = \{v_6\}$ is another tuner set of $G$ determined by $U$. However, there are subsets of $L$ which are not tuner sets. In this example, $\{v_1, v_3, v_7\}$ is a subset of $L$ but it is not a tuner set of $G$. Figure 3 displays a graph which does not have any tuner set. If we enumerate all subsets of $L$ we can verify that none of them satisfies the equality (4).

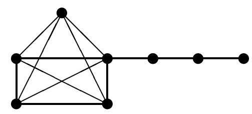

Figure 3. $G$ does not have any tuner set.

The next theorem characterizes all graphs with zero gap as a function of their respective tuner sets. Observe that a graph with zero gap has an unique tuner set.

**Theorem.** *Let $G$ be a graph and $L \neq \emptyset$ its lower vertex set. $G$ has the full tuner set $\Psi = L$ if and only if $h(G) = 0$.*

**Proof.** ($\Leftarrow$) Let $h(G) = 0$. From (3) we have

$$\mu(G) = \frac{\sum\limits_{i=1,\ldots,n} i\,\omega(i)}{n}.$$

First of all, it is necessary to know if $G$ has a tuner set $\Psi$. For this, we need to find a subset of $L$ satisfying the equality (4). From the definitions of the subsets $U$, $B$ and $L$ which determine a partition of $V(G)$, we obtain

$$\sum_{i=1,\ldots,n} i\,\omega(i) = \sum_{u\in U} d(u) + \sum_{b\in B} d(b) + \sum_{\ell\in L} d(\ell).$$

If $B = \emptyset$, from (4), $\Psi = L$. Suppose $B \neq \emptyset$. As $B$ is the balanced set, the equality below holds.

$$\mu(G) = \frac{\sum\limits_{b\in B} d(b)}{|B|}. \tag{5}$$

From (5), we get

$$\frac{\sum\limits_{b\in B} d(b)}{|B|} = \frac{\sum\limits_{u\in U} d(u) + \sum\limits_{b\in B} d(b) + \sum\limits_{\ell\in L} d(\ell)}{|U| + |B| + |L|}. \tag{6}$$

Since $h(G) = 0$, from (5) and using some algebraic manipulation on (6), we find

$$\mu(G) = \frac{\sum\limits_{u\in U} d(u) + \sum\limits_{\ell\in L} d(\ell)}{|U| + |L|}. \tag{7}$$

The equality (7) shows that $\Psi = L$. So, the full tuner set is the only subset of $L$ that satisfies (4).

($\Rightarrow$) Let us suppose that $G$ has the full tuner set $\Psi = L$. From (4),

$$\mu(G) = \frac{\sum_{u\in U} d(u) + \sum_{\ell\in L} d(\ell)}{|U| + |L|}. \tag{8}$$

If $B$ is an empty set, $|B| = 0$ and from (8), $\mu(G) = d(G)$. So, $h(G) = 0$.

Now, suppose $B \neq \emptyset$. Then, $|B| \neq 0$. Let us consider $h(G) \neq 0$. So, $\mu(G) - d(G) > 0$. As $B$ is the balanced set, $\mu(G) = \frac{\sum_{b\in B} d(b)}{|B|}$ and

$$\frac{\sum_{b\in B} d(b)}{|B|} - d(G) > 0. \tag{9}$$

We know that $\sum_{u\in U} d(u) + \sum_{b\in B} d(b) + \sum_{\ell\in L} d(\ell) = 2m$. From (9), we have

$$\frac{\sum_{b\in B} d(b)}{|B|} - \frac{\sum_{u\in U} d(u) + \sum_{b\in B} d(b) + \sum_{\ell\in L} d(\ell)}{n} > 0. \tag{10}$$

From (10), we have

$$\frac{\sum_{b\in B} d(b)}{|B|} - \frac{\sum_{u\in U} d(u) + \sum_{\ell\in L} d(\ell)}{n} - \frac{\sum_{b\in B} d(b)}{n} > 0.$$

After some algebraic manipulations we get

$$\frac{(n - |B|) \sum_{b\in B} d(b)}{|B|} > \sum_{u\in U} d(u) + \sum_{\ell\in L} d(\ell)$$

and so,

$$\frac{\sum_{b\in B} d(b)}{|B|} > \frac{\sum_{u\in U} d(u) + \sum_{\ell\in L} d(\ell)}{n - |B|}. \tag{11}$$

Applying (5) to inequality (11) we have

$$\mu(G) > \frac{\sum_{u\in U} d(u) + \sum_{\ell\in L} d(\ell)}{n - |B|}.$$

But, $n - |B| = |U| + |L|$. So,

$$\mu(G) > \frac{\sum_{u\in U} d(u) + \sum_{\ell\in L} d(\ell)}{|U| + |L|}. \tag{12}$$

Since the inequality (12) is incompatible with the equality (7), our admission $h(G) \neq 0$ does not hold. Consequently, for every graph $G$ such that $\Psi = L$, $G$ is a graph with zero gap. □

**Corollary 2.** *Let $G$ be a graph with zero gap. If $G$ is a non-balanced graph then the lower vertex set $L \neq \emptyset$.*

**Proof.** Let $G$ be a graph with zero gap. Suppose that $G$ is a non-balanced graph. So, $U \neq \emptyset$ and from Theorem 2, $\Psi = L$. Besides, from (4) we have

$$\mu(G) = \frac{\sum\limits_{u \in U} d(u) + \sum\limits_{t \in \Psi} d(t)}{|U| + |\Psi|}.$$

If $L = \emptyset$ and as $\Psi \subseteq L$, the equality above becomes $\mu(G) = \frac{\sum_{u \in U} d(u)}{|U|}$. But it is impossible, since each vertex $u \in U$ has $d(u) > \mu(u)$. Then, $L \neq \emptyset$. □

Finally, it is interesting to observe that Algorithm *GEN* presented in Section 3 can also be applied to generate graphs with full tuner sets.

## REFERENCES

1. R. Diestel: *Graph Theory.* Graduate Texts in Mathematics. **GMT 173**, Springer 1997.
2. J. L. Gross, J. Yellen: *Handbook of Graph Theory.* Series Discrete Mathematics and its Applications, CRC press, 2004.
3. R. M. N. D. Rodrigues, N. M. M. Abreu, L. Markenzon: *Maxregularity and Maximal Outerplanar Graphs.* Electronic Notes in Discrete Mathematics, **3** (1999), 171–175.

(Received October 28, 2006)

Maximiliano Pinto Damas
Programa de Engenharia de Produção - COPPE,
Universidade Federal do Rio de Janeiro,
P. O. Box: 68507, RJ, Brasil, 21.945-970
E–mail: maxdamas@hotmail.com

Lilian Markenzon
Núcleo de Computação Eletrônica,
Universidade Federal do Rio de Janeiro,
P. O. Box: 2324, RJ, Brasil, 20.010-974
E–mail: markenzon@nce.ufrj.br

Nair Maria Maia de Abreu
Programa de Engenharia de Produção - COPPE,
Universidade Federal do Rio de Janeiro,
P. O. Box: 68507, RJ, Brasil, 21.945-970
E–mail: nair@pep.ufrj.br