

FRACTIONAL INTEGRALS AND DERIVATIVES IN q -CALCULUS

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We generalize the notions of the fractional q -integral and q -derivative by introducing variable lower limit of integration. We discuss some properties and their relations. Finally, we give a q -TAYLOR-like formula which includes fractional q -derivatives of the function.

1. INTRODUCTION

In the theory of q -calculus (see [5] and [7]), for a real parameter $q \in \mathbb{R}^+ \setminus \{1\}$, we introduce a q -real number $[a]_q$ by

$$[a]_q := \frac{1 - q^a}{1 - q} \quad (a \in \mathbb{R}).$$

The q -analog of the POCHHAMMER symbol (q -shifted factorial) is defined by:

$$(a; q)_0 = 1, \quad (a; q)_k = \prod_{i=0}^{k-1} (1 - aq^i) \quad (k \in \mathbb{N} \cup \{\infty\}).$$

Also, the q -analog of the power $(a - b)^k$ is

$$(a - b)^{(0)} = 1, \quad (a - b)^{(k)} = \prod_{i=0}^{k-1} (a - bq^i) \quad (k \in \mathbb{N}; a, b \in \mathbb{R}).$$

There is the following relationship between them:

$$(a - b)^{(n)} = a^n (b/a; q)_n \quad (a \neq 0).$$

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Their natural expansions to the reals are

$$(1) \quad (a-b)^{(\alpha)} = a^\alpha \frac{(b/a; q)_\infty}{(q^\alpha b/a; q)_\infty}, \quad (a; q)_\alpha = \frac{(a; q)_\infty}{(aq^\alpha; q)_\infty} \quad (\alpha \in \mathbb{R}).$$

Notice that

$$(a-b)^{(\alpha)} = a^\alpha (b/a; q)_\alpha.$$

The following formulas (see, for example, [5] and [4]) will be useful:

$$(2) \quad (a; q)_n = (q^{1-n}/a; q)_n (-1)^n a^n q^{\binom{n}{2}};$$

$$(3) \quad \frac{(aq^{-n}; q)_n}{(bq^{-n}; q)_n} = \frac{(q/a; q)_n}{(q/b; q)_n} \left(\frac{a}{b}\right)^n;$$

$$(4) \quad (a-b)^{(\alpha)} = a^\alpha \sum_{k=0}^{\infty} (-1)^k \begin{bmatrix} \alpha \\ k \end{bmatrix}_q q^{\binom{k}{2}} \left(\frac{b}{a}\right)^k.$$

The q -gamma function is defined by

$$(5) \quad \Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty} (1-q)^{1-x} = (1-q)^{(x-1)} (1-q)^{1-x},$$

where $x \in \mathbb{R} \setminus \{0, -1, -2, \dots\}$. Obviously,

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x).$$

We can define q -binomial coefficients with

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}_q = \frac{\Gamma_q(\alpha+1)}{\Gamma_q(\beta+1) \Gamma_q(\alpha-\beta+1)} = \frac{(q^{\beta+1}; q)_\infty (q^{\alpha-\beta+1}; q)_\infty}{(q; q)_\infty (q^{\alpha+1}; q)_\infty}$$

$\alpha, \beta, \alpha-\beta \in \mathbb{R} \setminus \{-1, -2, \dots\}$. Particularly,

$$(6) \quad \begin{bmatrix} \alpha \\ k \end{bmatrix}_q = \frac{(q^{-\alpha}; q)_k}{(q; q)_k} (-1)^k q^{\alpha k} q^{-\binom{k}{2}} \quad (k \in \mathbb{N}).$$

The q -hypergeometric function is defined as

$${}_2\phi_1\left(\begin{matrix} a, & b \\ c \end{matrix} \middle| q; x\right) = \sum_{n=0}^{\infty} \frac{(a; q)_n (b; q)_n}{(c; q)_n (q; q)_n} x^n.$$

The famous HEINE transformation formula [5] is

$$(7) \quad {}_2\phi_1\left(\begin{matrix} a, & b \\ c \end{matrix} \middle| q; x\right) = \frac{(abx/c; q)_\infty}{(x; q)_\infty} {}_2\phi_1\left(\begin{matrix} c/a, & c/b \\ c \end{matrix} \middle| q; abx/c\right).$$

We define a q -derivative of a function $f(x)$ by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{x - qx} \quad (x \neq 0), \quad (D_q f)(0) = \lim_{x \rightarrow 0} (D_q f)(x)$$

and q -derivatives of higher order:

$$(8) \quad D_q^0 f = f, \quad D_q^n f = D_q(D_q^{n-1} f) \quad (n = 1, 2, 3, \dots).$$

For an arbitrary pair of functions $u(x)$ and $v(x)$ and constants $\alpha, \beta \in \mathbb{R}$, we have linearity and product rules

$$\begin{aligned} D_q(\alpha u(x) + \beta v(x)) &= \alpha(D_q u)(x) + \beta(D_q v)(x), \\ D_q(u(x) \cdot v(x)) &= u(qx)(D_q v)(x) + v(x)(D_q u)(x). \end{aligned}$$

The q -integral is defined by

$$(I_{q,0} f)(x) = \int_0^x f(t) d_q t = x(1-q) \sum_{k=0}^{\infty} f(xq^k) q^k \quad (0 \leq |q| < 1),$$

and

$$(9) \quad (I_{q,a} f)(x) = \int_a^x f(t) d_q t = \int_0^x f(t) d_q t - \int_0^a f(t) d_q t.$$

However, these definitions cause troubles in research as they include the points outside of the interval of integration (see [6] and [10]). In the case when the lower limit of integration is $a = xq^n$, i.e., when it is determined for some choice of x , q and positive integer n , the q -integral (9) becomes

$$(10) \quad \int_{xq^n}^x f(t) d_q t = x(1-q) \sum_{k=0}^{n-1} f(xq^k) q^k.$$

As for q -derivative, we can define an operator $I_{q,a}^n$ by

$$I_{q,a}^0 f = f, \quad I_{q,a}^n f = I_{q,a}(I_{q,a}^{n-1} f) \quad (n = 1, 2, 3, \dots).$$

For operators defined in this manner, the following is valid:

$$(11) \quad (D_q I_{q,a} f)(x) = f(x), \quad (I_{q,a} D_q f)(x) = f(x) - f(a).$$

The formula for q -integration by parts is

$$\int_a^b u(x)(D_q v)(x) d_q x = [u(x)v(x)]_a^b - \int_a^b v(qx)(D_q u)(x) d_q x.$$

W. A. AL-SALAM [2] and R. P. Agarwal [1] introduced several types of fractional q -integral operators and fractional q -derivatives. Here, we will only mention the fractional q -integral with the lower limit of integration $a = 0$, defined by

$$(I_q^{\eta,\alpha} f)(x) = \frac{x^{-(\eta+\alpha)}}{\Gamma_q(\alpha)} \int_0^x (x-tq)^{(\alpha-1)} t^\eta f(t) d_q t \quad (\eta, \alpha \in \mathbb{R}^+).$$

On the other hand, the solution of n th order q -differential equation

$$(D_q^n y)(x) = f(x), \quad (D_q^k y)(a) = 0 \quad (k = 0, 1, \dots, n-1),$$

can be written in the form of a multiple q -integral

$$y(x) = (I_{q,a}^n f)(x) = \int_a^x d_q t \int_a^t d_q t_{n-1} \int_a^{t_{n-1}} d_q t_{n-2} \cdots \int_a^{t_2} f(t_1) d_q t_1.$$

The reduction of the multiple q -integral to a single one was considered by AL-SALAM [3]. He thought of it as a q -analog of Cauchy's formula:

$$(12) \quad y(x) = (I_{q,a}^n f)(x) = \frac{1}{[n-1]_q!} \int_a^x (x-qt)^{(n-1)} f(t) d_q t \quad (n \in \mathbb{N}).$$

In this paper, our purpose is to consider fractional q -integrals with the parametric lower limit of integration. After preliminaries, in the third section we define the fractional q -integral in that sense. On the basis of that, the fractional q -derivative is introduced in the fourth section. Finally, in the last section, we give a q -TAYLOR-like formula using these fractional q -derivatives.

2. PRELIMINARIES

We will first specify some results which are useful in the sequel and which can be proved easily.

Lemma 1. For $a, b, \alpha \in \mathbb{R}^+$ and $k, n \in \mathbb{N}$, the following properties are valid:

$$(13) \quad (a - bq^k)^{(\alpha)} = a^\alpha (1 - q^k b/a)^{(\alpha)},$$

$$(14) \quad \frac{(a - bq^k)^{(\alpha)}}{(a - b)^{(\alpha)}} = \frac{(q^\alpha b/a; q)_k}{(b/a; q)_k},$$

$$(15) \quad (q^n - q^k)^{(\alpha)} = 0 \quad (k \leq n).$$

The next result will have an important role in proving the semigroup property of the fractional q -integral.

Lemma 2. For $\mu, \alpha, \beta \in \mathbb{R}^+$, the following identity is valid

$$(16) \quad \sum_{n=0}^{\infty} \frac{(1 - \mu q^{1-n})^{(\alpha-1)} (1 - q^{1+n})^{(\beta-1)}}{(1 - q)^{(\alpha-1)} (1 - q)^{(\beta-1)}} q^{\alpha n} = \frac{(1 - \mu q)^{(\alpha+\beta-1)}}{(1 - q)^{(\alpha+\beta-1)}}.$$

Proof. According to the formulas (1) and (3), we have

$$\begin{aligned} (1 - \mu q^{1-n})^{(\alpha-1)} &= \frac{(\mu q^{1-n}; q)_\infty}{(\mu q^{\alpha-n}; q)_\infty} = \frac{(\mu q^{1-n}; q)_n (\mu q; q)_\infty}{(\mu q^{\alpha-n}; q)_n (\mu q^\alpha; q)_\infty} \\ &= (1 - \mu q)^{(\alpha-1)} \frac{(\mu^{-1}; q)_n}{(\mu^{-1} q^{1-\alpha}; q)_n} q^{(1-\alpha)n}. \end{aligned}$$

Applying the identity (14) to the expression $(1 - q^{1+n})^{(\beta-1)}/(1 - q)^{(\beta-1)}$, the sum on the left side of (16) can be written as

$$\begin{aligned} LS &= \frac{(1 - \mu q)^{(\alpha-1)}}{(1 - q)^{(\alpha-1)}} \sum_{n=0}^{\infty} \frac{(q^{\beta}; q)_n}{(q; q)_n} \frac{(\mu^{-1}; q)_n}{(\mu^{-1} q^{1-\alpha}; q)_n} q^{(1-\alpha)n} q^{\alpha n} \\ &= \frac{(1 - \mu q)^{(\alpha-1)}}{(1 - q)^{(\alpha-1)}} {}_2\phi_1 \left(\begin{matrix} \mu^{-1}, & q^{\beta} \\ \mu^{-1} q^{1-\alpha} \end{matrix} \middle| q; q \right). \end{aligned}$$

Using (7), we get

$$\begin{aligned} LS &= \frac{(1 - \mu q)^{(\alpha-1)}}{(1 - q)^{(\alpha-1)}} \frac{(q^{\alpha+\beta}; q)_{\infty}}{(q; q)_{\infty}} {}_2\phi_1 \left(\begin{matrix} q^{1-\alpha}, & \mu^{-1} q^{1-\alpha-\beta} \\ \mu^{-1} q^{1-\alpha} \end{matrix} \middle| q; q^{\alpha+\beta} \right) \\ &= \frac{(1 - \mu q)^{(\alpha-1)}}{(1 - q)^{(\alpha-1)}} \frac{1}{(1 - q)^{(\alpha+\beta-1)}} \sum_{n=0}^{\infty} \frac{(q^{1-\alpha}; q)_n}{(q; q)_n} \frac{(\mu^{-1} q^{1-\alpha-\beta}; q)_n}{(\mu^{-1} q^{1-\alpha}; q)_n} q^{(\alpha+\beta)n}. \end{aligned}$$

According to (2) and (1), the following is valid:

$$\begin{aligned} \frac{(\mu^{-1} q^{1-\alpha-\beta}; q)_n}{(\mu^{-1} q^{1-\alpha}; q)_n} &= \frac{(\mu q^{\alpha+\beta-n}; q)_n}{(\mu q^{\alpha-n}; q)_n} q^{-\beta n} = \frac{(\mu q^{\alpha+\beta-n}; q)_{\infty}}{(\mu q^{\alpha+\beta}; q)_{\infty}} \frac{(\mu q^{\alpha}; q)_{\infty}}{(\mu q^{\alpha-n}; q)_{\infty}} q^{-\beta n} \\ &= \frac{(\mu q^{\alpha}; q)_{\infty}}{(\mu q^{\alpha+\beta}; q)_{\infty}} \frac{(\mu q^{\alpha+\beta-n}; q)_{\infty}}{(\mu q^{\alpha-n}; q)_{\infty}} q^{-\beta n} \\ &= \frac{(\mu q^{\alpha}; q)_{\infty}}{(\mu q^{\alpha+\beta}; q)_{\infty}} (1 - \mu q^{\alpha+\beta-n})^{(-\beta)} q^{-\beta n}. \end{aligned}$$

Hence

$$LS = \frac{(1 - \mu q)^{(\alpha+\beta-1)}}{(1 - q)^{(\alpha-1)} (1 - q)^{(\alpha+\beta-1)}} \sum_{n=0}^{\infty} \frac{(q^{1-\alpha}; q)_n}{(q; q)_n} q^{\alpha n} (1 - \mu q^{\alpha+\beta-n})^{(-\beta)}.$$

If we use formulas (6) and (4) and change the order of the summation, the last sum becomes

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(q^{1-\alpha}; q)_n}{(q; q)_n} q^{\alpha n} (1 - \mu q^{\alpha+\beta-n})^{(-\beta)} \\ &= \sum_{n=0}^{\infty} \left[\begin{matrix} \alpha - 1 \\ n \end{matrix} \right]_q (-1)^n q^{-(\alpha-1)n} q^{\binom{n}{2}} q^{\alpha n} \sum_{k=0}^{\infty} (-1)^k \left[\begin{matrix} -\beta \\ k \end{matrix} \right]_q q^{\binom{k}{2}} (\mu q^{\alpha+\beta-n})^k \\ &= \sum_{k=0}^{\infty} (-1)^k \left[\begin{matrix} -\beta \\ k \end{matrix} \right]_q q^{\binom{k}{2}} (\mu q^{\alpha+\beta})^k \sum_{n=0}^{\infty} (-1)^n \left[\begin{matrix} \alpha - 1 \\ n \end{matrix} \right]_q q^{\binom{n}{2}} (q^{1-k})^n \\ &= \sum_{k=0}^{\infty} (-1)^k \left[\begin{matrix} -\beta \\ k \end{matrix} \right]_q q^{\binom{k}{2}} (\mu q^{\alpha+\beta})^k (1 - q^{1-k})^{(\alpha-1)} = (1 - q)^{(\alpha-1)}. \end{aligned}$$

The last relation is valid because of $(1 - q^{1-k})^{(\alpha-1)} = 0$ for $k = 1, 2, \dots$. Finally, the identity holds:

$$LS = \frac{(1 - \mu q)^{(\alpha+\beta-1)}}{(1 - q)^{(\alpha-1)} (1 - q)^{(\alpha+\beta-1)}} (1 - q)^{(\alpha-1)} = \frac{(1 - \mu q)^{(\alpha+\beta-1)}}{(1 - q)^{(\alpha+\beta-1)}}. \quad \square$$

3. THE FRACTIONAL q -INTEGRAL

In all further considerations we assume that the functions are defined in an interval $(0, b)$ ($b > 0$), and $a \in (0, b)$ is an arbitrary fixed point. Also, the required q -derivatives and q -integrals exist and the convergence of the series mentioned in the proofs is assumed.

Generalizing the formula (12), we can define the *fractional q -integral of the Riemann-Liouville type* by

$$(17) \quad (I_{q,a}^\alpha f)(x) = \frac{1}{\Gamma_q(\alpha)} \int_a^x (x - qt)^{(\alpha-1)} f(t) d_q t \quad (\alpha \in \mathbb{R}^+).$$

Using formula (4), this integral can be written as

$$(I_{q,a}^\alpha f)(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \sum_{k=0}^{\infty} (-1)^k \begin{bmatrix} \alpha-1 \\ k \end{bmatrix}_q q^{\binom{k+1}{2}} x^{-k} \int_a^x t^k f(t) d_q t \quad (\alpha \in \mathbb{R}^+).$$

Lemma 3. For $\alpha \in \mathbb{R}^+$, the following is valid:

$$(I_{q,a}^\alpha f)(x) = (I_{q,a}^{\alpha+1} D_q f)(x) + \frac{f(a)}{\Gamma_q(\alpha+1)} (x-a)^{(\alpha)} \quad (0 < a < x < b).$$

Proof. Since the q -derivative over the variable t is

$$D_q((x-t)^{(\alpha)}) = -[\alpha]_q (x-qt)^{(\alpha-1)},$$

and using the q -integration by parts, we obtain

$$\begin{aligned} (I_{q,a}^\alpha f)(x) &= -\frac{1}{[\alpha]_q \Gamma_q(\alpha)} \int_a^x D_q((x-t)^{(\alpha)}) f(t) d_q t \\ &= \frac{1}{\Gamma_q(\alpha+1)} \left((x-a)^{(\alpha)} f(a) + \int_a^x (x-qt)^{(\alpha)} (D_q f)(t) d_q t \right) \\ &= (I_{q,a}^{\alpha+1} D_q f)(x) + \frac{f(a)}{\Gamma_q(\alpha+1)} (x-a)^{(\alpha)}. \end{aligned} \quad \square$$

Lemma 4. For $\alpha, \beta \in \mathbb{R}^+$, the following is valid:

$$\int_0^a (x-qt)^{(\beta-1)} (I_{q,a}^\alpha f)(t) d_q t = 0 \quad (0 < a < x < b).$$

Proof. Using Lemma 1 and formula (10), for $n \in \mathbb{N}_0$, we have

$$\begin{aligned} (I_{q,a}^\alpha f)(aq^n) &= \frac{1}{\Gamma_q(\alpha)} \int_a^{aq^n} (aq^n - qu)^{(\alpha-1)} f(u) d_q u \\ &= \frac{-a^\alpha(1-q)}{\Gamma_q(\alpha)} \sum_{j=0}^{n-1} (q^n - q^{j+1})^{(\alpha-1)} f(aq^j) q^j = 0. \end{aligned}$$

Then, according to the definition of q -integral, it follows

$$\int_0^a (x - qt)^{(\beta-1)} (I_{q,a}^\alpha f)(t) d_q t = a(1 - q) \sum_{n=0}^{\infty} (x - aq^{n+1})^{(\beta-1)} (I_{q,a}^\alpha f)(aq^n) q^n = 0. \quad \square$$

Theorem 5. *Let $\alpha, \beta \in \mathbb{R}^+$. The q -fractional integration has the following semi-group property*

$$(I_{q,a}^\beta I_{q,a}^\alpha f)(x) = (I_{q,a}^{\alpha+\beta} f)(x) \quad (0 < a < x < b).$$

Proof. By previous lemma, we have

$$(I_{q,a}^\beta I_{q,a}^\alpha f)(x) = \frac{1}{\Gamma_q(\beta)} \int_0^x (x - qt)^{(\beta-1)} (I_{q,a}^\alpha f)(t) d_q t,$$

i.e.,

$$\begin{aligned} (I_{q,a}^\beta I_{q,a}^\alpha f)(x) &= \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_0^x (x - qt)^{(\beta-1)} \int_0^t (t - qu)^{(\alpha-1)} f(u) d_q u \\ &\quad - \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_0^x (x - qt)^{(\beta-1)} \int_0^a (t - qu)^{(\alpha-1)} f(u) d_q u. \end{aligned}$$

Using the result from [1],

$$(I_{q,0}^\beta I_{q,0}^\alpha f)(x) = (I_{q,0}^{\alpha+\beta} f)(x),$$

we conclude that

$$(I_{q,a}^\beta I_{q,a}^\alpha f)(x) = (I_{q,0}^{\alpha+\beta} f)(x) - \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_0^x (x - qt)^{(\beta-1)} \int_0^a (t - qu)^{(\alpha-1)} f(u) d_q u.$$

Furthermore, we can write

$$\begin{aligned} (I_{q,a}^\beta I_{q,a}^\alpha f)(x) &= (I_{q,a}^{\alpha+\beta} f)(x) + \frac{1}{\Gamma_q(\alpha + \beta)} \int_0^a (x - qt)^{(\alpha+\beta-1)} f(t) d_q t \\ &\quad - \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_0^x (x - qt)^{(\beta-1)} \int_0^a (t - qu)^{(\alpha-1)} f(u) d_q u, \end{aligned}$$

wherefrom it follows

$$(I_{q,a}^\beta I_{q,a}^\alpha f)(x) = (I_{q,a}^{\alpha+\beta} f)(x) + a(1 - q) \sum_{j=0}^{\infty} c_j f(aq^j) q^j,$$

with

$$c_j = \frac{(x - aq^{j+1})^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)} - \frac{x(1-q)}{\Gamma_q(\alpha)\Gamma_q(\beta)} \sum_{n=0}^{\infty} (x - xq^{n+1})^{(\beta-1)} (xq^n - aq^{j+1})^{(\alpha-1)} q^n.$$

By using the formulas from Lemma 1 and (5), we get

$$c_j = ((1-q)x)^{\alpha+\beta-1} \times \left\{ \frac{\left(1 - \frac{a}{x} q^{j+1}\right)^{(\alpha+\beta-1)}}{(1-q)^{(\alpha+\beta-1)}} - \sum_{n=0}^{\infty} \frac{(1-q^{n+1})^{(\beta-1)}}{(1-q)^{(\beta-1)}} \frac{\left(1 - \frac{a}{x} q^{j+1-n}\right)^{(\alpha-1)}}{(1-q)^{(\alpha-1)}} q^{n\alpha} \right\}.$$

Putting $\mu = q^j a/x$ into (16), we see that $c_j = 0$ for all $j \in \mathbb{N}$, which completes the proof. $\square$

Lemma 6. For $\alpha \in \mathbb{R}^+$, $\lambda \in (-1, \infty)$, the following is valid

$$(18) \quad I_{q,a}^\alpha ((x-a)^{(\lambda)}) = \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\alpha+\lambda+1)} (x-a)^{(\alpha+\lambda)} \quad (0 < a < x < b).$$

Proof. For $\lambda \neq 0$, according to the definition (17), we have

$$I_{q,a}^\alpha ((x-a)^{(\lambda)}) = \frac{1}{\Gamma_q(\alpha)} \left(\int_0^x (x-qt)^{(\alpha-1)} (t-a)^{(\lambda)} d_q t - \int_0^a (x-qt)^{(\alpha-1)} (t-a)^{(\lambda)} d_q t \right).$$

Also, the following is valid:

$$\int_0^a (x-qt)^{(\alpha-1)} (t-a)^{(\lambda)} d_q t = a^{\lambda+1} (1-q) \sum_{k=0}^{\infty} (x - aq^{k+1})^{(\alpha-1)} (q^k - 1)^{(\lambda)} q^k = 0.$$

Therefrom, by using (16), we get

$$\begin{aligned} & \int_0^x (x-qt)^{(\alpha-1)} (t-a)^{(\lambda)} d_q t \\ &= x^{\alpha+\lambda} (1-q) \sum_{k=0}^{\infty} (1-q^{1+k})^{(\alpha-1)} \left(1 - \frac{a}{qx} q^{1-k}\right)^{(\lambda)} q^{(\lambda+1)k} \\ &= (1-q) \frac{(1-q)^{(\alpha-1)} (1-q)^{(\lambda)}}{(1-q)^{(\alpha+\lambda)}} (x-a)^{(\alpha+\lambda)}. \end{aligned}$$

Using (5), we obtain the required formula.

Particularly, for $\lambda = 0$, using a q -integration by parts, we have

$$\begin{aligned} (I_{q,a}^\alpha \mathbf{1})(x) &= \frac{1}{\Gamma_q(\alpha)} \int_a^x (x-qt)^{(\alpha-1)} d_q t = \frac{1}{\Gamma_q(\alpha)} \int_a^x \frac{D_q((x-t)^{(\alpha)})}{-[\alpha]_q} d_q t \\ &= \frac{-1}{\Gamma_q(\alpha+1)} \int_a^x D_q((x-t)^{(\alpha)}) d_q t = \frac{1}{\Gamma_q(\alpha+1)} (x-a)^{(\alpha)}. \quad \square \end{aligned}$$

4. THE FRACTIONAL q -DERIVATIVE

We define the *fractional q -derivative* by

$$(19) \quad (D_{q,a}^\alpha f)(x) = \begin{cases} (I_{q,a}^{-\alpha} f)(x), & \alpha < 0 \\ f(x), & \alpha = 0 \\ (D_q^{[\alpha]} I_{q,a}^{[\alpha]-\alpha} f)(x), & \alpha > 0, \end{cases}$$

where $[\alpha]$ denotes the smallest integer greater or equal to α .

Notice that $(D_{q,a}^\alpha f)(x)$ has subscript a to emphasize that it depends on the lower limit of integration used in definition (19). Since $[\alpha]$ is a positive integer for $\alpha \in \mathbb{R}^+$, then for $(D_q^{[\alpha]} f)(x)$ we apply definition (8).

Lemma 7. *For $\alpha \in \mathbb{R} \setminus \mathbb{N}_0$, the following is valid:*

$$(D_q D_{q,a}^\alpha f)(x) = (D_{q,a}^{\alpha+1} f)(x) \quad (0 < a < x < b) .$$

Proof. We will consider three cases. For $\alpha \leq -1$, according to Theorem 5, we have

$$\begin{aligned} (D_q D_{q,a}^\alpha f)(x) &= (D_q I_{q,a}^{-\alpha} f)(x) = (D_q I_{q,a}^{1-\alpha-1} f)(x) \\ &= (D_q I_{q,a} I_{q,a}^{-\alpha-1} f)(x) = (I_{q,a}^{-(\alpha+1)} f)(x) = (D_{q,a}^{\alpha+1} f)(x). \end{aligned}$$

In the case $-1 < \alpha < 0$, i.e., $0 < \alpha + 1 < 1$, we obtain

$$(D_q D_{q,a}^\alpha f)(x) = (D_q I_{q,a}^{-\alpha} f)(x) = (D_q I_{q,a}^{1-(\alpha+1)} f)(x) = (D_{q,a}^{\alpha+1} f)(x).$$

For $\alpha > 0$, we get

$$(D_q D_{q,a}^\alpha f)(x) = (D_q D_q^{[\alpha]} I_{q,a}^{[\alpha]-\alpha} f)(x) = (D_q^{[\alpha]+1} I_{q,a}^{[\alpha]-\alpha} f)(x) = (D_{q,a}^{\alpha+1} f)(x). \quad \square$$

Theorem 8. *For $\alpha \in \mathbb{R} \setminus \mathbb{N}_0$, the following is valid:*

$$(D_q D_{q,a}^\alpha f)(x) - (D_{q,a}^\alpha D_q f)(x) = \frac{f(a)}{\Gamma_q(-\alpha)} (x-a)^{(-\alpha-1)} \quad (0 < a < x < b) .$$

Proof. We will use formulas (11), Theorem 5, and Lemma 6, to prove the statement. Let us consider two cases. If $\alpha < 0$, then

$$\begin{aligned} (D_q D_{q,a}^\alpha f)(x) &= (D_q I_{q,a}^{-\alpha} f)(x) = D_q I_{q,a}^{-\alpha} ((I_{q,a} D_q f)(x) + f(a)) \\ &= (D_q I_{q,a}^{-\alpha} I_{q,a} D_q f)(x) + f(a) (D_q I_{q,a}^{-\alpha} \mathbf{1})(x) \\ &= (D_q I_{q,a}^{-\alpha+1} D_q f)(x) + f(a) D_q \left(\frac{(x-a)^{(-\alpha)}}{\Gamma_q(-\alpha+1)} \right) \\ &= (D_q I_{q,a} I_{q,a}^{-\alpha} D_q f)(x) + f(a) \frac{[-\alpha]_q (x-a)^{(-\alpha-1)}}{\Gamma_q(-\alpha+1)} \\ &= (D_{q,a}^\alpha D_q f)(x) + \frac{f(a)}{\Gamma_q(-\alpha)} (x-a)^{(-\alpha-1)}. \end{aligned}$$

If $\alpha > 0$, there exists $l \in \mathbb{N}_0$, such that $\alpha \in (l, l+1)$. Then, applying a similar procedure, we get

$$\begin{aligned}
 (D_q D_{q,a}^\alpha f)(x) &= (D_q D_q^{l+1} I_{q,a}^{l+1-\alpha} f)(x) \\
 &= D_q^{l+2} I_{q,a}^{l+1-\alpha} ((I_{q,a} D_q f)(x) + f(a)) \\
 &= (D_q^{l+1} D_q I_{q,a} I_{q,a}^{l+1-\alpha} D_q f)(x) + \frac{f(a)}{\Gamma_q(l+2-\alpha)} D_q^{l+1} ((x-a)^{(l+1-\alpha)}) \\
 &= (D_{q,a}^\alpha D_q f)(x) + \frac{f(a)}{\Gamma_q(-\alpha)} (x-a)^{(-\alpha-1)}. \quad \square
 \end{aligned}$$

5. THE FRACTIONAL q -TAYLOR-LIKE FORMULA

Many authors tried to generalize the ordinary TAYLOR formula in different manners. The use of the fractional calculus is of special interest in that area (see, for example [11] and [8]). Here, we will present one more generalization, based on the use of the fractional q -derivatives.

Lemma 9. *Let $f(x)$ be a function defined on an interval $(0, b)$ and $\alpha \in \mathbb{R}^+$. Then the following is valid:*

$$(D_{q,a}^\alpha I_{q,a}^\alpha f)(x) = f(x) \quad (0 < a < x < b).$$

Proof. For $\alpha > 0$, we have

$$\begin{aligned}
 (D_{q,a}^\alpha I_{q,a}^\alpha f)(x) &= (D_q^{[\alpha]} I_{q,a}^{[\alpha]-\alpha} I_{q,a}^\alpha f)(x) = (D_q^{[\alpha]} I_{q,a}^{[\alpha]-\alpha+\alpha} f)(x) \\
 &= (D_q^{[\alpha]} I_{q,a}^{[\alpha]} f)(x) = f(x). \quad \square
 \end{aligned}$$

Lemma 10. *Let $\alpha \in (0, 1)$. Then*

$$(I_{q,a}^\alpha D_{q,a}^\alpha f)(x) = f(x) + K(a)(x-a)^{(\alpha-1)} \quad (0 < a < x < b),$$

where $K(a)$ does not depend on x .

Proof. Let

$$A(x) = (I_{q,a}^\alpha D_{q,a}^\alpha f)(x) - f(x).$$

Applying $D_{q,a}^\alpha$ to the both sides of the above expression, and using Lemma 9, we get

$$\begin{aligned}
 (D_{q,a}^\alpha A)(x) &= (D_{q,a}^\alpha I_{q,a}^\alpha D_{q,a}^\alpha f)(x) - D_{q,a}^\alpha f(x) \\
 &= ((D_{q,a}^\alpha I_{q,a}^\alpha) D_{q,a}^\alpha f)(x) - D_{q,a}^\alpha f(x) = 0.
 \end{aligned}$$

On the other hand, according to Lemma 6, we obtain

$$D_{q,a}^\alpha ((x-a)^{(\alpha-1)}) = D_q I_{q,a}^{1-\alpha} ((x-a)^{(\alpha-1)}) = (D_q \mathbf{1})(x) = 0.$$

Hence, we conclude that $A(x)$ is a function of the form

$$A(x) = K(a)(x-a)^{(\alpha-1)}. \quad \square$$

Lemma 11. *Let $0 < a \leq c < x < b$ and $\alpha \in (0, 1)$. Then the following is valid:*

$$(I_{q,c}^{\alpha+k} D_{q,a}^{\alpha+k} f)(x) = \frac{(x-c)^{(\alpha+k)}}{\Gamma_q(\alpha+k+1)} (D_{q,a}^{\alpha+k} f)(c) + (I_{q,c}^{\alpha+k+1} D_{q,a}^{\alpha+k+1} f)(x), \quad (k \in \mathbb{N}_0).$$

Proof. According to Lemma 3 and Lemma 4, we have

$$\begin{aligned} (I_{q,c}^{\alpha+k} D_{q,a}^{\alpha+k} f)(x) &= (I_{q,c}^{\alpha+k+1} D_q D_{q,a}^{\alpha+k} f)(x) + \frac{(D_{q,a}^{\alpha+k} f)(c)}{\Gamma_q(\alpha+k+1)} (x-c)^{(\alpha+k)} \\ &= \frac{(D_{q,a}^{\alpha+k} f)(c)}{\Gamma_q(\alpha+k+1)} (x-c)^{(\alpha+k)} + (I_{q,c}^{\alpha+k+1} D_{q,a}^{\alpha+k+1} f)(x). \quad \square \end{aligned}$$

Now, we are ready to prove a TAYLOR type formula with fractional q -derivatives, which is the main result of this section.

Theorem 12. *Let $f(x)$ be defined on $(0, b)$ and $\alpha \in (0, 1)$. For $0 < a < c < x < b$, the following is true:*

$$(20) \quad f(x) = \sum_{k=0}^{n-1} \frac{(D_{q,a}^{\alpha+k} f)(c)}{\Gamma_q(\alpha+k+1)} (x-c)^{(\alpha+k)} + R_n(f),$$

with $R_n(f) = R_0(f) - K(a)(x-a)^{(\alpha-1)} + E_n(f)$, where

$$R_0(f) = \frac{1}{\Gamma_q(\alpha)} \int_a^c (x-qt)^{(\alpha-1)} (D_{q,a}^{\alpha} f)(t) d_q t,$$

and $E_n(f)$ can be represented in either of the following forms:

$$(21) \quad E_n(f) = (I_{q,c}^{\alpha+n} D_{q,a}^{\alpha+n} f)(x),$$

$$(22) \quad E_n(f) = \frac{(D_{q,a}^{\alpha+n} f)(\xi)}{\Gamma_q(\alpha+n+1)} (x-c)^{(\alpha+n)} \quad (c < \xi < x).$$

Proof. We will deduce the proof of (21) by mathematical induction. Since

$$(I_{q,a}^{\alpha} D_{q,a}^{\alpha} f)(x) = \frac{1}{\Gamma_q(\alpha)} \int_a^c (x-qt)^{(\alpha-1)} (D_{q,a}^{\alpha} f)(t) d_q t + (I_{q,c}^{\alpha} D_{q,a}^{\alpha} f)(x),$$

using Lemma 10, we obtain

$$f(x) = (I_{q,c}^{\alpha} D_{q,a}^{\alpha} f)(x) + R_0(f) - K(a)(x-a)^{(\alpha-1)}.$$

According to Lemma 11, for $k = 0$, we have

$$\begin{aligned} (I_{q,c}^\alpha D_{q,a}^\alpha f)(x) &= \frac{(D_{q,a}^\alpha f)(c)}{\Gamma_q(\alpha + 1)}(x - c)^{(\alpha)} + (I_{q,c}^{\alpha+1} D_{q,a}^{\alpha+1} f)(x) \\ &= \frac{(D_{q,a}^\alpha f)(c)}{\Gamma_q(\alpha + 1)}(x - c)^{(\alpha)} + E_1(f), \end{aligned}$$

which completes the expression for $R_1(f)$ and proves (21) for $n = 1$.

Assume that (21) is valid for any $n \in \mathbb{N}$. Then, again from Lemma 11, the following holds:

$$\begin{aligned} E_n(f) &= (I_{q,c}^{\alpha+n} D_{q,a}^{\alpha+n} f)(x) = \frac{(D_{q,a}^{\alpha+n} f)(c)}{\Gamma_q(\alpha + n + 1)}(x - c)^{(\alpha+n)} + (I_{q,c}^{\alpha+n+1} D_{q,a}^{\alpha+n+1} f)(x) \\ &= \frac{(D_{q,a}^{\alpha+n} f)(c)}{\Gamma_q(\alpha + n + 1)}(x - c)^{(\alpha+n)} + E_{n+1}(f). \end{aligned}$$

Hence the formula (21) is valid for $n + 1$. So, it is valid for each $n \in \mathbb{N}$.

The second form of remainder, (22), can be obtained by using a mean-value theorem for q -integrals [9]. Indeed, there exists $\xi \in (c, x)$, such that

$$\begin{aligned} E_n(f) &= (I_{q,c}^{\alpha+n} D_{q,a}^{\alpha+n} f)(x) = \frac{1}{\Gamma_q(\alpha + n)} \int_c^x (x - qt)^{(\alpha+n-1)} (D_{q,a}^{\alpha+n} f)(t) d_q t \\ &= \frac{(D_{q,a}^{\alpha+n} f)(\xi)}{\Gamma_q(\alpha + n)} \int_c^x (x - qt)^{(\alpha+n-1)} d_q t = \frac{(D_{q,a}^{\alpha+n} f)(\xi)}{\Gamma_q(\alpha + n)} (I_{q,c}^{\alpha+n} \mathbf{1})(x) \\ &= \frac{(D_{q,a}^{\alpha+n} f)(\xi)}{\Gamma_q(\alpha + n + 1)} (x - c)^{(\alpha+n)}. \quad \square \end{aligned}$$

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