APPLICABLE ANALYSIS AND DISCRETE MATHEMATICS, **1** (2007), 92–110.
Available electronically at **http://pefmath.etf.bg.ac.yu**

Presented at the conference: *Topics in Mathematical Analysis and Graph Theory*,
Belgrade, September 1–4, 2006.

INEQUALITIES FOR NORMAL OPERATORS IN HILBERT SPACES

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Some inequalities for normal operators in HILBERT spaces are established. For this purpose, classical and new vector inequalities due to BUZANO, DUNKL-WILLIAMS, HILE, GOLDSTEIN-RYFF-CLARKE, DRAGOMIR-SÁNDOR and the author are employed.

1. INTRODUCTION

Let $(H; \langle \cdot, \cdot \rangle)$ be a complex HILBERT space and $T : H \to H$ a bounded linear operator on H. Recall that T is a *normal operator* if $T^*T = TT^*$. Normal operators may be regarded as a generalisation of self-adjoint operator T in which T^* need not be exactly T but commutes with T [**11**, p. 15].

The *numerical range* of an operator T is the subset of the complex numbers $\mathbb{C}$ given by [**11**, p. 1]:

$$W(T) = \{\langle Tx, x \rangle, \ x \in H, \ \|x\| = 1\}.$$

For various properties of the numerical range see [**11**].

We recall here some of the ones related to normal operators.

Theorem 1. *If $W(T)$ is a line segment, then T is normal.*

We denote by $r(T)$ the operator *spectral radius* [**11**, p. 10] and by $w(T)$ its *numerical radius* [**11**, p. 8]. The following result may be stated as well [**11**, p. 15].

Theorem 2. *If T is normal, then $\|T^n\| = \|T\|^n$, $n = 1, \ldots$. Moreover, we have*:

$$r(T) = w(T) = \|T\|. \tag{1.1}$$

2000 Mathematics Subject Classification. 47A12.
Key Words and Phrases. Bounded linear operators, normal operators, Hilbert spaces, Schwarz inequality, reverse inequalities.

An important fact about the normal operators that will be used frequently in the sequel is the following one [**12**, p. 42]:

Theorem 3. *A necessary and sufficient condition that an operator T be normal is that $\|Tx\| = \|T^*x\|$ for every vector $x \in H$.*

We observe that, if one uses the SCHWARZ inequality

$$|\langle u, v\rangle| \le \|u\| \|v\|, u, v \in H,$$

for the choices $u = Tx, v = T^*x$ with $x \in H$, then that one gets the following simple inequality for the normal operator T :

$$\|Tx\|^2 \ge \left|\left\langle T^2x, x\right\rangle\right|, \ x \in H. \tag{1.2}$$

It is then natural to look for upper bounds for the quantity $\|Tx\|^2 - \left|\left\langle T^2x, x\right\rangle\right|$, $x \in H$ under various assumptions for the normal operator T, which would give a measure of the closeness of the terms involved in the inequality (1.2).

Motivated by this problem, the aim of the paper is to establish some reverse inequalities for (1.2). Norm inequalities for various expressions with normal operators and their adjoints are also provided. For both purposes, some inequalities for vectors in inner product spaces due to BUZANO, DUNKL-WILLIAMS, HILE, GOLDSTEIN-RYFF-CLARKE, DRAGOMIR-SÁNDOR and the author, are employed.

2. INEQUALITIES FOR VECTORS

The following result may be stated.

Theorem 4. *Let $(H; \langle\cdot,\cdot\rangle)$ be a Hilbert space and $T : H \to H$ a normal linear operator on H. Then*

$$\left(\|Tx\|^2\right) \ge \frac{1}{2}\left(\|Tx\|^2 + \left|\left\langle T^2x, x\right\rangle\right|\right) \ge |\langle Tx, x\rangle|^2, \tag{2.1}$$

for any $x \in H$, $\|x\| = 1$. The constant $\frac{1}{2}$ is best possible in (2.1).

Proof. The first inequality is obvious.

For the second inequality, we need the following refinement of SCHWARZ's inequality obtained by the author in 1985 [**2**, Theorem 2] (see also [**8**] and [**4**]):

$$\|a\| \|b\| \ge |\langle a, b\rangle - \langle a, e\rangle \langle e, b\rangle| + |\langle a, e\rangle \langle e, b\rangle| \ge |\langle a, b\rangle|, \tag{2.2}$$

provided a, b, e are vectors in H and $\|e\| = 1$.

Observing that

$$|\langle a, b\rangle - \langle a, e\rangle \langle e, b\rangle| \ge |\langle a, e\rangle \langle e, b\rangle| - |\langle a, b\rangle|,$$

then by the first inequality in (2.2) we deduce

$$\frac{1}{2}\left(\|a\|\|b\|+|\langle a,b\rangle|\right)\geq|\langle a,e\rangle\langle e,b\rangle|. \tag{2.3}$$

This inequality was obtained in a different way earlier by M. L. Buzano in [**1**].

Now, choose in (2.3), $e=x$, $\|x\|=1$, $a=Tx$ and $b=T^{*}x$ to get

$$\frac{1}{2}\left(\|Tx\|\|T^{*}x\|+\left|\left\langle T^{2}x,x\right\rangle\right|\right)\geq|\langle Tx,x\rangle|^{2} \tag{2.4}$$

for any $x\in H$, $\|x\|=1$. Since T is normal, then $\|Tx\|=\|T^{*}x\|$, and by (2.4) we deduce the desire result (2.1).

The fact that, the constant $\frac{1}{2}$ is best possible in (2.1) is obvious since for $T=I$, the identity operator, we get equality in (2.1). □

From a different perspective, we can state the following result:

Theorem 5. *Let $T:H\to H$ be a normal operator on the Hilbert space $(H;\langle\cdot,\cdot\rangle)$. If $\lambda\in\mathbb{C}$, then*

$$(0\leq)\|Tx\|^{2}-\left|\left\langle T^{2}x,x\right\rangle\right|\leq\frac{2}{(1+|\lambda|)^{2}}\|Tx-\lambda T^{*}x\|^{2} \tag{2.5}$$

for any $x\in H$, $\|x\|=1$.

Proof. We use the following inequality [**9**]:

$$\|a-b\|\geq\frac{1}{2}\left(\|a\|+\|b\|\right)\left\|\frac{a}{\|a\|}-\frac{b}{\|b\|}\right\|,\quad a,b\in H\setminus\{0\},$$

which is well known in the literature as the *Dunkl-Williams inequality*.

This inequality, by taking the square, is clearly equivalent to

$$\frac{4\|a-b\|^{2}}{(\|a\|+\|b\|)^{2}}\geq\left\|\frac{a}{\|a\|}-\frac{b}{\|b\|}\right\|^{2}=2-2\cdot\frac{\operatorname{Re}\langle a,b\rangle}{\|a\|\|b\|},$$

which shows that (see [**3**, Eq. (2.5)])

$$\frac{\|a\|\|b\|-|\langle a,b\rangle|}{\|a\|\|b\|}\leq\frac{2\|a-b\|^{2}}{(\|a\|+\|b\|)^{2}}.$$

Now, for $x\in H\setminus\ker(T)$, $\|x\|=1$, choose $a=Tx$ and $b=\lambda T^{*}x$ $(\lambda\neq0)$ to obtain

$$\|Tx\|\|T^{*}x\|-\left|\left\langle T^{2}x,x\right\rangle\right|\leq\frac{2\|Tx\|\|T^{*}x\|}{(\|Tx\|+|\lambda|\|T^{*}x\|)^{2}}\|Tx-\lambda T^{*}x\|^{2}. \tag{2.6}$$

Since $\|Tx\|=\|T^{*}x\|$, T being a normal operator, we get from (2.6) that (2.5) holds true for any $x\in H\setminus\ker(T)$, $\|x\|=1$.

For $\lambda = 0$ the inequality (2.5) is obvious.

Since for normal operators $\ker(T) = \ker(T^*)$ then for $x \in \ker(T)$, $\|x\| = 1$ the inequality (2.5) also holds true. □

The following result which provides a different upper bound for the nonnegative quantity

$$\|Tx\|^2 - \left|\left\langle T^2x, x\right\rangle\right|, x \in H, \|x\| = 1$$

may be stated as well:

Theorem 6. *Let $T : H \to H$ be a normal operator on the Hilbert space H and $\alpha, \lambda \in \mathbb{C}\setminus\{0\}$. Then*

$$\begin{aligned} (2.7) \qquad & (0 \le)\, \|Tx\|^2 - \left|\left\langle T^2x, x\right\rangle\right| \\ & \le \frac{1}{2} \cdot \frac{\left[|\mathrm{Re}\,\alpha| \left\|Tx - \frac{\alpha}{\bar{\alpha}}\lambda T^*x\right\| + |\mathrm{Im}\,\alpha| \left\|Tx + \frac{\alpha}{\bar{\alpha}}\lambda T^*x\right\|\right]^2}{|\lambda|\,|\alpha|^2} \end{aligned}$$

for any $x \in H$, $\|x\| = 1$.

Proof. We use the following inequality (see [**3**, Theorem 2.11]):

$$(2.8) \qquad \|a\|\,\|b\| - \mathrm{Re}\left[\frac{\alpha^2}{|\alpha|^2}\langle a, b\rangle\right] \le \frac{1}{2} \cdot \frac{\left[|\mathrm{Re}\,\alpha|\,\|a - b\| + |\mathrm{Im}\,\alpha|\,\|a + b\|\right]^2}{|\alpha|^2}$$

for the choices:

$$a = \frac{Tx}{\alpha}, \quad b = \frac{\lambda}{\bar{\alpha}}T^*x, \quad x \in H$$

to obtain:

$$\begin{aligned} (2.9) \qquad & \frac{|\lambda|\,\|Tx\|\,\|T^*x\|}{|\alpha|^2} - \mathrm{Re}\left[\frac{\alpha^2}{|\alpha|^2} \cdot \frac{\bar{\lambda}}{\alpha^2}\langle Tx, T^*x\rangle\right] \\ & \le \frac{1}{2} \cdot \frac{\left[|\mathrm{Re}\,\alpha| \left\|\frac{Tx}{\alpha} - \frac{\lambda}{\alpha}T^*x\right\| + |\mathrm{Im}\,\alpha| \left\|\frac{Tx}{\alpha} + \frac{\lambda}{\alpha}T^*x\right\|\right]^2}{|\alpha|^2}. \end{aligned}$$

Since T is normal, we get from (2.9) the desired result (2.7). The details are omitted. □

Another result of this type is incorporated in:

Theorem 7. *Let $T : H \to H$ be a normal operator on the Hilbert space H, $s \in [0, 1]$ and $t \in \mathbb{R}$. Then*

$$(2.10) \quad (0 \le)\, \|Tx\|^4 - \left|\left\langle T^2x, x\right\rangle\right|^2 \le \|Tx\|^2 \left[s\,\|tT^*x - Tx\|^2 + (1 - s)\,\|T^*x - tTx\|^2\right].$$

In particular

$$(0 \le)\, \|Tx\|^4 - \left|\left\langle T^2x, x\right\rangle\right|^2 \le \frac{1}{2}\,\|Tx\|^2 \inf_{t\in\mathbb{R}} \left[\|tT^*x - Tx\|^2 + \|T^*x - tTx\|^2\right].$$

Proof. We use the inequality obtained in [**4**, Theorem 2], to state that

$$\left[(1-s)\|a\|^2+s\|b\|^2\right]\left[(1-s)\|b\|^2+s\|a\|^2\right]-|\langle a,b\rangle|^2 \leq \left[(1-s)\|a\|^2+s\|b\|^2\right]\left[(1-s)\|b-ta\|^2+s\|tb-a\|^2\right] \tag{2.11}$$

for any $s\in[0,1]$, $t\in\mathbb{R}$ and $a,b\in H$.

If in (2.11) we choose $a=Tx$, $b=T^*x$, $x\in H$ and $\|x\|=1$, then we get

$$\|Tx\|^4-\left|\left\langle T^2x,x\right\rangle\right|^2 \leq \|Tx\|^2\left[s\|tT^*x-Tx\|^2+(1-s)\|T^*x-tTx\|^2\right]$$

for any $s\in[0,1]$, $t\in\mathbb{R}$, from where we deduce the desired inequality (2.10). □

From a different perspective, we can state the following result as well.

Theorem 8. *Let $T:H\to H$ be a normal operator on the Hilbert space $(H;\langle\cdot,\cdot\rangle)$. If $\lambda\in\mathbb{C}\setminus\{0\}$ and $r>0$ are such that*

$$\|T-\lambda T^*\|\leq r, \tag{2.12}$$

then:

$$(0\leq)\,\|Tx\|^4-\left|\left\langle T^2x,x\right\rangle\right|^2\leq\frac{r^2}{|\lambda|^2}\|Tx\|^2 \tag{2.13}$$

for any $x\in H$, $\|x\|=1$.

Proof. We use the following reverse of the quadratic SCHWARZ inequality obtained by the author in [**4**]

$$(0\leq)\,\|a\|^2\|b\|^2-|\langle a,b\rangle|^2\leq\frac{1}{|\alpha|^2}\|a\|^2\|a-\alpha b\|^2 \tag{2.14}$$

provided $a,b\in H$ and $\alpha\in\mathbb{C}\setminus\{0\}$.

Choosing in (2.14) $a=Tx$, $\alpha=\lambda$, $b=T^*x$, we get

$$\|Tx\|^4\leq\left|\left\langle T^2x,x\right\rangle\right|^2+\frac{1}{|\lambda|^2}\|Tx\|^2\|Tx-\lambda T^*x\|^2 \leq\left|\left\langle T^2x,x\right\rangle\right|^2+\frac{1}{|\lambda|^2}r^2\|Tx\|^2 \tag{2.15}$$

which is the desired result (2.13). □

Finally, on utilising the following result obtained in [**4**]:

Lemma 1. *Let $a,b\in H\setminus\{0\}$ and $\varepsilon\in(0,1/2]$. If*

$$(0\leq)\,1-\varepsilon-\sqrt{1-2\varepsilon}\leq\frac{\|a\|}{\|b\|}\leq1-\varepsilon+\sqrt{1-2\varepsilon}, \tag{2.16}$$

then

$$(0 \leq) \|a\| \|b\| - Re \langle a, b \rangle \leq \varepsilon \|a - b\|^2 . \tag{2.17}$$

We can state:

Theorem 9. *Let $T : H \to H$ be a normal operator on H. If $\lambda \in \mathbb{C}$ is such that*

$$(0 \leq) 1 - \varepsilon - \sqrt{1 - 2\varepsilon} \leq |\lambda| \leq 1 - \varepsilon + \sqrt{1 - 2\varepsilon}, \quad \varepsilon \in (0, 1/2] \tag{2.18}$$

then

$$(0 \leq) \|Tx\|^2 - \left|\left\langle T^2 x, x \right\rangle\right| \leq \frac{\varepsilon}{|\lambda|} \|Tx - \lambda T^* x\|^2 \tag{2.19}$$

for any $x \in H$, $\|x\| = 1$.

Proof. Utilising Lemma 1 for $a = \lambda T^* x$, $b = Tx$, $x \in H \setminus \ker(T)$, $\|x\| = 1$, we have

$$|\lambda| \|Tx\|^2 - |\lambda| \left|\left\langle T^2 x, x \right\rangle\right| \leq \varepsilon \|Tx - \lambda T^* x\|^2 . \tag{2.20}$$

For $x \in \ker(T)$, $\|x\| = 1$ the inequality (2.19) also holds, and the proof is completed. □

3. INEQUALITIES FOR OPERATOR NORM

The purpose of this section is to point out some norm inequalities for normal operators that can be naturally obtained from various vector inequalities in inner product spaces, such as the ones due to Hile, Goldstein-Ryff-Clarke, Dragomir-Sándor and the author.

Theorem 10. *Let $T : H \to H$ be a normal operator on the Hilbert space H. If $\lambda \in \mathbb{C}$, $|\lambda| \neq 1$, then:*

$$\left\| T - |\lambda|^{v+1} T^* \right\| \leq \frac{1 - |\lambda|^{v+1}}{1 - |\lambda|} \|T - \lambda T^*\| , \tag{3.1}$$

for any $v > 0$.

Proof. We use the following inequality:

$$\left\| \|a\|^v a - \|b\|^v b \right\| \leq \frac{\|a\|^{v+1} - \|b\|^{v+1}}{\|a\| - \|b\|} \|a - b\| \tag{3.2}$$

provided $v > 0$ and $\|a\| \neq \|b\|$, which is known in the literature as the *Hile inequality* [**13**].

Now, if we choose in (3.2) $a = Tx$, $b = \lambda T^* x$, since T is normal, we have $\|a\| = \|Tx\|$, $\|b\| = |\lambda| \|Tx\|$ and by (3.2) we get

$$\|Tx\|^v \left\| Tx - |\lambda|^{v+1} T^* x \right\| \leq \|Tx\|^v \frac{\left(1 - |\lambda|^{v+1}\right)}{1 - |\lambda|} \|Tx - \lambda T^* x\| \tag{3.3}$$

for any $x \in H \setminus \ker(T)$.

If $x \notin \ker(T)$, then from (3.3) we get

$$\left\| Tx - |\lambda|^{v+1} T^* x \right\| \leq \frac{1 - |\lambda|^{v+1}}{1 - |\lambda|} \|Tx - \lambda T^* x\|. \tag{3.4}$$

If $x \in \ker(T)$ and since $\ker(T) = \ker(T^*)$, T being normal, then the inequality (3.4) is also valid. Therefore, (3.4) holds for any $x \in H$.

Taking the supremum over $x \in H$, $\|x\| = 1$, we get the desired inequality (3.1). $\square$

Remark 1. For $v = 1$, we get the inequality:

$$\left\| T - |\lambda|^2 T^* \right\| \leq (1 + |\lambda|) \|T - \lambda T^*\|. \tag{3.5}$$

Utilising the second inequality due to Hile (see [**13**, Eq. (5.2)]):

$$\left\| \frac{a}{\|a\|^{v+2}} - \frac{b}{\|b\|^{v+2}} \right\| \leq \frac{\|a\|^{v+2} - \|b\|^{v+2}}{\|a\| - \|b\|} \cdot \frac{\|a - b\|}{\|a\|^{v+1} \cdot \|b\|^{v+1}}$$

for $a, b \in H$, $a, b \neq 0$ and $\|a\| \neq \|b\|$, and making use of an argument similar to the one in the proof of the above theorem, we can state the following result:

Theorem 11. *Let $T : H \to H$ be a normal operator on the Hilbert space H. If $\lambda \in \mathbb{C}$, $|\lambda| \neq 0, 1$, then*:

$$\left\| T - \frac{\lambda}{|\lambda|^{v+2}} T^* \right\| \leq \frac{1 - |\lambda|^{v+1}}{(1 - |\lambda|) |\lambda|^{v+1}} \|T - \lambda T^*\|, \tag{3.6}$$

where $v > 0$.

The following result may be stated as well.

Theorem 12. *Let $T : H \to H$ be a normal operator on the Hilbert space H. If $|\lambda| \leq 1$, then*

$$\left(1 - |\lambda|^\rho\right)^2 \|T\|^2 \leq \begin{cases} \rho^2 \|T - \lambda T^*\|^2 & \text{if } \rho \geq 1, \\ |\lambda|^{2\rho - 2} \|T - \lambda T^*\|^2 & \text{if } \rho < 1. \end{cases} \tag{3.7}$$

Proof. We use the following inequality due to Goldstein, Ryff and Clarke [**10**]

$$\|a\|^{2\rho}+\|b\|^{2\rho}-2\|a\|^{\rho-1}\|b\|^{\rho-1}\operatorname{Re}\langle a,b\rangle \le \begin{cases}\rho^2\|a\|^{2\rho-2}\|a-b\|^2 & \text{if } \rho\ge 1,\\ \|b\|^{2\rho-2}\|a-b\|^2 & \text{if } \rho<1,\end{cases} \tag{3.8}$$

provided $\rho\in\mathbb{R}$ and $a,b\in H$ with $\|a\|\ge\|b\|$.

Since $\operatorname{Re}\langle a,b\rangle\le|\langle a,b\rangle|$, then, from (3.8), we have the inequality

$$\|a\|^{2\rho}+\|b\|^{2\rho}\le 2\|a\|^{\rho-1}\|b\|^{\rho-1}|\langle a,b\rangle| + \begin{cases}\rho^2\|a\|^{2\rho-2}\|a-b\|^2 & \text{if } \rho\ge 1,\\ \|b\|^{2\rho-2}\|a-b\|^2 & \text{if } \rho<1.\end{cases} \tag{3.9}$$

We choose $a=Tx$, $b=\lambda T^*x$ and since $|\lambda|\le 1$, we have $\|a\|\ge\|b\|$. From (3.9), on taking into account that $\|Tx\|=\|T^*x\|$, we deduce

$$\|Tx\|^{2\rho}+|\lambda|^{2\rho}\|Tx\|^{2\rho}\le 2\|Tx\|^{2\rho-2}|\lambda|^{\rho}\left|\left\langle T^2x,x\right\rangle\right| + \begin{cases}\rho^2\|Tx\|^{2\rho-2}\|Tx-\lambda T^*x\|^2 & \text{if } \rho\ge 1,\\ |\lambda|^{2\rho-2}\|Tx\|^{2\rho-2}\|Tx-\lambda T^*x\|^2 & \text{if } \rho<1,\end{cases}$$

which implies that:

$$\left(1+|\lambda|^{2\rho}\right)\|Tx\|^2 \le 2|\lambda|^{\rho}\left|\left\langle T^2x,x\right\rangle\right| + \begin{cases}\rho^2\|Tx-\lambda T^*x\|^2 & \text{if } \rho\ge 1,\\ |\lambda|^{2\rho-2}\|Tx-\lambda T^*x\|^2 & \text{if } \rho<1,\end{cases} \tag{3.10}$$

for any $x\in H$, $\|x\|=1$.

This inequality is of interest in itself.

Taking the supremum over $x\in H$, $\|x\|=1$, and using the fact that

$$\sup_{\|x\|=1}\left|\left\langle T^2x,x\right\rangle\right| = w\left(T^2\right)=\|T\|^2,$$

we get the desired inequality (3.7). □

Remark 2. If $|\lambda|>1$, on choosing in (3.9) $a=\lambda T^*x$, $b=Tx$ we get:

$$\left(|\lambda|^{2\rho}+1\right)\|Tx\|^2\le 2|\lambda|^{\rho}\left|\left\langle T^2x,x\right\rangle\right| + \begin{cases}\rho^2|\lambda|^{2\rho-2}\|Tx-\lambda T^*x\|^2 & \text{if } \rho\ge 1,\\ \|Tx-\lambda T^*x\|^2 & \text{if } \rho<1,\end{cases}$$

which implies the "dual" inequality:

$$\left(1-|\lambda|^{\rho}\right)^2\|T\|^2\le\begin{cases}\rho^2|\lambda|^{2\rho-2}\|T-\lambda T^*\|^2 & \text{if } \rho\ge 1,\\ \|T-\lambda T^*\|^2 & \text{if } \rho<1,\end{cases} \tag{3.11}$$

for any $\lambda \in \mathbb{C}$, $|\lambda| > 1$.

The following result concerning operator norm inequalities may be stated as well:

Theorem 13. *Let $T : H \to H$ be a normal operator on the Hilbert space $(H; \langle \cdot, \cdot \rangle)$ and $\alpha, \beta \in \mathbb{C}$. Then:*

$$\|T\|^p \left[(|\alpha| + |\beta|)^p + ||\alpha| - |\beta||^p\right] \le \|\alpha T + \beta T^*\|^p + \|\alpha T - \beta T^*\|^p \tag{3.12}$$

if $p \in (1, 2)$ and

$$\|\alpha T + \beta T^*\|^p + \|\alpha T - \beta T^*\|^p \ge 2\left(|\alpha|^p + |\beta|^p\right)\|T\|^p \tag{3.13}$$

if $p \ge 2$.

Proof. We use the following result obtained by Dragomir and Sándor in [**8**]:

$$\|a + b\|^p + \|a - b\|^p \ge (\|a\| + \|b\|)^p + |\|a\| - \|b\||^p \tag{3.14}$$

if $p \in (1, 2)$ and

$$\|a + b\|^p + \|a - b\|^p \ge 2\left(\|a\|^p + \|b\|^p\right) \tag{3.15}$$

if $p \ge 2$, where a, b are arbitrary vectors in the inner product space $(H; \langle \cdot, \cdot \rangle)$.

We choose $a = \alpha T x$, $b = \beta T^* x$ to get:

$$\begin{aligned} \|(\alpha T + \beta T^*)(x)\|^p &+ \|(\alpha T - \beta T^*)(x)\|^p \\ &\ge (|\alpha| + |\beta|)^p \|Tx\|^p + ||\alpha| - |\beta||^p \|Tx\|^p \\ &= \left[(|\alpha| + |\beta|)^p + ||\alpha| - |\beta||^p\right]\|Tx\|^p \end{aligned} \tag{3.16}$$

if $p \in (1, 2)$ and

$$\|(\alpha T + \beta T^*)(x)\|^p + \|(\alpha T - \beta T^*)(x)\|^p \ge 2\left(|\alpha|^p + |\beta|^p\right)\|Tx\|^p \tag{3.17}$$

if $p \ge 2$.

Taking the supremum over $x \in H$, $\|x\| = 1$, we deduce (3.12) and (3.13). □

Remark 3. The case $p = 2$ produces the following inequality:

$$\|\alpha T + \beta T^*\|^2 + \|\alpha T - \beta T^*\|^2 \ge 2\left(|\alpha|^2 + |\beta|^2\right)\|T\|^2,$$

that can also be obtained by utilising the parallelogram identity.

The following general result may be stated as well:

Theorem 14. *Let $T : H \to H$ be a normal operator on the Hilbert space H. If $\alpha, \beta \in \mathbb{C}$ and $r, \rho > 0$ are such that*

$$\|T - \bar{\alpha} I\| \le r \quad \text{and} \quad \|T^* - \beta I\| \le \rho, \tag{3.18}$$

then

$$\|T\|^2 + \frac{1}{2}\left(|\alpha|^2 + |\beta|^2\right) \le \frac{1}{2}\left(r^2 + \rho^2\right) + \|\alpha T + \beta T^*\| . \tag{3.19}$$

Proof. The condition (3.18) obviously implies that

$$\|Tx\|^2 + |\alpha|^2 \le 2\operatorname{Re}\langle(\alpha T)x, x\rangle + r^2 \tag{3.20}$$

and

$$\|T^*x\|^2 + |\beta|^2 \le 2\operatorname{Re}\left\langle(\beta T)^* x, x\right\rangle + \rho^2 \tag{3.21}$$

for any $x \in H$, $\|x\| = 1$.

Adding (3.20) and (3.21) and taking into account that $\|Tx\| = \|T^*x\|$, we obtain

$$\begin{aligned} 2\|Tx\|^2 + |\alpha|^2 + |\beta|^2 &\le 2\operatorname{Re}\langle(\alpha T + \beta T^*)x, x\rangle + r^2 + \rho^2 \\ &\le 2|\langle(\alpha T + \beta T^*)x, x\rangle| + r^2 + \rho^2. \end{aligned} \tag{3.22}$$

Taking the supremum on (3.22) over $x \in H$, $\|x\| = 1$, and utilising the fact that for the normal operator T we have

$$w(\alpha T + \beta T^*) = \|\alpha T + \beta T^*\|$$

then we get the desired inequality (3.19). □

Remark. If $\alpha, \beta \in \mathbb{C}$ and $r, \rho > 0$ are such that $|\alpha|^2 + |\beta|^2 = \rho^2 + r^2$, then from (3.19) we have:

$$\|T\|^2 \le \|\alpha T + \beta T^*\| . \tag{3.23}$$

4. SOME REVERSE INEQUALITIES

The following result may be stated.

Theorem 15. *Let $(H; \langle\cdot,\cdot\rangle)$ be a Hilbert space and $T : H \to H$ a normal operator on H. If $\lambda \in \mathbb{C}\setminus\{0\}$ and $r > 0$ are such that*

$$\|T - \lambda T^*\| \le r, \tag{4.1}$$

then

$$\frac{1 + |\lambda|^2}{2|\lambda|}\|Tx\|^2 \le \left|\left\langle T^2 x, x\right\rangle\right| + \frac{r^2}{2|\lambda|} \tag{4.2}$$

for any $x \in H, \|x\| = 1.$

Proof. The inequality (4.1) is obviously equivalent to

$$\|Tx\|^2 + |\lambda|^2 \|T^*x\|^2 \le 2\operatorname{Re}\left[\overline{\lambda}\left\langle Tx, T^*x\right\rangle\right] + r^2 \tag{4.3}$$

for any $x \in H$, $\|x\| = 1$.

Since T is a normal operator, then $\|Tx\| = \|T^*x\|$ for any $x \in H$ and by (4.3) we get

$$\left(1 + |\lambda|^2\right)\|Tx\|^2 \le 2\operatorname{Re}\left[\overline{\lambda}\left\langle T^2x, x\right\rangle\right] + r^2 \tag{4.4}$$

for any $x \in H$, $\|x\| = 1$.

Now, on observing that $\operatorname{Re}\left[\overline{\lambda}\left\langle T^2x, x\right\rangle\right] \le |\lambda|\left|\left\langle T^2x, x\right\rangle\right|$, then by (4.4) we deduce (4.2). □

Remark. Observe that, since $|\lambda|^2 + 1 \ge 2|\lambda|$ for any $\lambda \in \mathbb{C}\setminus\{0\}$, hence by (4.2) we get the simpler (yet coarser) inequality:

$$(0 \le)\,\|Tx\|^2 - \left|\left\langle T^2x, x\right\rangle\right| \le \frac{r^2}{2|\lambda|}, \quad x \in H, \quad \|x\| = 1, \tag{4.5}$$

provided $\lambda \in \mathbb{C}\setminus\{0\}$, $r > 0$ and T satisfy (4.1).

If $r > 0$ and $\|T - \lambda T^*\| \le r$, with $|\lambda| = 1$, then by (4.2) we have

$$(0 \le)\,\|Tx\|^2 - \left|\left\langle T^2x, x\right\rangle\right| \le \frac{1}{2}r^2, \quad x \in H, \quad \|x\| = 1. \tag{4.6}$$

The following improvement of (2.5) should be noted:

Corollary 1. *With the assumptions of Theorem* 15, *we have the inequality*

$$(0 \le)\,\|Tx\|^2 - \left|\left\langle T^2x, x\right\rangle\right| \le \frac{r^2}{1 + |\lambda|^2}\left(\le \frac{2r^2}{(1 + |\lambda|)^2}\right) \tag{4.7}$$

for any $x \in H, \|x\| = 1.$

Proof. The inequality (4.2) is obviously equivalent to:

$$\|Tx\|^2 \le \frac{2|\lambda|}{1 + |\lambda|^2}\left|\left\langle T^2x, x\right\rangle\right| + \frac{r^2}{1 + |\lambda|^2} \le \left|\left\langle T^2x, x\right\rangle\right| + \frac{r^2}{1 + |\lambda|^2}$$

and the first part of the inequality (4.7) is obtained. The second part is obvious. □

For a normal operator T we observe that

$$\left|\left\langle T^2x, x\right\rangle\right| = |\langle Tx, T^*x\rangle| \le \|Tx\|\,\|T^*x\| = \|Tx\|^2$$

for any $x \in H$, hence

$$\|Tx\| - |\langle Tx, T^*x\rangle|^{\frac{1}{2}} \ge 0$$

for any $x \in H$.

Define $\delta(T) := \inf_{\|x\|=1} \left[\|Tx\| - \left|\left\langle T^2 x, x \right\rangle\right|^{1/2} \right] \geq 0$. The following inequality may be stated:

Theorem 16. *With the assumptions of Theorem* 15, *we have the inequality*:

$$(0 \leq) \|Tx\|^2 - \left|\left\langle T^2 x, x \right\rangle\right| \leq r^2 - 2|\lambda| \, \delta(T) \, \mu(T), \tag{4.8}$$

for any $x \in H, \|x\| = 1$, *where* $\mu(T) = \inf_{\|x\|=1} \left|\left\langle T^2 x, x \right\rangle\right|^{1/2}$.

Proof. From the inequality (4.3) we obviously have

$$\|Tx\|^2 - \left|\left\langle T^2 x, x \right\rangle\right| \leq 2 \operatorname{Re}\left[\overline{\lambda} \left\langle T^2 x, x \right\rangle\right] - \left|\left\langle T^2 x, x \right\rangle\right| - |\lambda|^2 \|Tx\|^2 + r^2 \tag{4.9}$$

for any $x \in H$, $\|x\| = 1$.

Now, observe that the right hand side of (4.9) can be written as:

$$I := r^2 + 2 \operatorname{Re}\left[\overline{\lambda} \left\langle T^2 x, x \right\rangle\right] - 2|\lambda| \left|\left\langle T^2 x, x \right\rangle\right|^{1/2} \|Tx\| - \left(\left|\left\langle T^2 x, x \right\rangle\right|^{1/2} - |\lambda| \|Tx\| \right)^2.$$

Since, obviously,

$$\operatorname{Re}\left[\overline{\lambda} \left\langle T^2 x, x \right\rangle\right] \leq |\lambda| \left|\left\langle T^2 x, x \right\rangle\right|$$

and

$$\left(\left|\left\langle T^2 x, x \right\rangle\right|^{1/2} - |\lambda| \|Tx\| \right)^2 \geq 0,$$

then

$$\begin{aligned} I &\leq r^2 - 2|\lambda| \left|\left\langle T^2 x, x \right\rangle\right|^{1/2} \left(\|Tx\| - \left|\left\langle T^2 x, x \right\rangle\right|^{1/2} \right) \\ &\leq r^2 - 2|\lambda| \, \delta(T) \left|\left\langle T^2 x, x \right\rangle\right|^{1/2}. \end{aligned}$$

Utilising (4.9) we get

$$\|Tx\|^2 \leq \left|\left\langle T^2 x, x \right\rangle\right| - 2|\lambda| \, \delta(T) \left|\left\langle T^2 x, x \right\rangle\right|^{1/2} + r^2$$

for any $x \in H$, $\|x\| = 1$, which implies the desired result. □

5. INEQUALITIES UNDER MORE RESTRICTIONS

Now, observe that, for a normal operator $T : H \to H$ and for $\lambda \in \mathbb{C} \setminus \{0\}$, $r > 0$, the following two conditions are equivalent

$$\|Tx - \lambda T^* x\| \leq r \leq |\lambda| \|T^* x\| \quad \text{for any } \; x \in H, \; \|x\| = 1 \tag{c}$$

and

$$\|T - \lambda T^*\| \leq r \quad \text{and} \quad \xi(T) := \inf_{\|x\|=1} \|Tx\| \geq \frac{r}{|\lambda|}. \tag{cc}$$

We can state the following result.

Theorem 17. *Assume that the normal operator $T : H \to H$ satisfies either* (c) *or, equivalently,* (cc) *for a given $\lambda \in \mathbb{C} \setminus \{0\}$ and $r > 0$. Then*:

$$(0 \leq) \left\| Tx \right\|^4 - \left| \left\langle T^2 x, x \right\rangle \right|^2 \leq r^2 \left\| Tx \right\|^2 \tag{5.1}$$

and

$$\left\| Tx \right\| \left(\xi^2 (T) - \frac{r^2}{\left| \lambda \right|^2} \right)^{1/2} \leq \left| \left\langle T^2 x, x \right\rangle \right|, \tag{5.2}$$

for any $x \in H$, $\|x\| = 1$.

Proof. We use the following elementary reverse of Schwarz's inequality for vectors in inner product spaces (see [**6**] or [**5**]):

$$\left\| y \right\|^2 \left\| a \right\|^2 - \left[\operatorname{Re} \left\langle y, a \right\rangle \right]^2 \leq r^2 \left\| y \right\|^2 \tag{5.3}$$

provided $\|y - a\| \leq r \leq \|a\|$.

If in (5.3) we choose $x \in H$, $\|x\| = 1$ and $y = Tx$, $a = \lambda T^* x$, then we have:

$$\left\| Tx \right\|^2 \left\| \lambda T^* x \right\|^2 - \left| \left\langle Tx, \lambda T^* x \right\rangle \right|^2 \leq r^2 \left\| \lambda T^* x \right\|^2$$

giving

$$\left\| Tx \right\|^4 \leq \left| \left\langle T^2 x, x \right\rangle \right|^2 + r^2 \left\| T^* x \right\|^2, \tag{5.4}$$

from where we deduce (5.1).

We also know that, if $\|y - a\| \leq r \leq \|a\|$, then (see [**6**] or [**5**])

$$\left\| y \right\| \left(\left\| a \right\|^2 - r^2 \right)^{1/2} \leq \operatorname{Re} \left\langle y, a \right\rangle,$$

which gives:

$$\left\| Tx \right\| \left(\left| \lambda \right|^2 \left\| Tx \right\|^2 - r^2 \right)^{1/2} \leq \operatorname{Re} \left\langle Tx, \lambda T^* x \right\rangle \leq \left| \lambda \right| \left| \left\langle T^2 x, x \right\rangle \right|$$

i.e.,

$$\left\| Tx \right\| \left(\left\| Tx \right\|^2 - \frac{r^2}{\left| \lambda \right|^2} \right)^{1/2} \leq \left| \left\langle T^2 x, x \right\rangle \right| \tag{5.5}$$

for any $x \in H$, $\|x\| = 1$. Since, obviously

$$\left(\left\| Tx \right\|^2 - \frac{r^2}{\left| \lambda \right|^2} \right)^{1/2} \geq \left(\xi^2 (T) - \frac{r^2}{\left| \lambda \right|^2} \right)^{1/2},$$

hence, by (5.5) we get (5.2). □

Theorem 18. *Assume that the normal operator $T : H \to H$ satisfies either* (c) *or, equivalently,* (cc) *for a given $\lambda \in \mathbb{C} \setminus \{0\}$ and $r > 0$. Then:*

$$(5.6) \qquad \begin{aligned} &(0 \le) \|Tx\|^4 - \left|\langle T^2x, x\rangle\right|^2 \\ &\le 2\left|\langle T^2x, x\rangle\right| \|Tx\| \left[|\lambda| \|T\| - \left(|\lambda|^2 \xi^2(T) - r^2\right)^{1/2}\right] \\ &\left(\le 2|\lambda| \left|\langle T^2x, x\rangle\right| \|T\|^2\right), \end{aligned}$$

for any $x \in H$, $\|x\| = 1$.

Proof. We use the following reverse of the Schwarz inequality obtained in [**5**]:

$$0 \le \|y\|^2 \|a\|^2 - |\langle y, a\rangle|^2 \le 2|\langle y, a\rangle| \|a\| \left(\|a\| - \sqrt{\|a\|^2 - r^2}\right)$$

provided $\|y - a\| \le r \le \|a\|$.

Now, let $x \in H$, $\|x\| = 1$ and choose $y = Tx$, $a = \lambda T^*x$ to get from (5.6) that:

$$\begin{aligned} &\|Tx\|^2 |\lambda|^2 \|T^*x\|^2 - |\lambda|^2 \left|\langle T^2x, x\rangle\right|^2 \\ &\le 2|\lambda|^2 \left|\langle T^2x, x\rangle\right| \|T^*x\| \left[|\lambda| \|T^*x\| - \left(|\lambda|^2 \|T^*x\|^2 - r^2\right)^{1/2}\right] \end{aligned}$$

giving

$$\|Tx\|^4 - \left|\langle T^2x, x\rangle\right|^2 \le 2\left|\langle T^2x, x\rangle\right| \|Tx\| \left[|\lambda| \|Tx\| - \left(|\lambda|^2 \|Tx\|^2 - r^2\right)^{1/2}\right],$$

which, by employing a similar argument to that used in the previous theorem, gives the desired inequality (5.6). □

6. OTHER RESULTS FOR ACCRETIVE OPERATORS

For a bounded linear operator $T : H \to H$ the following two statements are equivalent

$$\text{(d)} \qquad \operatorname{Re}\langle \Gamma T^*x - Tx, Tx - \gamma T^*x\rangle \ge 0 \quad \text{for any } \ x \in H, \ \|x\| = 1;$$

and

$$\text{(dd)} \qquad \left\|Tx - \frac{\gamma + \Gamma}{2} T^*x\right\| \le \frac{1}{2}|\Gamma - \gamma| \|T^*x\| \quad \text{for any } \ x \in H, \ \|x\| = 1.$$

This follows by the elementary fact that in any inner product space $(H;\langle\cdot,\cdot\rangle)$ we have, for $x,z,Z\in H$, that

$$\operatorname{Re}\langle Z-x,x-z\rangle\geq 0 \tag{6.1}$$

if and only if

$$\left\|x-\frac{z+Z}{2}\right\|\leq\frac{1}{2}\left\|Z-z\right\|. \tag{6.2}$$

An operator $B:H\to H$ is called *accretive* [**11**, p. 26] if $\operatorname{Re}\langle Bx,x\rangle\geq 0$ for any $x\in H$. We observe that, the condition (d) is in fact equivalent with the condition that

(ddd) *the operator* $(T^{*}-\bar{\gamma}T)(\Gamma T^{*}-T)$ *is accretive.*

Now, if $T:H\to H$ is a normal operator, then the following statements are equivalent

$$(T^{*}-\bar{\gamma}T)(\Gamma T^{*}-T)\geq 0 \tag{e}$$

and

$$\Gamma\left[T^{*}\right]^{2}-(\bar{\gamma}\Gamma+1)\,T^{*}T+\bar{\gamma}T^{2}\geq 0. \tag{ee}$$

This is obvious since for T a normal operator we have $T^{*}T=TT^{*}$.

We also must remark that (e) implies that

$$0\leq\langle\Gamma T^{*}x-Tx,Tx-\bar{\gamma}T^{*}x\rangle\quad\text{for any}\quad x\in H,\|x\|=1.$$

Therefore, (e) (or equivalently (ee)) is a sufficient condition for (d) (or equivalently (dd) [or (ddd)]) to hold true.

The following result may be stated.

Theorem 19. *Let $\gamma,\Gamma\in\mathbb{C}$ with $\Gamma\neq-\gamma$. For a normal operator $T:H\to H$ assume that* (ddd) *holds true. Then*:

$$(0\leq)\left\|Tx\right\|^{2}-\left|\left\langle T^{2}x,x\right\rangle\right|\leq\frac{1}{4}\cdot\frac{\left|\Gamma-\gamma\right|^{2}}{\left|\Gamma+\gamma\right|}\left\|Tx\right\|^{2} \tag{6.3}$$

for any $x\in H$, $\|x\|=1$.

Proof. We use the following reverse of the Schwarz inequality established in [**7**] (see also [**5**]):

$$\left\|z\right\|\left\|y\right\|-\frac{\operatorname{Re}(\Gamma+\gamma)\operatorname{Re}\langle z,y\rangle+\operatorname{Im}(\Gamma+\gamma)\operatorname{Im}\langle z,y\rangle}{\left|\Gamma+\gamma\right|}\leq\frac{1}{4}\cdot\frac{\left|\Gamma-\gamma\right|^{2}}{\left|\Gamma+\gamma\right|}\left\|y\right\|^{2}, \tag{6.4}$$

provided $\gamma, \Gamma \in \mathbb{C}$, $\Gamma \neq -\gamma$ and $z, y \in H$ satisfy either the condition

$$(\ell) \qquad \operatorname{Re}\langle \Gamma y - z, z - \gamma y\rangle \geq 0,$$

or, equivalently the condition

$$(\ell\ell) \qquad \left\| z - \frac{\gamma + \Gamma}{2} y \right\| \leq \frac{1}{2} |\Gamma - \gamma| \, \|y\| \, .$$

Now, if in (6.4) we choose $z = Tx$, $y = T^*x$ for $x \in H$, $\|x\| = 1$, then we obtain

$$\|Tx\| \, \|T^*x\| - |\langle Tx, T^*x\rangle| \leq \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \|T^*x\|^2 ,$$

which is equivalent with (6.3). □

Remark 6. The second inequality in (6.3) is equivalent with

$$\|Tx\|^2 \left(1 - \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|}\right) \leq \left|\left\langle T^2 x, x\right\rangle\right|$$

for any $x \in H$, $\|x\| = 1$. This inequality is of interest if $4 |\Gamma + \gamma| \geq |\Gamma - \gamma|^2$.

The following result may be stated as well.

Theorem 20. *Let $\gamma, \Gamma \in \mathbb{C}$ with $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$. If $T : H \to H$ is a normal operator such that* (ddd) *holds true, then*:

$$(6.5) \qquad \|Tx\|^2 \leq \frac{|\Gamma + \gamma|}{2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}} \left|\left\langle T^2 x, x\right\rangle\right|$$

for any $x \in H$, $\|x\| = 1$.

Proof. We can use the following reverse of the Schwarz inequality:

$$(6.6) \qquad \|z\| \, \|y\| \leq \frac{|\Gamma + \gamma|}{2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}} |\langle z, y\rangle| ,$$

provided $\gamma, \Gamma \in \mathbb{C}$ with $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$ and $z, y \in H$ are satisfying either the condition (ℓ) or, equivalently the condition $(\ell\ell)$.

Now, if in (6.6) we choose $z = Tx$, $y = T^*x$ for $x \in H$, $\|x\| = 1$, then we get

$$\|Tx\| \, \|T^*x\| \leq \frac{|\Gamma + \gamma|}{2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}} |\langle Tx, T^*x\rangle|$$

which is equivalent with (6.5). □

Also, we have:

Theorem 21. *If γ, Γ, T satisfy the hypothesis of Theorem* 20, *then we have the inequality*:

$$(0 \leq) \|Tx\|^4 - \left|\left\langle T^2x, x\right\rangle\right|^2 \leq \left[|\Gamma + \gamma| - 2\sqrt{Re\,(\Gamma\bar{\gamma})}\right] \left|\left\langle T^2x, x\right\rangle\right| \|Tx\|^2, \tag{6.7}$$

for any $x \in H$, $\|x\| = 1$.

Proof. We make use of the following inequality [**5**]:

$$(0 \leq) \|z\|^2 \|y\|^2 - |\langle z, y\rangle|^2 \leq \left[|\Gamma + \gamma| - 2\sqrt{\mathrm{Re}\,(\Gamma\bar{\gamma})}\right] |\langle z, y\rangle| \|y\|^2 \tag{6.8}$$

that holds for $\gamma, \Gamma \in \mathbb{C}$ with $\mathrm{Re}\,(\Gamma\bar{\gamma}) > 0$ and provided the vectors $z, y \in H$ satisfy either the condition (ℓ) or, equivalently the condition $(\ell\ell)$.

Now, if in (6.8) we choose $z = Tx$, $y = T^*x$ with $x \in H$, $\|x\| = 1$, then we get the desired result (6.7). □

Remark. If we choose $\Gamma = M \geq m = \gamma > 0$, then, obviously

$$\mathrm{Re}\,\langle MT^*x - Tx, Tx - mT^*x\rangle \geq 0 \quad \text{for any} \quad x \in H, \|x\| = 1 \tag{6.9}$$

is equivalent with

$$\left\|Tx - \frac{m+M}{2}T^*x\right\| \leq \frac{1}{2}(M - m) \quad \text{for any} \quad x \in H,\ \|x\| = 1, \tag{6.10}$$

or with the fact that

$$\text{the operator } (T^* - mT)(MT^* - T) \text{ is accretive.} \tag{6.11}$$

If T is normal, then the above are implied by the following two conditions that are equivalent between them:

$$(T^* - mT)(MT^* - T) \geq 0 \tag{6.12}$$

and

$$M[T^*]^2 - (mM + 1)T^*T + mT^2 \geq 0. \tag{6.13}$$

Now, if (6.11) holds, then

$$(0 \leq) \|Tx\|^2 - \left|\left\langle T^2x, x\right\rangle\right| \leq \frac{1}{4} \cdot \frac{(M - m)^2}{M + m} \|Tx\|^2, \tag{6.14}$$

$$\|Tx\|^2 \leq \frac{M + m}{2\sqrt{mM}} \left|\left\langle T^2x, x\right\rangle\right| \tag{6.15}$$

or, equivalently

$$(0 \leq) \|Tx\|^2 - \left|\left\langle T^2x, x\right\rangle\right| \leq \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{mM}} \left|\left\langle T^2x, x\right\rangle\right| \tag{6.16}$$

and

$$(0 \leq) \|Tx\|^4 - \left|\langle T^2x, x\rangle\right|^2 \leq \left(\sqrt{M} - \sqrt{m}\right)^2 \left|\langle T^2x, x\rangle\right| \|Tx\|^2, \tag{6.17}$$

for any $x \in H$, $\|x\| = 1$.

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(Received October 17, 2006)