

## SIXTH ORDER DIFFERENTIAL OPERATORS WITH EIGENVALUE DEPENDENT BOUNDARY CONDITIONS

*Manfred Möller, Bertin Zinsou*

We consider eigenvalue problems for sixth-order ordinary differential equations. Such differential equations occur in mathematical models of vibrations of curved arches. With suitably chosen eigenvalue dependent boundary conditions, the problem is realized by a quadratic operator pencil. It is shown that the operators in this pencil are self-adjoint, and that the spectrum of the pencil consists of eigenvalues of finite multiplicity in the closed upper half-plane, except for finitely many eigenvalues on the negative imaginary axis.

### 1. INTRODUCTION

The spectral theory of higher order ordinary linear differential operators, in particular those with eigenvalue parameter dependent boundary conditions, is much less investigated and understood than the spectral theory of Sturm-Liouville operators. Like the spectral theory of Sturm-Liouville operators, (quasi-)regular and singular problems of higher order differential operators are distinguished by their spectral properties. Amongst known fundamental results are characterizations of symmetry of the minimal operator for  $n$ th order differential expressions, see [20, 32]. General characterizations of self-adjoint boundary conditions have been obtained in [30, 31]. Various aspects of higher order differential operators whose boundary conditions depend on the eigenvalue parameter, including spectral asymptotics and basis properties, have been investigated in [10, 11, 17, 28].

In order to motivate the subject of this paper we recall that the generalized Regge problem is realised by a second order differential operator which depends

---

2010 Mathematics Subject Classification. 34B07, 34B09, 34L05.

Keywords and Phrases. Sixth order differential operator, self-adjoint, boundary condition, eigenvalue, quadratic operator pencil.

quadratically on the eigenvalue parameter and which has eigenvalue parameter dependent boundary conditions, see [27]. The particular feature of the Regge problem is that the coefficient operators of the corresponding quadratic operator pencil are self-adjoint, and it is shown that this gives some a priori knowledge about the location of the spectrum. In [19] this approach has been extended to a fourth order differential equation describing small transversal vibrations of a homogeneous beam compressed or stretched by a force  $g$ . Separation of variables leads to an ordinary fourth order differential equation with eigenvalue parameter dependent boundary conditions, where the differential equation depends quadratically on the eigenvalue parameter. For the same differential operator as in [19], we have investigated a more general class of eigenvalue parameter dependent boundary conditions. Necessary and sufficient conditions for the associated operator pencil to consist of self-adjoint operators have been obtained in [21], while in [22, 23] we have continued the work in the direction of [19] to find the asymptotic distribution of eigenvalues for boundary conditions which lead to self-adjoint operator representations. In this paper we start to extend this investigation to a corresponding problem for a sixth order differential equation. In a forthcoming paper we will investigate the asymptotic distribution of the eigenvalues of these operator pencils.

The general spectral theory of sixth-order differential operators is (almost) unknown. In this paper, we will therefore investigate the location of the spectrum of quadratic self-adjoint sixth-order differential operator pencils. Numerical methods and other techniques for the investigation of sixth-order boundary value problems can be found in [2, 5, 7, 15, 24, 25].

Before introducing our operator pencil we will briefly discuss physical configurations which are described by sixth order linear differential equations. A quite extensive literature deals with curved arches. The corresponding mathematical models give a sixth order differential equation if all but one independent variables are eliminated, see e.g. WUEST [33], WALKING [29], AURICELLI and DE ROSA [1]. Here, one is often interested in the stability of the underlying system, which, in general, is determined by the location of the smallest eigenvalue of the differential equation. We observe that, as above, sixth order differential equations occur when one eliminates two of the three unknown functions in systems of second order differential equations. But in particular for numerical methods it might be more convenient to use systems of differential equations, first order systems as well as higher order systems. For this and numerical results, we refer the reader to [8, 12, 13, 14, 26] and related publications.

WUEST [33, page 266] derived a model for beams and pipes whose central axis is a circular arc between the angles  $\varphi = 0$  and  $\varphi = \varphi^*$ . The resulting differential equation for the tangential movement  $v$  of the pipe is

$$\frac{\partial^6 v}{\partial \varphi^6} + 2 \frac{\partial^4 v}{\partial \varphi^4} + \frac{\partial^2 v}{\partial \varphi^2} = c \frac{\partial^2}{\partial t^2} \left( - \frac{\partial^2 v}{\partial \varphi^2} + v \right)$$

with a constant  $c$  depending on the geometry and the physical properties of the configuration. The end at  $\varphi = 0$  is clamped, whereas the other end is free. Using

separation of variables, both ends give three boundary conditions each, where one of the boundary conditions at the free end depends quadratically on the eigenvalue parameter. The same differential equation has been considered by AURICELLO and DE ROSA in [1, page 435], where also hinged end-points are considered. Separation of variables leads to a sixth order ordinary differential equation

$$y^{(6)} + 2y^{(4)} + y'' = -c\lambda^2(-y'' + y),$$

and the boundary conditions at a fixed end are such that  $y$ ,  $y'$  and  $y''$  are zero there, see [33, page 267] and [1, page 437], whereas the boundary conditions at the hinged end are such that  $y$ ,  $y'$  and  $y' + y'''$  are zero, see [1, page 437]. A more general configuration can lead to a differential equation where the eigenvalue occurs as  $\lambda^2$  and as  $\lambda^4$  or where the  $\lambda$ -dependent part contains a fourth order derivative, see FEDERHOFER [4, page 279] and WALKING [29, page 435].

Keeping in mind that in [19] the hinged undamped condition  $y''(a) = 0$  leads to the boundary condition  $y''(a) + i\alpha\lambda y'(a) = 0$ , we will replace the hinged boundary condition  $y'''(a) = 0$  by the boundary condition  $y'''(a) + i\alpha\lambda y''(a) = 0$ . We can now describe the problem which will be considered in this paper. On the interval  $[0, a]$ , the boundary eigenvalue problem is defined by the differential equation

$$(1) \quad -y^{(6)} + (g_2 y'')'' - (g_1 y')' + g_0 y = \lambda^2((h_2 y'')'' - (h_1 y')' + h_0 y)$$

and the boundary conditions

$$(2) \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0,$$

$$(3) \quad y(a) = 0, \quad y'(a) = 0, \quad y'''(a) + i\alpha\lambda y''(a) = 0,$$

where  $g_j, h_j \in C^j[0, a]$  are real-valued with  $h_j \geq 0$  and  $h_0 + h_1 + h_2 > 0$ ,  $a > 0$  and  $\alpha > 0$ .

We associate a quadratic operator pencil

$$(4) \quad L(\lambda, \alpha) = \lambda^2 M - i\alpha\lambda K - A$$

in the space  $L_2(0, a) \oplus \mathbb{C}$  with this problem, where  $K$  is the operator with domain  $\mathcal{D}(K) = L_2(0, a) \oplus \mathbb{C}$  given by

$$(5) \quad K = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The operator  $A$  acting in  $L_2(0, a) \oplus \mathbb{C}$  with domain

$$(6) \quad \mathcal{D}(A) = \left\{ \tilde{y} = \begin{pmatrix} y \\ y''(a) \end{pmatrix} : y \in W_6^2(0, a), y(0) = y'(0) = y''(0) = y(a) = y'(a) = 0 \right\},$$

is given by

$$(7) \quad A\tilde{y} = \begin{pmatrix} -y^{(6)} + (g_2 y'')'' - (g_1 y')' + g_0 y \\ y'''(a) \end{pmatrix} \text{ for } \tilde{y} \in \mathcal{D}(A),$$

where  $W_6^2(0, a)$  is the Sobolev space of order 6 associated with  $L_2(0, a)$ . Finally,  $M$  will be the Friedrichs extension of the operator  $M_0$  which is defined by  $\mathcal{D}(M_0) = \mathcal{D}(A)$  and

$$(8) \quad M_0 \tilde{y} = \begin{pmatrix} (h_2 y''')'' - (h_1 y')' + h_0 y \\ 0 \end{pmatrix} \text{ for } \tilde{y} \in \mathcal{D}(A),$$

see Corollary 2.4.

We are going to use that  $L_2(0, a) \oplus \mathbb{C}$  is a Hilbert space with respect to the inner product

$$\langle \tilde{v}, \tilde{w} \rangle = (v, w) + c\bar{d} = \int_0^a v(x)\overline{w(x)} dx + c\bar{d}, \quad \tilde{v} = \begin{pmatrix} v \\ c \end{pmatrix}, \tilde{w} = \begin{pmatrix} w \\ d \end{pmatrix} \in L_2(0, a).$$

The domain of the operator pencil  $L(\lambda, \alpha)$  is  $\mathcal{D}(L(\lambda, \alpha)) = \mathcal{D}(A)$ , and for  $\tilde{y} \in \mathcal{D}(A)$ ,  $L(\lambda, \alpha)\tilde{y} = 0$  holds if and only if the differential equation (1) and the boundary conditions (2) and (3) hold. Hence the operator pencil (4) describes the eigenvalue problem (1)-(3).

We show in Section 2 that the operator  $A$  is self-adjoint and that the operator  $M_0$  has a self-adjoint extension. In Section 3 we show that the spectrum of the pencil  $L(\cdot, \alpha)$  consists of isolated eigenvalues of finite multiplicity. The eigenvalues are located in the closed upper half-plane, with the possible exception of finitely many eigenvalues on the negative imaginary axis, inside an interval  $[0, -i\nu^{1/2}]$ , where  $\nu$  is independent of  $\alpha$ .

## 2. SELF-ADJOINTNESS OF THE PENCIL $L$

We are going to show that  $L(\lambda, \alpha)$  is a self-adjoint operator pencil, that is, the operators  $A$ ,  $K$ ,  $M$  are self-adjoint. Clearly,  $K$  is a self-adjoint bounded operator in  $L_2(0, a) \oplus \mathbb{C}$ .

**Proposition 2.1.** *The operator  $A$  is symmetric.*

**Proof.** We first show that  $A$  is densely defined. Let  $\tilde{w} = \begin{pmatrix} w \\ c \end{pmatrix} \in L_2(0, a) \oplus \mathbb{C}$  such that  $\langle \tilde{y}, \tilde{w} \rangle = 0$  for all  $\tilde{y} \in \mathcal{D}(A)$ , i. e.,

$$\int_0^a y(x)\overline{w(x)} dx + y''(a)\bar{c} = 0.$$

If  $y \in C_0^\infty(0, a)$ , then  $y''(a) = 0$  and  $\tilde{y} = \begin{pmatrix} y \\ 0 \end{pmatrix} \in \mathcal{D}(A)$ , where

$$\int_0^a y(x)\overline{w(x)} dx = 0 \quad \text{for all } y \in C_0^\infty(0, a).$$

It follows that  $w = 0$ .

The polynomial  $y(x) = x^3(x - a)^2$  satisfies  $y(0) = y'(0) = y''(0) = y(a) = y'(a) = 0$  and  $y''(a) = 2a^3 \neq 0$ . Hence

$$\tilde{y} = \begin{pmatrix} y \\ y''(a) \end{pmatrix} \in \mathcal{D}(A).$$

Since  $w = 0$ , it follows that  $0 = \langle \tilde{y}, \tilde{w} \rangle = y''(a)\bar{c} = 2a^3\bar{c}$  and therefore  $c = 0$ , showing that  $\tilde{w} = 0$ . Hence  $\mathcal{D}(A)^\perp = \{0\}$ , that is,  $A$  is densely defined.

For  $\tilde{y}, \tilde{z} \in \mathcal{D}(A)$  we have

$$\begin{aligned} \langle A\tilde{y}, \tilde{z} \rangle &= - \int_0^a y^{(6)}(x)\overline{z(x)} \, dx + \int_0^a (g_2 y'')''(x)\overline{z(x)} \, dx \\ &\quad - \int_0^a (g_1 y')'(x)\overline{z(x)} \, dx + \int_0^a (g_0 y)(x)\overline{z(x)} \, dx + y'''(a)\overline{z''(a)}. \end{aligned}$$

Integrating by parts and observing the boundary conditions satisfied by elements in  $\mathcal{D}(A)$ , it follows that

$$\begin{aligned} - \int_0^a y^{(6)}(x)\overline{z(x)} \, dx &= \int_0^a y'''(x)\overline{z'''(x)} \, dx - y'''(a)\overline{z''(a)}, \\ \int_0^a (g_2 y'')''(x)\overline{z(x)} \, dx &= \int_0^a g_2(x)y''(x)\overline{z''(x)} \, dx, \\ - \int_0^a (g_1 y')'(x)\overline{z(x)} \, dx &= \int_0^a g_1(x)y'(x)\overline{z'(x)} \, dx. \end{aligned}$$

Hence

$$(9) \quad \begin{aligned} \langle A\tilde{y}, \tilde{z} \rangle &= \int_0^a y'''(x)\overline{z'''(x)} \, dx + \int_0^a g_2(x)y''(x)\overline{z''(x)} \, dx \\ &\quad + \int_0^a g_1(x)y'(x)\overline{z'(x)} \, dx + \int_0^a g_0(x)y(x)\overline{z(x)} \, dx, \end{aligned}$$

which shows that

$$\langle A\tilde{y}, \tilde{z} \rangle = \overline{\langle A\tilde{z}, \tilde{y} \rangle} = \langle \tilde{y}, A\tilde{z} \rangle.$$

Since  $A$  is densely defined this shows that  $A$  is symmetric.

**Theorem 2.2.** *The operator  $A$  is self-adjoint.*

**Proof.** Since the operator  $A$  is symmetric, it is sufficient to show that its deficiency indices are zero, that is, that  $A - \mu I$  is surjective for all  $\mu \in \mathbb{C} \setminus \mathbb{R}$ . To show this, we will consider the operators  $B_0$  and  $B_3$  in  $L_2(0, a)$  defined by

$$\begin{aligned} \mathcal{D}(B_0) &= \{y \in W_6^2(0, a) : y(0) = y'(0) = y''(0) = y(a) = y'(a) = 0\}, \\ \mathcal{D}(B_3) &= \{y \in \mathcal{D}(B_0) : y'''(a) = 0\}, \\ B_0 y &= -y^{(6)} + (g_2 y'')'' - (g_1 y')' + g_0 y \text{ for } y \in \mathcal{D}(B_0), \\ B_3 y &= B_0 y \text{ for } y \in \mathcal{D}(B_3). \end{aligned}$$

It is well-known that the operator  $B_3$  is self-adjoint, see e. g. [20, Theorem 2.4]. Hence the operator  $B_3 - \mu I$  is bijective. Let  $f \in L_2(0, a)$  and  $c \in \mathbb{C}$ . Then there is  $u \in \mathcal{D}(B_3)$  such that  $(B_3 - \mu I)u = f$ . Also, since  $B_0$  is a proper extension of  $B_3$ ,  $B_0 - \mu I$  is not injective, so that there is a nontrivial  $v \in \mathcal{D}(B_0)$  such that  $(B_0 - \mu I)v = 0$ . Because  $B_3 - \mu I$  is injective,  $v \notin \mathcal{D}(B_3)$ , which implies  $v'''(a) \neq 0$ . Then  $y = u + \gamma v$  satisfies  $(B_0 - \mu I)y = f$  for all  $\gamma \in \mathbb{C}$  and therefore

$$(A - \mu I)\tilde{y} = \begin{pmatrix} f \\ \gamma v'''(a) \end{pmatrix} = \begin{pmatrix} f \\ c \end{pmatrix}$$

for a suitable choice of  $\gamma$ . Hence we have shown that  $A - \mu I$  is surjective for all  $\mu \in \mathbb{C} \setminus \mathbb{R}$ , and the self-adjointness of the symmetric operator  $A$  follows.

**Proposition 2.3.** *The operator  $M_0$  is symmetric and positive.*

**Proof.** The symmetry follows as in the proof of Proposition 2.1, and in particular

$$(10) \quad \langle M_0 \tilde{y}, \tilde{y} \rangle = \int_0^a h_2(x) |y''(x)|^2 dx + \int_0^a h_1(x) |y'(x)|^2 dx + \int_0^a h_0(x) |y(x)|^2 dx$$

for  $\tilde{y} = \begin{pmatrix} y \\ y''(a) \end{pmatrix} \in \mathcal{D}(M_0)$  shows that  $M_0 \geq 0$ . Assume there is  $\tilde{y} \neq 0$  such that  $M_0 \tilde{y} = 0$ . Then there is a largest  $b \in [0, a)$  such that  $y = 0$  on  $[0, b]$ . Since  $h_0 + h_1 + h_2 > 0$ , it follows from  $\langle M_0 \tilde{y}, \tilde{y} \rangle = 0$  that at least one of  $y$ ,  $y'$  or  $y''$  must be zero near  $b$ , and  $y(b) = y'(b) = y''(b) = 0$  then gives the contradiction  $y = 0$  near  $b$ .  $\square$

Since  $M_0 \geq 0$ , its Friedrichs extension  $M$  is defined, see e. g. [9, Section VI.3], and we have

**Corollary 2.4.** *The operator  $M$  is self-adjoint and non-negative.*

We observe that the operator  $M$  is more general than the corresponding operator in [19, 21], where  $M = I - K$ .

### 3. SPECTRAL PROPERTIES OF THE OPERATOR PENCIL $L$

**Proposition 3.5.** *For all  $\alpha \geq 0$ , the operator pencil  $L(\cdot, \alpha)$  is a Fredholm valued operator function with index 0. The spectrum of the pencil  $L(\cdot, \alpha)$  consists of discrete eigenvalues of finite multiplicities and all eigenvalues of  $L(\cdot, \alpha)$  lie in the closed upper half-plane and on the imaginary axis and are symmetric with respect to the imaginary axis.*

**Proof.** As in [19, Section 3] we can argue that for all  $\lambda \in \mathbb{C}$ ,  $L(\lambda, \alpha)$  is a relatively compact perturbation of  $L(0, 0)$ , where  $L(0, 0)$  is well known to be a Fredholm operator. The statement on the location of the spectrum now follows as in [19, Lemma 3.1].  $\square$

The proofs of the two following lemmas are similar to the proofs of [19, Lemma 3.2 and Lemma 3.3].

**Lemma 3.6.** *All nonzero real eigenvalues of  $L(\cdot, \alpha)$ ,  $\alpha > 0$ , (if any) are semi-simple, i.e., the corresponding eigenvectors do not possess associated vectors. All real eigenvalues of  $L(\cdot, \alpha)$ ,  $\alpha > 0$ , are independent of  $\alpha$ .*

**Lemma 3.7.** *Let  $\lambda = -i\tau$ ,  $\tau > 0$ , be an eigenvalue of  $L(\cdot, \alpha)$ ,  $\alpha \geq 0$ . Then  $\lambda$  is semi-simple.*

**Proposition 3.8.** *Let  $\lambda$  be an eigenvalue of  $L(\cdot, \alpha)$ ,  $\alpha \geq 0$ . Then the geometric multiplicity of  $\lambda$  is at most 3.*

**Proof.** Let  $N(L(\lambda, \alpha))$  be the null space of the operator  $L(\lambda, \alpha)$  and define the operator  $f_\lambda : N(L(\lambda, \alpha)) \rightarrow \mathbb{C}^3$  by  $f_\lambda y = (y^{(3)}(0), y^{(4)}(0), y^{(5)}(0))^T$ . If  $y \in N(L(\lambda, \alpha))$  and  $f_\lambda(y) = 0$ , then  $y^{(j)}(0) = 0$  for  $j = 0, \dots, 5$ . Hence the function  $y = 0$  is the unique solution of an initial value problem, which shows that  $f_\lambda$  is injective. Hence the dimension of  $N(L(\lambda, \alpha))$  is at most 3.

REMARK 3.9. Writing the boundary conditions (2), (3) in the form  $B_j(\lambda)y(\lambda, \cdot) = 0$  and letting  $y_k(\lambda, \cdot)$  be the solutions of (1) satisfying  $y_k^{(\ell)}(\lambda, 0) = \delta_{k,\ell}$  for  $k, \ell = 0, \dots, 5$ , the characteristic determinant  $m(\lambda, \alpha)$  of (1)-(3) is the determinant of the given matrix  $(B_j(\lambda)y_k(\lambda, \cdot))_{j,k=0}^5$ . The zeros of  $m(\cdot, \alpha)$  are the eigenvalues of (1)-(3), see e.g. [18, Section 6.3]. It is well-known that the functions  $y_k$  depend analytically on  $\lambda$  and  $\alpha$ . Therefore, for each  $\alpha_0 \in \mathbb{R}$  an eigenvalue  $\lambda(\alpha_0)$  of (1)-(3) leads to a continuous eigenvalue branch  $\lambda(\alpha)$  for  $\alpha$  near  $\alpha_0$ , see e.g. [3, Appendix A 5.4, Theorem 3], [6, Section A.1, Lemma A.1.3] or [16, Section 45, Corollary, p. 303]. In the following, we will always choose such continuous branches. If we have multiple eigenvalues, there is some ambiguity. However, if the eigenvalue  $\lambda(\alpha_0)$  is semi-simple, say of multiplicity  $n$ , then the space of eigenfunctions satisfying  $y''(a) = 0$  has dimension  $n - 1$  or  $n$ . But these eigenfunctions are then eigenfunctions for all  $\alpha$ , so that there are  $n - 1$  or  $n$  constant eigenvalue branches  $\lambda = \lambda(\alpha_0)$ , and the remaining eigenvalue branch in case  $n - 1$  would depend analytically on  $\alpha$  since it is the unique solution of  $\lambda \mapsto m(\lambda, \alpha)(\lambda - \lambda(\alpha_0))^{-n+1} = 0$  near  $\lambda(\alpha_0)$ .

**Lemma 3.10.** 1. *Let  $\lambda(\alpha) = -i\tau$ ,  $\tau > 0$ , be an eigenvalue of  $L(\cdot, \alpha)$ ,  $\alpha \geq 0$ . Then  $\operatorname{Re} \dot{\lambda}(\alpha) = 0$  and  $\operatorname{Im} \dot{\lambda}(\alpha) \geq 0$  for all  $\alpha \geq 0$ ; here  $\cdot$  means derivative with respect to  $\alpha$ .*

2. *If 0 is an eigenvalue of  $L(\cdot, \alpha)$  for some  $\alpha \geq 0$ , then it is an eigenvalue for all  $\alpha \geq 0$ , its geometric multiplicity  $m \leq 3$  is the same for all  $\alpha \geq 0$ ,  $m = \dim N(A)$ , whereas its algebraic multiplicity  $p$  is the same for all  $\alpha > 0$  and satisfies  $m \leq p \leq \min\{2m, m + 2\}$ .*

**Proof.** 1. If  $\lambda(\alpha) = -i\tau$  is an eigenvalue of  $L(\cdot, \alpha)$ ,  $\alpha \geq 0$  with corresponding eigenvector  $Y$ , then

$$(11) \quad \langle L(-i\tau, \alpha)Y, Y \rangle = \tau^2 \langle MY, Y \rangle + \tau\alpha \langle KY, Y \rangle + \langle AY, Y \rangle = 0.$$

Since the eigenvalue  $\lambda(\alpha)$  is semi-simple by Lemma 3.7, it depends analytically on  $\alpha$  by Remark 3.9, and the eigenvector  $Y$  corresponding to  $\lambda$  can be chosen to

depend analytically on  $\alpha$ . Differentiating (11) with respect to  $\alpha$  we obtain

$$(12) \quad \langle L(-i\tau, \alpha)Y, \dot{Y} \rangle + \langle L(-i\tau, \alpha)\dot{Y}, Y \rangle - 2i\tau\dot{\lambda}\langle MY, Y \rangle - i\alpha\dot{\lambda}\langle KY, Y \rangle - \tau\langle KY, Y \rangle = 0.$$

Obviously,

$$\langle L(-i\tau, \alpha)Y, \dot{Y} \rangle = 0, \quad \langle L(-i\tau, \alpha)\dot{Y}, Y \rangle = \langle \dot{Y}, L(-i\tau, \alpha)Y \rangle = 0.$$

Substituting these equations into (12) we obtain

$$(13) \quad (2\tau\langle MY, Y \rangle + \alpha\langle KY, Y \rangle)\dot{\lambda} = i\tau\langle KY, Y \rangle.$$

Since  $\tau > 0$ ,  $\alpha \geq 0$ ,  $M|_{\mathcal{D}(A)} > 0$ , and  $K \geq 0$ , it follows that

$$(14) \quad \dot{\lambda} = \frac{i\tau\langle KY, Y \rangle}{2\tau\langle MY, Y \rangle + \alpha\langle KY, Y \rangle},$$

which completes the proof of statement 1.

2. Since  $L(0, \alpha) = L(0, 0) = -A$  is independent of  $\alpha$ , it follows that if 0 is an eigenvalue for some  $\alpha \geq 0$ , then it is eigenvalue for all  $\alpha \geq 0$ ; also the statement about the geometric multiplicity is obvious. On the other hand, since  $L(\cdot, 0)$  is a function of  $\lambda^2$ , each eigenvector of  $L(\cdot, 0)$  corresponding to the eigenvalue 0 has a chain with at least one associated vector zero. Assume there is an eigenvector  $Y_0$  corresponding to the eigenvalue 0 of  $L(\cdot, 0)$  which has a chain of associated vectors  $Y_1, Y_2$ , i. e.,

$$(15) \quad -AY_0 = 0, \quad -AY_1 = 0, \quad MY_0 - AY_2 = 0.$$

Taking the scalar product with  $Y_0$  in the last equation and observing the first equation and the self-adjointness of  $A$ , we infer

$$(16) \quad 0 = \langle MY_0, Y_0 \rangle - \langle AY_2, Y_0 \rangle = \langle MY_0, Y_0 \rangle,$$

which gives  $Y_0 = 0$  since  $M|_{\mathcal{D}(A(U))} > 0$ ; a contradiction as  $Y_0$  is an eigenvector. The assertions for  $\alpha = 0$  are proved.

Now let  $\alpha > 0$ . If 0 is an eigenvalue of  $L(\cdot, \alpha)$  with an eigenvector  $Y_0$  which has an associated vector  $Y_1$ , then

$$(17) \quad -AY_0 = 0, \quad -i\alpha KY_0 - AY_1 = 0.$$

It follows that

$$(18) \quad 0 = -i\alpha\langle KY_0, Y_0 \rangle - \langle AY_1, Y_0 \rangle = -i\alpha\langle KY_0, Y_0 \rangle,$$

and  $K \geq 0$  implies

$$(19) \quad KY_0 = 0,$$



which gives

$$0 = KY_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_0 \\ y_0''(a) \end{pmatrix} = y_0''(a).$$

This has two consequences: firstly,  $Y_0$  and  $Y_1$  are independent of  $\alpha$ , and secondly,  $y_0$  satisfies  $y_0(a) = y_0'(a) = y_0''(a) = y_0'''(a) = 0$ . Thus at most two linearly independent eigenvectors can have an associated vector. Assume such an eigenvector  $Y_0$  has a chain  $Y_1, Y_2$  of associated vectors, i. e., (17) and

$$(20) \quad MY_0 - i\alpha KY_1 - AY_2 = 0$$

hold. This leads to

$$(21) \quad \begin{aligned} 0 &= \langle MY_0, Y_0 \rangle - i\alpha \langle KY_1, Y_0 \rangle - \langle AY_2, Y_0 \rangle \\ &= \langle MY_0, Y_0 \rangle - i\alpha \langle Y_1, KY_0 \rangle - \langle Y_2, AY_0 \rangle. \end{aligned}$$

By (17) and (19) we thus arrive at the contradiction  $\langle MY_0, Y_0 \rangle = 0$ . This completes the proof of the assertions for  $\alpha > 0$ .

**Lemma 3.11.** *Let  $V = \{y \in W_3^2(0, a) : y(0) = y'(0) = y''(0) = y(a) = y'(a) = 0\}$ . For  $\nu \geq 0$  and  $y \in V$  define*

$$(22) \quad \kappa(\nu, y) = \sum_{j=0}^2 \int_0^a g_j(x) |y^{(j)}(x)|^2 dx + \nu \sum_{j=0}^2 \int_0^a h_j(x) |y^{(j)}(x)|^2 dx.$$

*Then there exists  $\nu \geq 0$  such that  $\int_0^a |y'''(x)|^2 dx + \kappa(\nu, y) \geq 0$  for all  $y \in V$ .*

**Proof.** Assume the statement is false. Then there is a sequence  $(y_n)$  in  $V$  such that

$$(23) \quad \int_0^a |y_n'''(x)|^2 dx + \kappa(n, y_n) < 0.$$

We may choose  $y_n$  such that

$$(24) \quad \sum_{j=0}^3 \int_0^a |y_n^{(j)}(x)|^2 dx = 1.$$

Since the embedding  $W_3^2(0, a) \hookrightarrow C^2[0, a]$  is compact, see e. g. [18, Lemma 2.4.1], we may assume without loss of generality that  $(y_n)$  converges to some  $y \in C^2[0, a]$ . The inequality (23) implies that  $\kappa(n, y_n) < 0$ , so that

$$(25) \quad \sum_{j=0}^2 \int_0^a h_j(x) |y^{(j)}(x)|^2 dx = \lim_{n \rightarrow \infty} \sum_{j=0}^2 \int_0^a h_j(x) |y_n^{(j)}(x)|^2 dx = 0.$$

For  $j = 0, 1, 2$  let  $U_j = \{x \in [0, a] : h_j(x) > 0\}$ . From (25) we conclude that  $y^{(j)}|_{U_j} = 0$  a. e. The openness of  $U_j$  and the continuity of  $y^{(j)}$  imply that  $y^{(j)} = 0$  on  $\overline{U_j}$ . The continuity of  $h_0, h_1, h_2, h_0 + h_1 + h_2 > 0$ , and a compactness argument give the existence of numbers  $0 = a_0 < a_1 < \dots < a_n = a$  and  $j_k \in \{0, 1, 2\}$  for  $k = 1, \dots, n$  such that  $[a_{k-1}, a_k] \subset U_{j_k}$  for these  $k$ . By the above,  $y^{(j_k)} = 0$  on  $[a_{k-1}, a_k]$ , and an easy induction argument, starting with  $y(0) = y'(0) = 0$ , shows that  $y = 0$  on all of these intervals, so that  $y = 0$  on  $[0, a]$ .

From (23) it follows that

$$\int_0^a |y_n'''(x)|^2 dx + \sum_{j=0}^2 \int_0^a g_j(x) |y_n^{(j)}(x)|^2 dx < \int_0^a |y_n'''(x)|^2 dx + \kappa(\nu, y_n) < 0,$$

and therefore

$$\limsup_{n \rightarrow \infty} \int_0^a |y_n'''(x)|^2 dx = 0,$$

whereas (24) would give

$$\lim_{n \rightarrow \infty} \int_0^a |y_n'''(x)|^2 dx = 1.$$

This contradiction completes the proof.

**Theorem 3.12.** *The operator pencil  $L(\cdot, \alpha)$  has at most finitely many eigenvalues on the negative imaginary axis, their total multiplicity does not exceed the corresponding total multiplicity for  $\alpha = 0$ , and the spectrum of  $L(\cdot, \alpha)$  on the negative imaginary axis lies in  $[0, -i\nu^{1/2}]$  for all  $\alpha \geq 0$ , where the number  $\nu$  is as in Lemma 3.11.*

**Proof.** In view of (9) and (10),

$$\langle -L(-i\tau, 0)\tilde{y}, \tilde{y} \rangle = \int_0^a |y'''(x)|^2 dx + \kappa(\tau^2, y) > \int_0^a |y'''(x)|^2 dx + \kappa(\nu, y) \geq 0$$

for all  $y \in \mathcal{D}(L) \setminus \{0\}$  and  $\tau^2 > \nu$ , where we have used that  $M|_{\mathcal{D}(L)} > 0$ . Hence  $L(\cdot, 0)$  has no eigenvalues  $-i\tau$  with  $\tau^2 > \nu$ . This proves the statement for  $\alpha = 0$ , and the statement for  $\alpha > 0$  follows from Lemma 3.10.

**Acknowledgements.** This work is based on part of the second author's PhD thesis. This work has been partially supported by the National Research Foundation of South Africa, GUN 69659. Any findings and conclusions expressed in this material are those of the authors and therefore the NRF do not accept any liability in regard thereto.

## REFERENCES

1. N. M. AUCIELLO, M. A. DE ROSA: *Free vibrations of circular arches: A review*. J. Sound Vibrat., **176** (1994), 433–458.

2. W. AL-HAYANI: *Adomian decomposition method with Green's function for sixth-order boundary value problems*. *Comput. Math. Appl.*, **61** (2011), 1567–1575.
3. H. BAUMGÄRTEL: *Analytic perturbation theory for matrices and operators*. Birkhäuser, Basel, 1985.
4. K. FEDERHOFER: *Berechnung der niedrigsten Eigenschwingzahl des radial belasteten Kreisbogens*. *Ing. Arch.*, **4** (1933), 276–286.
5. C. I. GHEORGHU, F.-I. DRAGOMIRESCU: *Spectral methods in linear stability. Applications to thermal convection with variable gravity field*. *Appl. Numer. Math.*, **59** (2009), 1290–1302.
6. L. HÖRMANDER: *The Analysis of Linear Partial Differential Operators II*. Springer-Verlag, Berlin, 1983.
7. R. JALILIAN, J. RASHIDINIA: *Convergence analysis of nonic-spline solutions for special nonlinear sixth-order boundary value problems*. *Commun. Nonlinear Sci. Numer. Simul.*, **15** (2010), 3805–3813.
8. B. KANG, C. H. RIEDEL, C. A. TAN: *Free vibration analysis of planar curved beams by wave propagation*. *J. Sound Vibrat.*, **260** (2003), 19–44.
9. T. KATO: *Perturbation theory for linear operators*. Springer, Berlin, 1966.
10. N. B. KERIMOV, Z. S. ALIEV: *Basis properties of a spectral problem with a spectral parameter in the boundary condition*. (Russian) *Mat. Sb.*, **197** (10) (2006), 65–86. Translation in *Sb. Math.*, **197** (2006), 1467–1487.
11. N. B. KERIMOV, Z. S. ALIEV: *On the basis property of the system of eigenfunctions of a spectral problem with a spectral parameter in the boundary condition*. (Russian) *Differ. Uravn.*, **43** (2007), 886–895, 1004. Translation in *Differ. Equ.*, **43** (2007), 905–915.
12. N.-I. KIM, C. C. FU, M.-Y. KIM: *Exact thin-walled curved beam element considering shear deformation effects and non-symmetric cross sections*. *Int. J. Steel Struct.*, **6** (2006), 191–207.
13. J. LEE: *In-plane free vibration analysis of curved Timoshenko beams by the pseudospectral method*. *KSME Int. J.*, **17** (2003), 1156–1163.
14. S. Y. LEE, J. Y. HSIAO: *Free in-plane vibrations of curved nonuniform beams*. *Acta Mech.*, **155** (2002), 173–189.
15. S. LIANG, D. J. JEFFREY: *Approximate solutions to a parametrized sixth order boundary value problem*. *Comput. Math. Appl.*, **59** (2010), 247–253.
16. A. I. MARKUSHEVICH: *Theory of functions of a complex variable, Vol. III*. Prentice-Hall, Englewood Cliffs, 1967.
17. M. MARLETTA, A. SHKALIKOV, C. TRETTER: *Pencils of differential operators containing the eigenvalue parameter in the boundary conditions*. *Proc. Royal Soc. Edinburgh*, **133A** (2003), 893–917.
18. R. MENNICKEN, M. MÖLLER: *Non-self-adjoint boundary eigenvalue problems*. North-Holland Mathematics Studies 192. Elsevier, Amsterdam, 2003.
19. M. MÖLLER, V. PIVOVARCHIK: *Spectral Properties of a Fourth Order Differential Equation*. *Z. Anal. Anwend.*, **25** (2006), 341–366.

20. M. Möller, A. Zettl: *Symmetric Differential Operators and their Friedrichs Extension*. J. Differential Equations, **115** (1995), 50–69.
21. M. MÖLLER, B. ZINSOU: *Self-adjoint fourth order differential operator with eigenvalue parameter dependent boundary conditions*. Quaest. Math., **34** (2011), 393–406, DOI 10.2989/16073606.2011.622913.
22. M. MÖLLER, B. ZINSOU: *Spectral asymptotics of self-adjoint fourth order differential operator with eigenvalue parameter dependent boundary conditions*. Complex Anal. Oper. Theory, **6** (2012), 799–818, DOI 10.1007/s11785-011-0162-1.
23. M. MÖLLER, B. ZINSOU: *Spectral asymptotics of self-adjoint fourth order boundary value problems with eigenvalue parameter dependent boundary conditions*. Bound. Value Probl., **2012** (106) (2012), DOI 10.1186/1687-2770-2012-106.
24. M. A. NOOR, S. T. MOHYUD-DIN: *Homotopy perturbation method for solving sixth-order boundary value problems*. Comput. Math. Appl., **55** (2008), 2953–2972.
25. M.A. NOOR, K. I. NOOR, S.T. MOHYUD-DIN: *Variational iteration method for solving sixth-order boundary value problems*. Commun. Nonlinear Sci. Numer. Simul., **14** (2009), 2571–2580.
26. S. J. OH, B. K. LEE, I. W. LEE: *Free vibrations of non-circular arches with non-uniform cross-section*. Int. J. Solids Struct., **37** (2000), 4871–4891.
27. V. PIVOVARCHIK, C. VAN DER MEE: *The inverse generalized Regge problem*. Inverse Probl., **17** (2001), 1831–1845.
28. A. A. SHKALIKOV: *Boundary problems for ordinary differential equations with parameter in the boundary conditions (Russian)*. Trudy Sem. Petrovsk. **9** (1983), 190–229, Engl. transl. J. Soviet Math., **33** (1986), 1311–1342.
29. F. W. WALKING: *Schwingungszahlen und Schwingungsformen von Kreisbogen-trägern*. Ing. Arch., **5** (1934), 429–449.
30. A. WANG, J. SUN, A. ZETTL: *The classification of self-adjoint boundary conditions: separated, coupled, and mixed*. J. Funct. Anal., **255** (2008), 1554–1573.
31. A. WANG, J. SUN, A. ZETTL: *Characterization of domains of self-adjoint ordinary differential operators*. J. Differential Equations, **246** (2009), 1600–1622.
32. J. WEIDMANN: *Spectral theory of ordinary differential operators*. Lect. Notes Math., Vol. 1258, Springer-Verlag, 1987.
33. W. Wuest: *Die Biegeschwingungen einseitig eingespannter gekrümmter Stäbe und Rohre*, Ing. Arch., **17** (1949), 265–272.

The John Knopfnacher Centre for  
 Applicable Analysis and Number Theory,  
 School of Mathematics,  
 University of the Witwatersrand, Johannesburg  
 South Africa  
 E-mails: manfred.moller@wits.ac.za  
 bertin\_zin@yahoo.fr

(Received January 7, 2013)  
 (Revised May 23, 2013)