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THE ROMAN DOMINATION NUMBER OF SOME SPECIAL CLASSES OF GRAPHS - CONVEX POLYTOPES

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In this paper we study the Roman domination number of some classes of planar graphs - convex polytopes: A_n, R_n and T_n. We establish the exact values of Roman domination number for: A_n, R_{3k}, R_{3k+1}, T_{8k}, T_{8k+2}, T_{8k+3}, T_{8k+5} and T_{8k+6}. For R_{3k+2}, T_{8k+1}, T_{8k+4} and T_{8k-1} we propose new upper and lower bounds, proving that the gap between the bounds is 1 for all cases except for the case of T_{8k+4}, where the gap is 2.

1. INTRODUCTION

The Roman domination problem is motivated by an interesting historical story about the Roman Empire's military strategies [**23, 27**]. In order to protect the territory of the Roman Empire, Constantine the Great, defined a new *defense in depth* strategy of locating and moving the legions over the Empire. In the period of reduced power of the Empire in the IV century, there was a necessity to provide protection for unsecured regions. The unsecured regions were those containing no legions, while the regions with at least one legion were considered secured. Unsecured locations could be secured by a legion sent from an adjacent region, but the movement of legions through different regions was allowed only if the old region remains secured, i.e. legion can be moved from one region to adjacent one only if another legion still remains there. Thus, at least two legions must be stationed at the secure location before a legion is sent to another region. Motivated

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by this military strategy proposed in the IV century in the ancient Roman Empire, Cockayne et al. [**7**] formally introduced the Roman domination problem (RDP) as follows.

Let $G = (V, E)$ be an undirected and simple graph with $|V| = n$ vertices. A labeling function $f : V \to \{0, 1, 2\}$ is called Roman dominating function (RDF) if for each vertex $u \in V$ with $f(u) = 0$, there is an adjacent vertex $v \in V$ with $f(v) = 2$.

The weight of a Roman dominating function f is defined as the sum of all vertex labels, i.e. $f(V) = \sum_{v \in V} f(v)$. The Roman domination number of a graph G, denoted by $\gamma_R(G)$, is defined as the minimum value of a Roman dominating function, i.e. $\gamma_R(G) = \min_{f \in \mathcal{F}} f(V)$, where $\mathcal{F}$ is the set of all Roman dominating functions.

The function f induces a partition of the set $V = (V_0,\ V_1,\ V_2)$, where $V_i = \{v \in V : f(v) = i\}$. Let us denote with $n_i = |V_i|$, $i \in \{0, 1, 2\}$. Clearly, $n_0 + n_1 + n_2 = n = |V|$. The weight of a Roman dominating function f can be written as

$$\begin{aligned} f(V) &= \sum_{v \in V} f(v) \\ &= 2 \cdot n_2 + 1 \cdot n_1 + 0 \cdot n_0 \\ &= 2 \cdot n_2 + n_1. \end{aligned}$$

Let $\delta(G)$ and $\Delta(G)$ be the minimum and the maximum degree of vertices in G, respectively. With $N(v),\ v \in V$ we denote the set of all neighbors of the vertex v in G, i.e. $N(v) = \{u \in V : \{u, v\} \in E\}$. In the latter text, edge $\{u, v\}$ notation will be shortened to uv.

Roman domination problem belongs to a wide class of domination set problems, which have been intensively studied for the last several decades. A set $S \subset V$ is a dominating set if every vertex in $V \setminus S$ is adjacent to at least one vertex in S. The domination number $\gamma(G)$ is then defined as the minimum cardinality of the dominating set in G. Cockayne et al. [**7**] studied basic properties of Roman dominating functions and calculated $\gamma_R(G)$ for some classes of graphs. For $k \leq 2$, they characterised the graphs for which $\gamma_R(G) = \gamma(G) + k$. This result was extended to arbitrary k in [**29**]. Song and Wang [**26**] characterized the trees T with $\gamma_R(T) = \gamma(T) + 3$.

Cockayne et al. [**7**] also proved that for any graph G it holds that $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$. Graphs with Roman domination number equal to twice their domination number are called Roman graphs. In [**30**], several classes of Roman graphs were presented: $P_{3k}, P_{3k+2}, C_{3k}, C_{3k+2}$ for $k \geq 1$, $K_{m,n}$ for $\min\{m, n\} \neq 2$. In the same paper some regular graphs were proved to be Roman graphs: some subclasses of circulant and generalized Petersen graphs, as well as Cartesian product graphs $C_{5m} \Box C_{5n}$, where $m \geq 1,\ n \geq 1$. Henning [**12**] also contributed to this concept of Roman graphs by characterizing Roman trees.

The Roman domination number was studied for a number of classes of graphs: interval graphs, cographs, asteroidal triple-free graphs and graphs with a d-octopus in [**17**], corona graphs in [**31**], grid graphs in [**8**], generalized Sierpiński graphs in [**22**], generalized Petersen graphs $GP(n, 2)$ in [**28**] and $GP(n, 3)$ and $GP(n, 4)$ in [**32**], cardinal product of paths and cycles in[**16, 15**], strongly chordal graphs in [**19**] and others.

A relation between the Roman domination number and the differential of a graph is studied in [**3**]. A 5-approximation algorithm of linear time and a polynomial-time approximation scheme are presented in [**24**]. A relation between the Roman domination number and 2-rainbow domination number $\gamma_{2r}(G)$ was given in [**6**] by the formula $\gamma_R(G)/\gamma_{r2}(G) \leq 3/2$. The Roman domination number in digraphs was studied in [**25**].

Although RDP is NP-hard in the general case [**9, 24**], for some classes of graphs it can be solved in polynomial time. For example, linear-time algorithms for computing the Roman domination number are known on interval graphs and cographs [**17**]. In the same paper polynomial-time algorithm is proposed on AT-free graphs.

The Roman domination problem has also been solved by integer linear programming (ILP). The first formulation was introduced in [**23**]. ILP formulations for five graph domination problems, including Roman domination were proposed in [**4**]. Two improved ILP formulations using less number of constraints than previous models were presented in [**14**].

Several upper and lower bounds were proposed in literature.

Chambers et al. [**5**] proved that $\gamma_R(G) \leq \frac{4n}{5}$ if $\delta(G) \geq 1$ and $\gamma_R(G) \leq 8n/11$ if $\delta(G) \geq 2$. In two subsequent papers [**18, 20**], Liu and Chang proved that $\gamma_R(G) \leq 2n/3$ when $\delta(G) \geq 3$ and $\gamma_R(G) \leq \max\{\lceil 2n/3 \rceil, \lceil 23n/34 \rceil\}$ when G is 2-connected. Favaron et al. [**10**] proved that $\gamma_R(G) + \gamma(G)/2 \leq n$ for any connected graph G of order $n \geq 3$. Lower and upper bounds for the Roman domination numbers in terms of the diameter and the girth were proposed in [**21**].

A lower bound on the Roman domination number is given by the total domination number, $\gamma_t(G) \leq \gamma_R(G)$ in [**11**]. An upper bound for the Roman domination number $\gamma_R(G) \leq 2\left(1 - \dfrac{2^{1/\delta(G)}\delta(G)}{(1+\delta(G))^{1+1/\delta(G)}}\right)n$ was presented in [**33**], with a proof that the proposed result is asymptotically best possible.

The following, tractable lower bound, established in [**7**], will be used in this work:

Proposition 1. *For any graph G,* $\gamma_R(G) \geq \dfrac{2 \cdot |V(G)|}{1 + \Delta(G)}$.

2. CONVEX POLYTOPES A_n

Convex polytope A_n, for $n \geq 5$, or antiprism was introduced in [**1**] and can be defined as a graph with the set of vertices

$$V(A_n) = \{a_i, b_i, c_i | i = 0, ..., n-1\},$$

and the set of edges

$$E(A_n) = \{a_i a_{i+1}, b_i b_{i+1}, c_i c_{i+1}, a_i b_i, b_i c_i, a_{i+1} b_i, b_{i+1} c_i | i = 0, ..., n-1\}.$$

It should be noted that vertex indices are taken modulo n throughout the whole paper. Also, note that the exact values of the RDF for smaller graphs were obtained by a total enumeration technique. These findings were a good starting point for estimating the bounds of RDF for the graphs of larger dimensions.

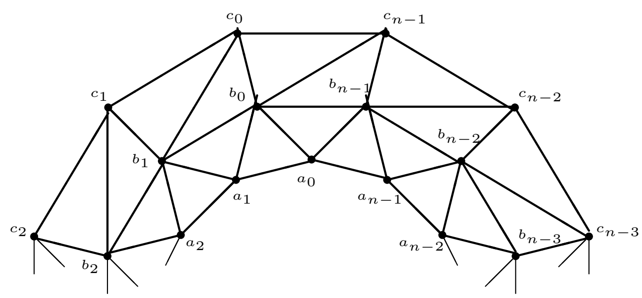

Figure 1: The graph of convex polytope A_n

Theorem 1. $\gamma_R(A_n) = \begin{cases} n, & n = 2k \wedge k \geq 3, \\ n+1, & n = 2k+1 \wedge k \geq 2. \end{cases}$

Proof. Step 1. <u>*Upper bound*</u>

Let us define a function $f : V \to \{0, 1, 2\}$ by the partition (V_0, V_1, V_2) of the set $V(A_n)$.

Let $V_2 = \{b_{2i} | i = 0, ..., \lfloor \frac{n-1}{2} \rfloor\}$, $V_1 = \emptyset$, while all other vertices are in V_0. Since $f(V(A_n)) = 2|V_2| = 2\lceil \frac{n}{2} \rceil$, it follows that $\gamma_R(A_n) \leq 2\lceil \frac{n}{2} \rceil$.

Let us prove that f is a RDF. From the definition of the convex polytope A_n, it holds that each a-vertex has exactly one adjacent b-vertex with an even index, with the exception of the vertex a_0 when n is odd. In that case, the vertex a_0 has two adjacent b-vertices with even indices (namely b_0 and b_{n-1}).

Similarly, each c-vertex is adjacent to exactly one b-vertex with an even index, except the vertex c_{n-1} when n is odd. In that case, c_{n-1} is adjacent with b_0 and b_{n-1}. Also, every b-vertex with an odd index has two adjacent b-vertices with even indices. In Table 1, this consideration is shown in a systematized view. So, we proved that f is a RDF.

We will now prove that the $\gamma_R(A_n) \geq 2\lceil \frac{n}{2} \rceil$.

Step 2.

<u>*Known lower bound*</u>

Proposition 1 introduces lower bound: $\gamma_R(A_n) \geq \dfrac{2 \cdot |V(A_n)|}{1 + \Delta(A_n)} = \dfrac{6 \cdot n}{7}$ so

$$\gamma_R(A_n) \geq \left\lceil \frac{6 \cdot n}{7} \right\rceil.$$

Table 1: Roman domination coverage for A_n

n	$v \in V_0$	$V_2 \cap N(v)$	range of i
$2k$	a_{2i}	$\{b_{2i}\}$	$0 \le i \le k-1$
	a_{2i+1}	$\{b_{2i}\}$	$0 \le i \le k-1$
	b_{2i+1}	$\{b_{2i}, b_{2i+2}\}$	$0 \le i \le k-1$
	c_{2i}	$\{b_{2i}\}$	$0 \le i \le k-1$
	c_{2i+1}	$\{b_{2i+2}\}$	$0 \le i \le k-1$
$2k+1$	a_{2i}	$\{b_{2i}\}$	$1 \le i \le k$
	a_0	$\{b_0, b_{2k}\}$	
	a_{2i+1}	$\{b_{2i}\}$	$0 \le i \le k-1$
	b_{2i+1}	$\{b_{2i}, b_{2i+2}\}$	$0 \le i \le k-1$
	c_{2i}	$\{b_{2i}\}$	$0 \le i \le k-1$
	c_{2k}	$\{b_0, b_{2k}\}$	
	c_{2i+1}	$\{b_{2i+2}\}$	$0 \le i \le k-1$

This lower bound is strictly lower than $2\lceil \frac{n}{2} \rceil$ for $n \ge 7$, so it cannot be used to prove the optimality.

Step 3.

<u>*Improved lower bound*</u>

Let f be a RDF, represented by (V_0,V_1,V_2) and $n_j = |V_j|$, for $j \in \{0, 1, 2\}$. Next, we define $n_{j,i}$, $i \in \{0, 1, 2\}$, as the number of the elements of V_j which are a–vertices, b–vertices, and c–vertices, respectively to i.

Therefore it holds:

$$f(V(A_n)) = \sum_{i=0}^{2} (n_{1,i} + 2 \cdot n_{2,i})$$

subject to:

$$\sum_{j=0}^{2} n_{j,i} = n, \quad \text{for each } i = 0, 1, 2. \tag{1}$$

Each a-vertex is a neighbour of exactly two a-vertices and two b-vertices. Since f is a RDF, then each a-vertex from V_0 must have at least one neighbour from V_2, so number of a-vertices with zero f value cannot exceed two times of number of a-vertices with f value equal to two plus two times of number of b-vertices with f value equal to two, i.e.:

$$n_{0,0} \le 2 \cdot n_{2,0} + 2 \cdot n_{2,1}$$

or equivalently:

$$(2) \qquad 2 \cdot n_{2,0} + 2 \cdot n_{2,1} - n_{0,0} \geq 0.$$

Similarly, each b-vertex is a neighbour of exactly two a-vertices, two b-vertices and two c-vertices, so we conclude that:

$$2 \cdot n_{2,0} + 2 \cdot n_{2,1} + 2 \cdot n_{2,2} - n_{0,1} \geq 0.$$

Finally, each c-vertex is a neighbour of exactly two b-vertices, and two c-vertices, so:

$$(3) \qquad 2 \cdot n_{2,1} + 2 \cdot n_{2,2} - n_{0,2} \geq 0.$$

Now, we sum:

- equality (1) for $i = 0$ multiplied with $\frac{2}{3}$;
- equality (1) for $i = 2$ multiplied with $\frac{1}{3}$;
- inequality (2) multiplied with $\frac{2}{3}$
- and inequality (3) multiplied with $\frac{1}{3}$.

Then, we obtain:

$$(4) \qquad \frac{2}{3} \cdot n_{1,0} + \frac{1}{3} \cdot n_{1,2} + 2 \cdot n_{2,0} + 2 \cdot n_{2,1} + n_{2,2} \geq n.$$

Since all $n_{1,i}$, for $i = 0, 1, 2$ are non-negative integers, then:

$$\begin{aligned} f(V(A_n)) = \sum_{i=0}^{2} (n_{1,i} + 2 \cdot n_{2,i}) &= n_{1,0} + n_{1,1} + n_{1,2} + 2 \cdot n_{2,0} + 2 \cdot n_{2,1} + 2 \cdot n_{2,2} \\ &\geq \frac{2}{3} \cdot n_{1,0} + \frac{1}{3} \cdot n_{1,2} + 2 \cdot n_{2,0} + 2 \cdot n_{2,1} + n_{2,2} \geq n \end{aligned}$$

then $f(V(A_n)) \geq n$ holds, implying $\gamma_R(A_n) \geq n$. If n is an even number, then we have obtained the optimal solution, since in Step 1 we have proved $\gamma_R(A_n) \leq n$.

If n is odd, we have two cases:

Case 1. $V_1 = \emptyset$. In this case, $f(V(A_n)) \geq |V_1| + 2 \cdot |V_2| = 2 \cdot |V_2|$. Combining the previous inequality $f(V(A_n)) \geq n$ with the facts that n is odd and $f(V(A_n))$ is even, we get that $f(V(A_n)) \geq n + 1$.

Case 2. $V_1 \neq \emptyset$. In this case $|V_1| = n_{1,0} + n_{1,1} + n_{1,2} > 0$. Since $n_{1,0}$, $n_{1,1}$ and $n_{1,2}$ are integers, we have $n_{1,0} + n_{1,1} + n_{1,2} \geq 1$.

We write the inequality (4) as:

$$\begin{aligned}
n &\leq \frac{2}{3}\cdot n_{1,0} + \frac{1}{3}\cdot n_{1,2} + 2\cdot n_{2,0} + 2\cdot n_{2,1} + n_{2,2} \\
&= f(V(A_n)) - \left(\frac{1}{3}\cdot n_{1,0} + n_{1,1} + \frac{2}{3}\cdot n_{1,2} + n_{2,2}\right) \\
&= f(V(A_n)) - \frac{1}{3}\cdot (n_{1,0} + n_{1,1} + n_{1,2}) - \left(\frac{2}{3}\cdot n_{1,1} + \frac{1}{3}\cdot n_{1,2} + n_{2,2}\right) \\
&\leq f(V(A_n)) - \frac{1}{3}\cdot (n_{1,0} + n_{1,1} + n_{1,2}) \\
&\leq f(V(A_n)) - \frac{1}{3}.
\end{aligned}$$

From $n \leq f(V(A_n)) - \frac{1}{3}$ we have $f(V(A_n)) \geq n + \frac{1}{3}$. Since n and $f(V(A_n))$ are integers, we have $f(V(A_n)) \geq n+1 \Rightarrow \gamma_R(A_n) \geq n+1$. Since in Step 1 we have proved that $\gamma_R(A_n) \leq n+1$ for odd n, it follows $\gamma_R(A_n) = n+1$ for odd n. □

3. CONVEX POLYTOPES R_n

The class of the convex polytopes R_n has been introduced by Bača [**2**]. The convex polytope R_n is defined as a combination of the graph of a prism and the graph of an antiprism. More formally, for $n \geq 5$, R_n is the graph with the set of vertices:

$$V(R_n) = \{a_i, b_i, c_i | i = 0, ..., n-1\}$$

and the set of edges:

$$E(R_n) = \{a_i a_{i+1}, b_i b_{i+1}, c_i c_{i+1}, a_i b_i, b_i c_i, a_{i+1} b_i | i = 0, ..., n-1\}.$$

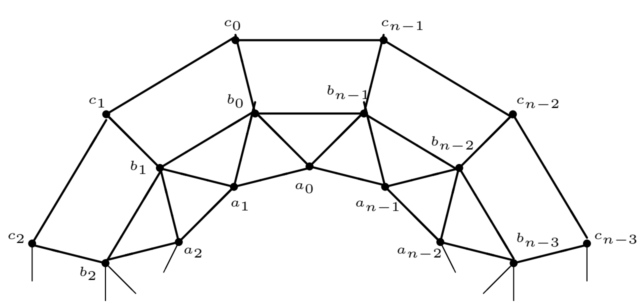

Figure 2: The graph of convex polytope R_n

Theorem 2. *For $k \geq 2$, $\gamma_R(R_{3k}) = 4k$, $\gamma_R(R_{3k+1}) = 4k+2$ and $\gamma_R(R_{3k+2}) \in \{4k+3, 4k+4\}$, while $\gamma_R(R_5) = 7$.*

Proof. Step 1. Upper bound

Let function f be defined as in Table 2. In order to prove that f is a RDF, we consider three possible cases:

Case 1: $n = 3k$. As it can be seen from the first part of Table 3, each a-vertex with the index having the remainder 1 or 2 when divided by 3 is adjacent to an a-vertex with an index divisible by 3. Each b-vertex with the index divisible by 3, or having the remainder 2 when divided by 3 is a neighbor of an a-vertex with an index divisible by 3. Further, each b-vertex with the index having the remainder 1 when divided by 3, each c-vertex with the index divisible by 3, or having the remainder 2 when divided by 3 is adjacent to an c-vertex with the index having the remainder 1 when divided by 3.

Case 2: $n = 3k+1$. From the middle part of Table 3 one can see that each vertex from V_0 has an adjacent vertex from V_2. Vertices a_0, b_0, b_{3k-1} and c_{3k} are adjacent to b_{3k}, which is in V_2. Similarly to Case 1, considering the remainder of the vertex index when divided by 3, it can be shown that all other vertices from V_0 have an adjacent vertex from V_2.

Case 3: $n = 3k+2$. From the last part of Table 3, one can see that a_{3k+1}, b_{3k+1} and c_0 have two adjacent vertices from V_2, while the rest of the vertices from V_0 have one adjacent vertex from V_2.

Therefore, f formally satisfies RDF requirements.

Table 2: Non-zero values of RDF on R_n

n	V_2	V_1	$f(V)$
$3k$	$\{a_{3i}, c_{3i+1} \mid i = 0, ..., k-1\}$	$\emptyset$	$4k$
$3k+1$	$\{c_1, b_{3k}\} \cup \{a_{3i}, c_{3i+1} \mid i = 1, ..., k-1\}$	$\{a_1, a_{3k-1}\}$	$4k+2$
$3k+2$	$\{a_{3i}, c_{3i+1} \mid i = 0, ..., k\}$	$\emptyset$	$4k+4$

Step 2.

Known lower bound

Since $\Delta(R_n) = deg(b_i) = 5$, then by applying Proposition 1 we obtain the lower bound $\gamma_R(R_n) \geq \frac{2 \cdot |V(R_n)|}{1+\Delta(R_n)} = \frac{6 \cdot n}{6} = n$. Combined with the upper bound obtained in Step 1, we get $n \leq \gamma_R(R_n) \leq 2 \cdot \lfloor \frac{2 \cdot n}{3} + \frac{1}{2} \rfloor$. By increasing n, the gap between these lower and upper bounds is also increasing, so it is needed for the proof to obtain a lower bound with a smaller gap.

Improved lower bound

Let f be a RDF, represented by (V_0, V_1,V_2). Similarly to the case of convex polytopes A_n, we introduce the numbers $n_{0,i}$, $n_{1,i}$, $n_{2,i}$, for $i = 0, 1, 2$. Therefore, it holds $f(V(R_n)) = \sum_{i=0}^{2} (n_{1,i} + 2 \cdot n_{2,i})$, subject to:

$$\sum_{j=0}^{2} n_{j,i} = n, \quad \text{for each } i = 0, 1, 2. \tag{5}$$

Table 3: Roman domination coverage for R_n

n	$v \in V_0$	$V_2 \cap N(v)$	range of i	$v \in V_0$	$V_2 \cap N(v)$	range of i
	a_{3i+1}	$\{a_{3i}\}$	$0 \leq i \leq k-1$	a_{3i+2}	$\{a_{3i+3}\}$	$0 \leq i \leq k-1$
$3k$	b_{3i}	$\{a_{3i}\}$	$0 \leq i \leq k-1$	b_{3i+1}	$\{c_{3i+1}\}$	$0 \leq i \leq k-1$
	b_{3i+2}	$\{a_{3i+3}\}$	$0 \leq i \leq k-1$	c_{3i}	$\{c_{3i+1}\}$	$0 \leq i \leq k-1$
	c_{3i+2}	$\{c_{3i+1}\}$	$0 \leq i \leq k-1$			
	a_0	$\{b_{3k}\}$		a_{3k}	$\{b_{3k}\}$	
	a_{3i+1}	$\{a_{3i}\}$	$1 \leq i \leq k-1$	a_{3i+2}	$\{a_{3i+3}\}$	$0 \leq i \leq k-2$
$3k+1$	b_0	$\{b_{3k}\}$		b_{3i}	$\{a_{3i}\}$	$1 \leq i \leq k-1$
	b_{3i+1}	$\{c_{3i+1}\}$	$0 \leq i \leq k-1$	b_{3i+2}	$\{a_{3i+3}\}$	$0 \leq i \leq k-2$
	b_{3k-1}	$\{b_{3k}\}$		c_{3i}	$\{c_{3i+1}\}$	$0 \leq i \leq k-1$
	c_{3k}	$\{b_{3k}\}$		c_{3i+2}	$\{c_{3i+1}\}$	$0 \leq i \leq k-1$
	a_{3i+1}	$\{a_{3i}\}$	$0 \leq i \leq k-1$	a_{3k+1}	$\{a_0, a_{3k}\}$	
	a_{3i+2}	$\{a_{3i+3}\}$	$0 \leq i \leq k-1$	b_{3i}	$\{a_{3i}\}$	$0 \leq i \leq k$
$3k+2$	b_{3i+1}	$\{c_{3i+1}\}$	$0 \leq i \leq k-1$	b_{3k+1}	$\{a_0, c_{3k+1}\}$	
	b_{3i+2}	$\{a_{3i+3}\}$	$0 \leq i \leq k-1$	c_0	$\{c_1, c_{3k+1}\}$	
	c_{3i}	$\{c_{3i+1}\}$	$1 \leq i \leq k$	c_{3i+2}	$\{c_{3i+1}\}$	$0 \leq i \leq k-1$

Each a-vertex is a neighbour of exactly two a-vertices and two b-vertices. Therefore, the total number of a-vertices with zero f value cannot exceed two times of the number of a-vertices with f value equal to two plus two times of the number of b-vertices with f value equal to two, i.e.:

$$2 \cdot n_{2,0} + 2 \cdot n_{2,1} - n_{0,0} \geq 0. \tag{6}$$

Further, each b-vertex is a neighbour of exactly two a-vertices, two b-vertices and one c-vertex, so we have:

$$2 \cdot n_{2,0} + 2 \cdot n_{2,1} + n_{2,2} - n_{0,1} \geq 0.$$

Finally, each c-vertex is a neighbour of exactly one b-vertex, and two c-vertices, so:

$$n_{2,1} + 2 \cdot n_{2,2} - n_{0,2} \geq 0. \tag{7}$$

We sum the following:

- equality (5) for $i = 0$ multiplied with $\frac{2}{3}$;
- equality (5) for $i = 2$ multiplied with $\frac{2}{3}$;
- inequality (6) multiplied with $\frac{2}{3}$
- and inequality (7) multiplied with $\frac{2}{3}$.

Then, we obtain:

$$\frac{2}{3} \cdot n_{1,0} + \frac{2}{3} \cdot n_{1,2} + 2 \cdot n_{2,0} + 2 \cdot n_{2,1} + 2 \cdot n_{2,2} \geq \frac{4n}{3}.$$

Since all $n_{1,i}$, for $i = 0, 1, 2$ are non-negative integers, then:

$$\begin{aligned} f(V(R_n)) = \sum_{i=0}^{2} (n_{1,i} + 2 \cdot n_{2,i}) &= n_{1,0} + n_{1,1} + n_{1,2} + 2 \cdot n_{2,0} + 2 \cdot n_{2,1} + 2 \cdot n_{2,2} \\ &\geq \frac{2}{3} \cdot n_{1,0} + \frac{2}{3} \cdot n_{1,2} + 2 \cdot n_{2,0} + 2 \cdot n_{2,1} + 2 \cdot n_{2,2} \geq \frac{4n}{3}. \end{aligned}$$

Thus, $f(V(R_n)) \geq \frac{4n}{3}$ holds, implying $\gamma_R(R_n) \geq \frac{4n}{3}$ so $\gamma_R(R_n) \geq \lceil \frac{4n}{3} \rceil$ holds.

In the case when $n = 3k$ the lower bound is $\lceil \frac{4}{3} \cdot 3k \rceil = 4k$. When $n = 3k + 1$, the lower bound is $\lceil \frac{4}{3} \cdot (3k + 1) \rceil = \lceil 4 \cdot k + \frac{4}{3} \rceil = 4k + 2$. Since lower bounds are equal to upper bounds obtained in Step 1, we conclude that in these cases, exact values are determined.

In the case when $n = 3k+2$ the lower bound $\lceil \frac{4}{3} \cdot (3k+2) \rceil = \lceil 4 \cdot k + \frac{8}{3} \rceil = 4k+3$ is equal to the upper bound obtained in Step 1 minus one. Therefore, in this case, we have a gap between the lower and the upper bound that equals 1, and since the value must be an integer, we have $\gamma_R(R_{3k+2}) \in \{4k + 3, 4k + 4\}$.

Total enumeration was able to obtain exact values for $n = 3k + 2$ and $|V(R_n)| \leq 33$ in a reasonable time. In those cases, it was interesting to see which bound was achieved as an exact value. For $k = 1$, $\gamma_R(R_5) = 7$, which represents the lower bound, while for $k = 2$, $\gamma_R(R_8) = 12$ and for $k = 3$, $\gamma_R(R_{11}) = 16$, which represent the upper bound.

□

4. CONVEX POLYTOPES T_n

A convex polytope T_n, $n \geq 5$ is obtained by combination of the graph of convex polytope R_n and the graph of an antiprism [**13**]. The explicite definition of this polytope is given with its set of vertices:

$$V(T_n) = \{a_i, b_i, c_i, d_i | i = 0, ..., n - 1\}$$

and its set of edges:

$$E(T_n) = \{a_i a_{i+1}, b_i b_{i+1}, c_i c_{i+1}, d_i d_{i+1}, a_i b_i, b_i c_i, c_i d_i, a_{i+1} b_i, c_{i+1} d_i | i = 0, ..., n - 1\}.$$

Theorem 3. *For $k \geq 1$, exact values are $\gamma_R(T_{8k}) = 12k$, $\gamma_R(T_{8k+2}) = 12k + 4$, $\gamma_R(T_{8k+3}) = 12k+5$, $\gamma_R(T_{8k+5}) = 12k+8$, and $\gamma_R(T_{8k+6}) = 12k+10$ and specially, $\gamma_R(T_5) = 8$ and $\gamma_R(T_6) = 10$. Moreover, for $k \geq 1$, bounds are $\gamma_R(T_{8k+1}) \in \{12k + 2, 12k + 3\}$, $\gamma_R(T_{8k+4}) \in \{12k + 6, 12k + 7, 12k + 8\}$, $\gamma_R(T_{8k-1}) \in \{12k - 1, 12k\}$.*

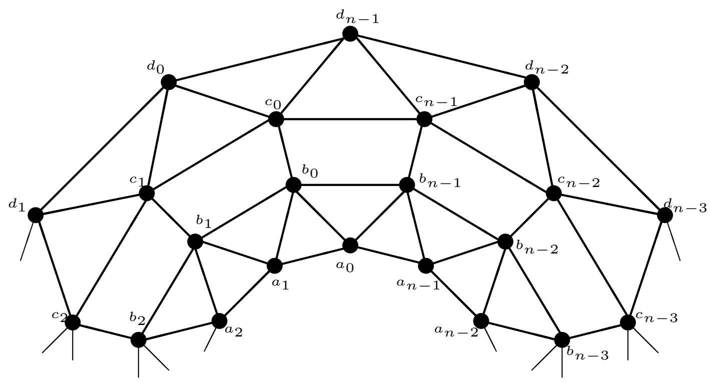

Figure 3: The graph of convex polytope T_n

Proof. Step 1. <u>*Upper bound.*</u>

Let function f be defined as in Table 4. As we can see from Table 4, definition of the function f depends on the remainder obtained by dividing n by 8. In Tables 5–7, for each vertex from V_0 we identify its neighbor from V_2. In Table 5 we consider the cases: $n = 8k$, $n = 8k - 1$, $n = 8k + 1$, in Table 6 are analyzed the cases: $n = 8k + 2$, $n = 8k + 3$, $n = 8k + 4$, and finally in Table 7 are shown two cases, $n = 8k + 5$ and $n = 8k + 6$. It should be noticed that in some cases a vertex from V_0 has more than one neighbor from V_2.

Table 4: Non-zero values of RDF on T_n

n	V_2	V_1	$f(V)$
$8k, 8k-1$	$\{a_{8i}, a_{8i+3}, b_{8i+5}, c_{8i+1}, d_{8i+3}, d_{8i+6} \| i = 0, ..., k-1\}$	$\emptyset$	$12 \cdot k$
$8k+1$	$\{c_{8k}\} \cup$ $\{a_{8i+1}, a_{8i+6}, b_{8i+3}, c_{8i+7}, d_{8i+1}, d_{8i+4} \| i = 0, ..., k-1\}$	$\{a_{8k}\}$	$12 \cdot k + 3$
$8k+2$	$\{b_{8k+1}, c_{8k}\} \cup$ $\{a_{8i+2}, a_{8i+7}, b_{8i+4}, c_{8i}, d_{8i+2}, d_{8i+5} \| i = 0, ..., k-1\}$	$\emptyset$	$12 \cdot k + 4$
$8k+3$	$\{b_{8k+2}, d_{8k}\} \cup$ $\{a_{8i+5}, a_{8i+8}, b_{8i+2}, c_{8i+6}, d_{8i}, d_{8i+3} \| i = 0, ..., k-1\}$	$\{a_1\}$	$12 \cdot k + 5$
$8k+4$	$\{a_{8k-2}, a_{8k+3}, b_{8k}, c_{8k}, d_{8k+2}\} \cup \{a_{8i+7} \| i = 0, ..., k-2\} \cup$ $\{a_{8i+2}, b_{8i+4}, c_{8i}, d_{8i+2}, d_{8i+5} \| i = 0, ..., k-1\}$	$\emptyset$	$12 \cdot k + 8$
$8k+5$	$\{a_{8k}, b_{8k+2}, d_{8k}, d_{8k+3}\} \cup$ $\{a_{8i}, a_{8i+5}, b_{8i+2}, c_{8i+6}, d_{8i}, d_{8i+3} \| i = 0, ..., k-1\}$	$\emptyset$	$12 \cdot k + 8$
$8k+6$	$\{a_{8k+3}, b_{8k}, c_{8k+4}, d_{8k+1}\} \cup$ $\{a_{8i+3}, a_{8i+6}, b_{8i}, c_{8i+4}, d_{8i+1}, d_{8i+6} \| i = 0, ..., k-1\}$	$\{a_{8k+5}, d_{8k+5}\}$	$12 \cdot k + 10$

Step 2.

<u>*Known lower bound*</u>

As it was the case with A_n and R_n, we first consider the lower bound given by Proposition 1. Since $\Delta(T_n) = deg(b_i) = deg(c_i) = 5$, consequently $\gamma_R(T_n) \geq \dfrac{2 \cdot |V(T_n)|}{1 + \Delta(T_n)} = \dfrac{8 \cdot n}{6}$ so

$$\gamma_R(T_n) \geq \left\lceil \frac{4 \cdot n}{3} \right\rceil.$$

It is easy to see that lower bound given by Proposition 1 is strictly less than the upper bound, so it is needed for the proof to obtain a better bound.

Step 3.

Improved lower bound

Let f be a RDF, represented by (V_0, V_1,V_2). Next, similarly to the cases of A_n and R_n, we introduce $n_{0,i}, n_{1,i}, n_{2,i}$, for $i = 0, 1, 2, 3$ as the numbers of elements in V_0, V_1 and V_2 respectively, where the index i is respectively related to a-vertices, b-vertices, c-vertices and d-vertices. Therefore it holds:

$$f(V(T_n)) = \sum_{i=0}^{3} (n_{1,i} + 2 \cdot n_{2,i})$$

subject to:

$$\sum_{j=0}^{2} n_{j,i} = n, \text{ for each } i = 0, 1, 2, 3. \tag{8}$$

Each a-vertex is a neighbour of exactly two a-vertices and two b-vertices. Neither a-vertex is adjacent to any c-vertex or d-vertex. Since f is a RDF, each a-vertex from V_0 must have at least one neighbour from V_2, so the number of a-vertices with zero f value cannot exceed two times the number of a-vertices with f value equal to 2 plus two times the number of b-vertices with f value equal to 2, i.e.:

$$2 \cdot n_{2,0} + 2 \cdot n_{2,1} - n_{0,0} \geq 0. \tag{9}$$

Since every b-vertex is a neighbour of exactly two a-vertices, two b-vertices and one c-vertex, we have:

$$2 \cdot n_{2,0} + 2 \cdot n_{2,1} + n_{2,2} - n_{0,1} \geq 0. \tag{10}$$

Similarly, every c-vertex is a neighbour of exactly one b-vertex, two c-vertices and two d-vertices, so:

$$n_{2,1} + 2 \cdot n_{2,2} + 2 \cdot n_{2,3} - n_{0,2} \geq 0. \tag{11}$$

Finally, every d-vertex is a neighbour of exactly two c-vertices and two d-vertices, so:

$$(12) \qquad 2 \cdot n_{2,2} + 2 \cdot n_{2,3} - n_{0,3} \geq 0.$$

We sum the following:

- equality (8) for $i = 0$ multiplied with $\frac{1}{2}$;
- equality (8) for $i = 1$ multiplied with $\frac{1}{4}$;
- equality (8) for $i = 2$ multiplied with $\frac{1}{4}$;
- equality (8) for $i = 3$ multiplied with $\frac{1}{2}$;
- inequality (9) multiplied with $\frac{1}{2}$;
- inequality (10) multiplied with $\frac{1}{4}$;
- inequality (11) multiplied with $\frac{1}{4}$
- and inequality (12) multiplied with $\frac{1}{2}$.

Then, we have:

$$(13) \quad \frac{1}{2} \cdot n_{1,0} + \frac{1}{4} \cdot n_{1,1} + \frac{1}{4} \cdot n_{1,2} + \frac{1}{2} \cdot n_{1,3} + 2 \cdot n_{2,0} + 2 \cdot n_{2,1} + 2 \cdot n_{2,2} + 2 \cdot n_{2,3} \geq \frac{3}{2} \cdot n.$$

Since all $n_{1,i}$, for $i = 0, 1, 2, 3$ are non-negative integers, we get:

$$\begin{aligned} f(V(T_n)) &= \sum_{i=0}^{3} (n_{1,i} + 2 \cdot n_{2,i}) \\ &= n_{1,0} + n_{1,1} + n_{1,2} + n_{1,3} + 2 \cdot n_{2,0} + 2 \cdot n_{2,1} + 2 \cdot n_{2,2} + 2 \cdot n_{2,3} \\ &\geq \frac{1}{2} \cdot n_{1,0} + \frac{1}{4} \cdot n_{1,1} + \frac{1}{4} \cdot n_{1,2} + \frac{1}{2} \cdot n_{1,3} + 2 \cdot n_{2,0} + 2 \cdot n_{2,1} + 2 \cdot n_{2,2} + 2 \cdot n_{2,3} \\ &\geq \frac{3}{2} \cdot n. \end{aligned}$$

Therefore, $f(V(T_n)) \geq \frac{3}{2} \cdot n$ holds, implying $\gamma_R(T_n) \geq \frac{3}{2} \cdot n$. Because $\gamma_R(T_n)$ must be an integer, we have $\gamma_R(T_n) \geq \lceil \frac{3}{2} \cdot n \rceil$.

Now we analyze 8 cases, depending on the remainder obtained when dividing n by 8.

In the cases $n = 8k$, $n = 8k+3$ and $n = 8k+5$, lower bounds are $\lceil \frac{3}{2} \cdot 8k \rceil = 12k$, $\lceil \frac{3}{2} \cdot (8k+3) \rceil = \lceil 12 \cdot k + \frac{9}{2} \rceil = 12k + 5$ and $\lceil \frac{3}{2} \cdot (8k+5) \rceil = \lceil 12 \cdot k + \frac{15}{2} \rceil = 12k + 8$ and they are equal to the upper bound given in Step 1. So, we can conclude that for these cases, exact value of $\gamma_R(T_n)$ is obtained.

In the cases $n = 8k+2$ and $n = 8k+6$ lower bounds $\lceil \frac{3}{2} \cdot (8k+2) \rceil = 12k+3$ and $\lceil \frac{3}{2} \cdot (8k+6) \rceil = 12k+9$ are odd, so they can be reached only if $V_1 \neq \emptyset$ i.e. $|V_1| \geq 1$, implying $|V_1| = n_{1,0} + n_{1,1} + n_{1,2} + n_{1,3} \geq 1$.

From (13) we have:

$$\begin{aligned}
\frac{3}{2}\cdot n \leq \frac{1}{2}\cdot n_{1,0} + \frac{1}{4}\cdot n_{1,1} + \frac{1}{4}\cdot n_{1,2} + \frac{1}{2}\cdot n_{1,3} + 2\cdot n_{2,0} + 2\cdot n_{2,1} + 2\cdot n_{2,2} + 2\cdot n_{2,3} \\
= f(V(T_n)) - \left(\frac{1}{2}\cdot n_{1,0} + \frac{3}{4}\cdot n_{1,1} + \frac{3}{4}\cdot n_{1,2} + \frac{1}{2}\cdot n_{1,3}\right) \\
= f(V(T_n)) - \frac{1}{2}(n_{1,0} + n_{1,1} + n_{1,2} + n_{1,3}) - \frac{1}{4}(n_{1,1} + n_{1,2}) \\
\leq f(V(T_n)) - \frac{1}{2}(n_{1,0} + n_{1,1} + n_{1,2} + n_{1,3}) \\
\leq f(V(T_n)) - \frac{1}{2}.
\end{aligned}$$

Thus, $f(V(T_n)) \geq \frac{3}{2}\cdot n + \frac{1}{2}$.

For $n = 8k+2$ we have:

$f(V(T_{8k+2})) \geq \frac{3}{2}\cdot(8k+2) + \frac{1}{2} = 12k+3+\frac{1}{2}$. Since $f(V(T_{8k+2}))$ is an integer, we get that $f(V(T_{8k+2}) \geq 12k+4$ and we now obtain that the lower bound is equal to the upper. Therefore, $\gamma_R(T_{8k+2}) = 12k+4$.

For $n = 8k+6$ we have:

$f(V(T_{8k+6})) \geq \frac{3}{2}\cdot(8k+6) + \frac{1}{2} = 12k+9+\frac{1}{2}$. Since $f(V(T_{8k+6}))$ is an integer, we get that $f(V(T_{8k+6}) \geq 12k+10$ and we obtain that the lower bound is equal to the upper in this case. Therefore, we have proved that $\gamma_R(T_{8k+6}) = 12k+10$.

In the cases $n = 8k+1$ and $n = 8k-1$, we have that $\lceil \frac{3}{2}\cdot(8k+1)\rceil = \lceil 12\cdot k + \frac{3}{2}\rceil = 12k+2$, $\lceil \frac{3}{2}\cdot(8k-1)\rceil = \lceil 12\cdot k - \frac{3}{2}\rceil = 12k-1$, and we have a gap between lower and upper bounds equal to one.

In the remaining case, $n = 8k+4$, $\lceil \frac{3}{2}\cdot(8k+4)\rceil = 12k+6$, we have a gap that equals two.

Similarly as for R_n, by using the total enumeration, we obtained exact values $\gamma_R(T_5) = 8$, $\gamma_R(T_6) = 10$ and $\gamma_R(T_7) = 12$. □

Table 5: Roman domination coverage for T_n - part 1

n	$v \in V_0$	$V_2 \cap N(v)$	range of i	$v \in V_0$	$V_2 \cap N(v)$	range of i
$8k$	a_{8i+1}	$\{a_{8i}\}$	$0 \le i \le k-1$	a_{8i+2}	$\{a_{8i+3}\}$	$0 \le i \le k-1$
	a_{8i+4}	$\{a_{8i+3}\}$	$0 \le i \le k-1$	a_{8i+5}	$\{b_{8i+5}\}$	$0 \le i \le k-1$
	a_{8i+6}	$\{b_{8i+5}\}$	$0 \le i \le k-1$	a_{8i+7}	$\{a_{8i+8}\}$	$0 \le i \le k-1$
	b_{8i}	$\{a_{8i}\}$	$0 \le i \le k-1$	b_{8i+1}	$\{c_{8i+1}\}$	$0 \le i \le k-1$
	b_{8i+2}	$\{a_{8i+3}\}$	$0 \le i \le k-1$	b_{8i+3}	$\{a_{8i+3}\}$	$0 \le i \le k-1$
	b_{8i+4}	$\{b_{8i+5}\}$	$0 \le i \le k-1$	b_{8i+6}	$\{b_{8i+5}\}$	$0 \le i \le k-1$
	b_{8i+7}	$\{a_{8i+8}\}$	$0 \le i \le k-1$	c_{8i}	$\{c_{8i+1}\}$	$0 \le i \le k-1$
	c_{8i+2}	$\{c_{8i+1}\}$	$0 \le i \le k-1$	c_{8i+3}	$\{d_{8i+3}\}$	$0 \le i \le k-1$
	c_{8i+4}	$\{d_{8i+3}\}$	$0 \le i \le k-1$	c_{8i+5}	$\{b_{8i+5}\}$	$0 \le i \le k-1$
	c_{8i+6}	$\{d_{8i+6}\}$	$0 \le i \le k-1$	c_{8i+7}	$\{d_{8i+6}\}$	$0 \le i \le k-1$
	d_{8i}	$\{c_{8i+1}\}$	$0 \le i \le k-1$	d_{8i+1}	$\{c_{8i+1}\}$	$0 \le i \le k-1$
	d_{8i+2}	$\{d_{8i+3}\}$	$0 \le i \le k-1$	d_{8i+4}	$\{d_{8i+3}\}$	$0 \le i \le k-1$
	d_{8i+5}	$\{d_{8i+6}\}$	$0 \le i \le k-1$	d_{8i+7}	$\{d_{8i+6}\}$	$0 \le i \le k-1$
$8k-1$	a_{8i+1}	$\{a_{8i}\}$	$0 \le i \le k-1$	a_{8i+2}	$\{a_{8i+3}\}$	$0 \le i \le k-1$
	a_{8i+4}	$\{a_{8i+3}\}$	$0 \le i \le k-1$	a_{8i+5}	$\{b_{8i+5}\}$	$0 \le i \le k-1$
	a_{8i+6}	$\{b_{8i+5}\}$	$0 \le i \le k-2$	a_{8k-2}	$\{b_{8k-3}, a_0\}$	
	a_{8i+7}	$\{a_{8i+8}\}$	$0 \le i \le k-2$	b_{8i}	$\{a_{8i}\}$	$0 \le i \le k-1$
	b_{8i+1}	$\{c_{8i+1}\}$	$0 \le i \le k-1$	b_{8i+2}	$\{a_{8i+3}\}$	$0 \le i \le k-1$
	b_{8i+3}	$\{a_{8i+3}\}$	$0 \le i \le k-1$	b_{8i+4}	$\{b_{8i+5}\}$	$0 \le i \le k-1$
	b_{8i+6}	$\{b_{8i+5}\}$	$0 \le i \le k-2$	b_{8k-2}	$\{b_{8k-3}, a_0\}$	
	b_{8i+7}	$\{a_{8i+8}\}$	$0 \le i \le k-2$	c_{8i}	$\{c_{8i+1}\}$	$1 \le i \le k-1$
	c_0	$\{c_1, d_{8k-2}\}$		c_{8i+2}	$\{c_{8i+1}\}$	$0 \le i \le k-1$
	c_{8i+3}	$\{d_{8i+3}\}$	$0 \le i \le k-1$	c_{8i+4}	$\{d_{8i+3}\}$	$0 \le i \le k-1$
	c_{8i+5}	$\{b_{8i+5}\}$	$0 \le i \le k-1$	c_{8i+6}	$\{d_{8i+6}\}$	$0 \le i \le k-1$
	c_{8i+7}	$\{d_{8i+6}\}$	$0 \le i \le k-2$	d_{8i}	$\{c_{8i+1}\}$	$1 \le i \le k-1$
	d_0	$\{c_1, d_{8k-2}\}$		d_{8i+1}	$\{c_{8i+1}\}$	$0 \le i \le k-1$
	d_{8i+2}	$\{d_{8i+3}\}$	$0 \le i \le k-1$	d_{8i+4}	$\{d_{8i+3}\}$	$0 \le i \le k-1$
	d_{8i+5}	$\{d_{8i+6}\}$	$0 \le i \le k-1$	d_{8i+7}	$\{d_{8i+6}\}$	$0 \le i \le k-2$
$8k+1$	a_{8i}	$\{a_{8i+1}\}$	$0 \le i \le k-1$	a_{8i+2}	$\{a_{8i+1}\}$	$0 \le i \le k-1$
	a_{8i+3}	$\{b_{8i+3}\}$	$0 \le i \le k-1$	a_{8i+4}	$\{b_{8i+3}\}$	$0 \le i \le k-1$
	a_{8i+5}	$\{a_{8i+6}\}$	$0 \le i \le k-1$	a_{8i+7}	$\{a_{8i+6}\}$	$0 \le i \le k-1$
	b_{8i}	$\{a_{8i+1}\}$	$0 \le i \le k-1$	b_{8k}	$\{c_{8k}\}$	
	b_{8i+1}	$\{a_{8i+1}\}$	$0 \le i \le k-1$	b_{8i+2}	$\{b_{8i+3}\}$	$0 \le i \le k-1$
	b_{8i+4}	$\{b_{8i+3}\}$	$0 \le i \le k-1$	b_{8i+5}	$\{a_{8i+6}\}$	$0 \le i \le k-1$
	b_{8i+6}	$\{a_{8i+6}\}$	$0 \le i \le k-1$	b_{8i+7}	$\{c_{8i+7}\}$	$0 \le i \le k-1$
	c_0	$\{c_{8k}\}$		c_{8i}	$\{c_{8i-1}\}$	$1 \le i \le k-1$
	c_{8i+1}	$\{d_{8i+1}\}$	$0 \le i \le k-1$	c_{8i+2}	$\{d_{8i+1}\}$	$0 \le i \le k-1$
	c_{8i+3}	$\{b_{8i+3}\}$	$0 \le i \le k-1$	c_{8i+4}	$\{d_{8i+4}\}$	$0 \le i \le k-1$
	c_{8i+5}	$\{d_{8i+4}\}$	$0 \le i \le k-1$	c_{8i+6}	$\{c_{8i+7}\}$	$0 \le i \le k-1$
	d_{8i}	$\{d_{8i+1}\}$	$0 \le i \le k-1$	d_{8k}	$\{c_{8k}\}$	
	d_{8i+2}	$\{d_{8i+1}\}$	$0 \le i \le k-1$	d_{8i+3}	$\{d_{8i+4}\}$	$0 \le i \le k-1$
	d_{8i+5}	$\{d_{8i+4}\}$	$0 \le i \le k-1$	d_{8i+6}	$\{c_{8i+7}\}$	$0 \le i \le k-1$
	d_{8i+7}	$\{c_{8i+7}\}$	$0 \le i \le k-2$	d_{8k-1}	$\{c_{8k-1}, c_{8k}\}$	

Table 6: Roman domination coverage for T_n - part 2

n	$v \in V_0$	$V_2 \cap N(v)$	range of i	$v \in V_0$	$V_2 \cap N(v)$	range of i
$8k+2$	a_0	$\{b_{8k+1}\}$		a_{8i}	$\{a_{8i-1}\}$	$1 \le i \le k$
	a_{8i+1}	$\{a_{8i+2}\}$	$0 \le i \le k-1$	a_{8k+1}	$\{b_{8k+1}\}$	
	a_{8i+3}	$\{a_{8i+2}\}$	$0 \le i \le k-1$	a_{8i+4}	$\{b_{8i+4}\}$	$0 \le i \le k-1$
	a_{8i+5}	$\{b_{8i+4}\}$	$0 \le i \le k-1$	a_{8i+6}	$\{a_{8i+7}\}$	$0 \le i \le k-1$
	b_0	$\{b_{8k+1}, c_0\}$		b_{8i}	$\{c_{8i}\}$	$1 \le i \le k-1$
	b_{8k}	$\{b_{8k+1}, c_{8k}\}$		b_{8i+1}	$\{a_{8i+2}\}$	$0 \le i \le k-1$
	b_{8i+2}	$\{a_{8i+2}\}$	$0 \le i \le k-1$	b_{8i+3}	$\{b_{8i+4}\}$	$0 \le i \le k-1$
	b_{8i+5}	$\{b_{8i+4}\}$	$0 \le i \le k-1$	b_{8i+6}	$\{a_{8i+7}\}$	$0 \le i \le k-1$
	b_{8i+7}	$\{a_{8i+7}\}$	$0 \le i \le k-1$	c_{8i+1}	$\{c_{8i}\}$	$0 \le i \le k-1$
	c_{8k+1}	$\{c_0, c_{8k}, b_{8k+1}\}$		c_{8i+2}	$\{d_{8i+2}\}$	$0 \le i \le k-1$
	c_{8i+3}	$\{d_{8i+2}\}$	$0 \le i \le k-1$	c_{8i+4}	$\{b_{8i+4}\}$	$0 \le i \le k-1$
	c_{8i+5}	$\{d_{8i+5}\}$	$0 \le i \le k-1$	c_{8i+6}	$\{d_{8i+5}\}$	$0 \le i \le k-1$
	c_{8i+7}	$\{c_{8i+8}\}$	$0 \le i \le k-1$	d_{8i}	$\{c_{8i}\}$	$0 \le i \le k$
	d_{8i+1}	$\{d_{8i+2}\}$	$0 \le i \le k-1$	d_{8k+1}	$\{c_0\}$	
	d_{8i+3}	$\{d_{8i+2}\}$	$0 \le i \le k-1$	d_{8i+4}	$\{d_{8i+5}\}$	$0 \le i \le k-1$
	d_{8i+6}	$\{d_{8i+5}\}$	$0 \le i \le k-1$	d_{8i+7}	$\{c_{8i+8}\}$	$0 \le i \le k-1$
$8k+3$	a_0	$\{b_{8k+2}\}$		a_{8i+1}	$\{a_{8i}\}$	$1 \le i \le k$
	a_{8i+2}	$\{b_{8i+2}\}$	$0 \le i \le k$	a_{8i+3}	$\{b_{8i+2}\}$	$0 \le i \le k-1$
	a_{8i+4}	$\{a_{8i+5}\}$	$0 \le i \le k-1$	a_{8i+6}	$\{a_{8i+5}\}$	$0 \le i \le k-1$
	a_{8i+7}	$\{a_{8i+8}\}$	$0 \le i \le k-1$	b_{8i}	$\{a_{8i}\}$	$1 \le i \le k$
	b_0	$\{b_{8k+2}\}$		b_{8i+1}	$\{b_{8i+2}\}$	$0 \le i \le k$
	b_{8i+3}	$\{b_{8i+2}\}$	$0 \le i \le k-1$	b_{8i+4}	$\{a_{8i+5}\}$	$0 \le i \le k-1$
	b_{8i+5}	$\{a_{8i+5}\}$	$0 \le i \le k-1$	b_{8i+6}	$\{c_{8i+6}\}$	$0 \le i \le k-1$
	b_{8i+7}	$\{a_{8i+8}\}$	$0 \le i \le k-1$	c_{8i}	$\{d_{8i}\}$	$0 \le i \le k$
	c_{8i+1}	$\{d_{8i}\}$	$0 \le i \le k$	c_{8i+2}	$\{b_{8i+2}\}$	$0 \le i \le k$
	c_{8i+3}	$\{d_{8i+3}\}$	$0 \le i \le k-1$	c_{8i+4}	$\{d_{8i+3}\}$	$0 \le i \le k-1$
	c_{8i+5}	$\{c_{8i+6}\}$	$0 \le i \le k-1$	c_{8i+7}	$\{c_{8i+6}\}$	$0 \le i \le k-1$
	d_{8i+1}	$\{d_{8i}\}$	$0 \le i \le k$	d_{8i+2}	$\{d_{8i+3}\}$	$0 \le i \le k$
	d_{8i+4}	$\{d_{8i+3}\}$	$0 \le i \le k-1$	d_{8i+5}	$\{c_{8i+6}\}$	$0 \le i \le k-1$
	d_{8i+6}	$\{c_{8i+6}\}$	$0 \le i \le k-1$	d_{8i+7}	$\{d_{8i+8}\}$	$0 \le i \le k-1$
$8k+4$	a_0	$\{a_{8k+3}\}$		a_{8i}	$\{a_{8i-1}\}$	$1 \le i \le k-1$
	a_{8k}	$\{b_{8k}\}$		a_{8i+1}	$\{a_{8i+2}\}$	$0 \le i \le k-1$
	a_{8k+1}	$\{b_{8k}\}$		a_{8k+2}	$\{a_{8k+3}\}$	
	a_{8i+3}	$\{a_{8i+2}\}$	$0 \le i \le k-1$	a_{8i+4}	$\{b_{8i+4}\}$	$0 \le i \le k-1$
	a_{8i+5}	$\{b_{8i+4}\}$	$0 \le i \le k-2$	a_{8k-3}	$\{a_{8k-2}, b_{8k-4}\}$	
	a_{8i+6}	$\{a_{8i+7}\}$	$0 \le i \le k-2$	a_{8k-1}	$\{a_{8k-2}\}$	
	b_{8i}	$\{c_{8i}\}$	$0 \le i \le k-1$	b_{8i+1}	$\{a_{8i+2}\}$	$0 \le i \le k-1$
	b_{8k+1}	$\{b_{8k}\}$		b_{8i+2}	$\{a_{8i+2}\}$	$0 \le i \le k-1$
	b_{8k+2}	$\{a_{8k+3}\}$		b_{8i+3}	$\{b_{8i+4}\}$	$0 \le i \le k-1$
	b_{8k+3}	$\{a_{8k+3}\}$		b_{8i+5}	$\{b_{8i+4}\}$	$0 \le i \le k-1$
	b_{8i+6}	$\{a_{8i+7}\}$	$0 \le i \le k-2$	b_{8k-2}	$\{a_{8k-2}\}$	
	b_{8i+7}	$\{a_{8i+7}\}$	$0 \le i \le k-2$	b_{8k-1}	$\{b_{8k}\}$	
	c_{8i+1}	$\{c_{8i}\}$	$0 \le i \le k$	c_{8i+2}	$\{d_{8i+2}\}$	$0 \le i \le k$
	c_{8i+3}	$\{d_{8i+2}\}$	$0 \le i \le k-1$	c_{8k+3}	$\{c_0, d_{8k+2}\}$	
	c_{8i+4}	$\{b_{8i+4}\}$	$0 \le i \le k-1$	c_{8i+5}	$\{d_{8i+5}\}$	$0 \le i \le k-1$
	c_{8i+6}	$\{d_{8i+5}\}$	$0 \le i \le k-1$	c_{8i+7}	$\{c_{8i+8}\}$	$0 \le i \le k-1$
	d_{8i}	$\{c_{8i}\}$	$0 \le i \le k$	d_{8i+1}	$\{d_{8i+2}\}$	$0 \le i \le k$
	d_{8i+3}	$\{d_{8i+2}\}$	$0 \le i \le k-1$	d_{8k+3}	$\{d_{8k+2}, c_0\}$	
	d_{8i+4}	$\{d_{8i+5}\}$	$0 \le i \le k-1$	d_{8i+6}	$\{d_{8i+5}\}$	$0 \le i \le k-1$
	d_{8i+7}	$\{c_{8i}\}$	$0 \le i \le k-1$			

Table 7: Roman domination coverage for T_n - part 3

n	$v \in V_0$	$V_2 \cap N(v)$	range of i	$v \in V_0$	$V_2 \cap N(v)$	range of i
$8k+5$	a_{8i+1}	$\{a_{8i}\}$	$0 \leq i \leq k$	a_{8i+2}	$\{b_{8i+2}\}$	$0 \leq i \leq k$
	a_{8i+3}	$\{b_{8i+2}\}$	$0 \leq i \leq k$	a_{8i+4}	$\{a_{8i+5}\}$	$0 \leq i \leq k$
	a_{8i+6}	$\{a_{8i+5}\}$	$0 \leq i \leq k-1$	a_{8i+7}	$\{a_{8i+8}\}$	$0 \leq i \leq k-1$
	b_{8i}	$\{a_{8i}\}$	$0 \leq i \leq k$	b_{8i+1}	$\{b_{8i+2}\}$	$0 \leq i \leq k$
	b_{8i+3}	$\{a_{8i+2}\}$	$0 \leq i \leq k$	b_{8i+4}	$\{a_{8i+5}\}$	$0 \leq i \leq k$
	b_{8i+5}	$\{a_{8i+5}\}$	$0 \leq i \leq k-1$	b_{8i+6}	$\{c_{8i+6}\}$	$0 \leq i \leq k-1$
	b_{8i+7}	$\{a_{8i+8}\}$	$0 \leq i \leq k-1$	c_{8i}	$\{d_{8i}\}$	$0 \leq i \leq k$
	c_{8i+1}	$\{d_{8i}\}$	$0 \leq i \leq k$	c_{8i+2}	$\{b_{8i+2}\}$	$0 \leq i \leq k$
	c_{8i+3}	$\{d_{8i+3}\}$	$0 \leq i \leq k$	c_{8i+4}	$\{d_{8i+3}\}$	$0 \leq i \leq k$
	c_{8i+5}	$\{c_{8i+6}\}$	$0 \leq i \leq k-1$	c_{8i+7}	$\{c_{8i+6}\}$	$0 \leq i \leq k-1$
	d_{8i+1}	$\{d_{8i}\}$	$0 \leq i \leq k$	d_{8i+2}	$\{d_{8i+3}\}$	$0 \leq i \leq k$
	d_{8i+4}	$\{d_{8i+3}\}$	$0 \leq i \leq k$	d_{8i+5}	$\{c_{8i+6}\}$	$0 \leq i \leq k-1$
	d_{8i+6}	$\{c_{8i+6}\}$	$0 \leq i \leq k-1$	d_{8i+7}	$\{d_{8i+8}\}$	$0 \leq i \leq k-1$
$8k+6$	a_{8i}	$\{b_{8i}\}$	$0 \leq i \leq k$	a_{8i+1}	$\{b_{8i}\}$	$0 \leq i \leq k$
	a_{8i+2}	$\{a_{8i+3}\}$	$0 \leq i \leq k$	a_{8i+4}	$\{a_{8i+3}\}$	$0 \leq i \leq k$
	a_{8i+5}	$\{a_{8i+6}\}$	$0 \leq i \leq k-1$	a_{8i+7}	$\{a_{8i+6}\}$	$0 \leq i \leq k-1$
	b_{8i+1}	$\{b_{8i}\}$	$0 \leq i \leq k$	b_{8i+2}	$\{a_{8i+3}\}$	$0 \leq i \leq k$
	b_{8i+3}	$\{a_{8i+3}\}$	$0 \leq i \leq k$	b_{8i+4}	$\{c_{8i+4}\}$	$0 \leq i \leq k$
	b_{8i+5}	$\{a_{8i+6}\}$	$0 \leq i \leq k-1$	b_{8k+5}	$\{b_0\}$	
	b_{8i+6}	$\{a_{8i+6}\}$	$0 \leq i \leq k-1$	b_{8i+7}	$\{b_{8i+8}\}$	$0 \leq i \leq k-1$
	c_{8i}	$\{b_{8i}\}$	$0 \leq i \leq k$	c_{8i+1}	$\{d_{8i+1}\}$	$0 \leq i \leq k$
	c_{8i+2}	$\{d_{8i+1}\}$	$0 \leq i \leq k$	c_{8i+3}	$\{c_{8i+4}\}$	$0 \leq i \leq k$
	c_{8i+5}	$\{c_{8i+4}\}$	$0 \leq i \leq k$	c_{8i+6}	$\{d_{8i+6}\}$	$0 \leq i \leq k-1$
	c_{8i+7}	$\{d_{8i+6}\}$	$0 \leq i \leq k-1$	d_{8i}	$\{d_{8i+1}\}$	$0 \leq i \leq k$
	d_{8i+2}	$\{d_{8i+1}\}$	$0 \leq i \leq k$	d_{8i+3}	$\{c_{8i+4}\}$	$0 \leq i \leq k$
	d_{8i+4}	$\{c_{8i+4}\}$	$0 \leq i \leq k$	d_{8i+5}	$\{d_{8i+6}\}$	$0 \leq i \leq k-1$
	d_{8i+7}	$\{d_{8i+6}\}$	$0 \leq i \leq k-1$			

5. CONCLUSIONS

In this paper we have studied Roman domination problem of convex polytopes A_n, R_n and T_n. We found closed formulas for all $A_n,\ n \geq 5$, for $R_n, n \geq 5$, if n is not congruent to 2 (mod 3) and $T_n,\ n \geq 5$, where n is congruent to x (mod 8), where $x \in \{0, 2, 3, 5, 6\}$. For the remaining cases, i.e. R_{3k+2}, T_{8k+1}, T_{8k+4} and T_{8k-1}, we improved the lower bound from the literature and found new upper and lower bounds. We proved that the gap between the bounds is 1 in all cases, with the exception of the case of T_{8k+4}, where the gap is 2.

Future work can be continued in terms of determination of Roman domination numbers for some other challenging classes of graphs. The other promising direction for future work is solving some other graph invariants on convex polytopes.

REFERENCES

1. M. Bača: *Labelings of 2 classes of convex polytopes.* Util. Math., **34** (1988), 24–31.
2. M. Bača: *On magic labellings of convex polytopes.* Ann. Discrete Math., **51** (1992), 13–16.
3. S. Bermudo, H. Fernau, J. M. Sigarreta: *The differential and the Roman domination number of a graph.* Ann. Discrete Math., (2014), 155–171.
4. A. P. Burger, A. P. De Villiers, J. H. Van Vuuren: *A binary programming approach towards achieving effective graph protection.* Proc. of 2013 ORSSA Ann. Conf., (2013), 19–30.
5. E. W. Chambers, B. Kinnersley, N. Prince, D. B. West: *Extremal problems for Roman domination.* SIAM J. Discret. Math., **23** (2009), 1575–1586.
6. M. Chellali, N. J. Rad: *On 2-rainbow domination and Roman domination in graphs.* Australas J. Combin., **56** (2013), 85–93.
7. E. J. Cockayne, P. A. Dreyer, S. M. Hedetniemi, S. T. Hedetniemi: *Roman domination in graphs.* Discrete Math., **278** (2004), 11–22.
8. V. Currò: *The Roman domination problem on grid graphs.* Università di Catania, 2014.
9. P. A. Dreyer: *Applications and variations of domination in graphs.* Rutgers University NJ, USA, 2000.
10. O. Favaron, H. Karami, R. Khoeilar, S. M. Sheikholeslami: *On the Roman domination number of a graph.* Discrete Math., **309** (2009), 3447–3451.
11. S. T. Hedetniemi, R. R. Rubalcaba, P. J. Slater, M. Walsh: *Few compare to the great Roman Empire.* Congr. Numer., **217** (2013), 129–136.
12. M. A. Henning: *A characterization of Roman trees.* Discuss. Math. Graph Theory, **22** (2002), 325–334.
13. M. Imran, S. A. U. H. Bokhary, A. Q. Baig: *On families of convex polytopes with constant metric dimension.* Comput. Math. Appl., **60** (2010), 2629–2638.
14. M. Ivanović: *Improved mixed integer linear programing formulations for Roman domination problem.* Publ. Inst. Math., **99** (2016), 51–58.
15. A. Klobučar, I. Puljić: *Some results for Roman domination number on cardinal product of paths and cycles.* Kragujev. J. Math., **38** (2014), 83–94.
16. A. Klobučar, I. Puljić: *Roman domination number on cardinal product of paths and cycles.* Croat. Oper. Res. Rev., **6** (2015), 71–78.
17. M. Liedloff, T. Kloks, J. Liu, S. L. Peng: *Roman domination over some graph classes.* Int. Workshop Graph-Theoretic Concepts Comput. Sci., (2005), 103–114.
18. C. H. Liu, G. J. Chang: *Roman domination on 2-connected graphs.* SIAM J. Discret. Math., **26** (2012), 193–205.
19. C. H. Liu, G. J. Chang: *Roman domination on strongly chordal graphs.* J. Comb. Optim., **26** (2013), 608–619.
20. C. H. Liu, G. J. Chang: *Upper bounds on Roman domination numbers of graphs.* Discrete Math., **312** (2012), 1386–1391.

21. B. P. Mobaraky, S. M. Sheikholeslami: *Bounds on Roman domination numbers of graphs.* Mat. Vesn., **60** (2008), 247–253.

22. F. Ramezani, E. D. Rodriguez-Bazan, J. A. Rodriguez-Velazquez: *On the Roman domination number of generalized Sierpinski graphs.* arXiv preprint arXiv:1605.06918, 2016.

23. C. S. ReVelle, K. E. Rosing: *Defendens Imperium Romanum: a classical problem in military strategy.* Am. Math. Mon., **107** (2000), 585–594.

24. W. Shang, X. Hu: *The Roman domination problem in unit disk graphs.* Int. Conf. Comput. Sci., (2007), 305–312.

25. S. M. Sheikholeslami, L. Volkmann: *The Roman domination number of a digraph.* Acta Univ. Apulensis Math. Inform., **27** (2011), 77–86.

26. X. Song, X. Wang: *Roman domination number and domination number of a tree.* Chinese Quart. J. Math., **21** (2006), 358.

27. I. Stewart: *Defend the Roman Empire!.* Sci. Am., **281** (1999), 136–138.

28. H. Wang, X. Xu, Y. Yang, C. Ji: *Roman domination number of eneralized Petersen graphs $P(n, 2)$.* arXiv preprint arXiv:1103.2419, 2011.

29. H. M. Xing, X. Chen, X. G. Chen: *A note on Roman domination in graphs.* Discrete Math., **306** (2006), 3338–3340.

30. F. Xueliang, Y. Yuansheng, J. Baoqi: *Roman domination in regular graphs.* Discrete Math., **309** (2009), 1528–1537.

31. I. G. Yero, D. Kuziak, A. R. Aguilar: *Coloring, location and domination of corona graphs.* Aequ. Math., **86** (2013), 1–21.

32. Z. Zhang, Z. Shao, X. Xu: *On the Roman domination numbers of generalized Petersen graphs.* J. Comb. Math. Comb. Comput., **89** (2014), 311–320.

33. V. Zverovich, A. Poghosyan: *On Roman, global and restrained domination in graphs.* Graphs Combin., **27** (2011), 755–768.

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