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PROTECTING A GRAPH WITH MOBILE GUARDS

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Mobile guards on the vertices of a graph are used to defend it against attacks on either its vertices or its edges. Various models for this problem have been proposed. In this survey we describe a number of these models when the attack sequence is infinitely long and the guards must induce some particular configuration before each attack, such as a dominating set or a vertex cover. Results from the literature concerning the number of guards needed to successfully defend a graph in each of these problems are surveyed.

1. INTRODUCTION

Graph protection involves the placement of mobile guards on the vertices of a graph to protect its vertices and edges against single or sequences of attacks and has its historical roots in the time of the ancient Roman Empire. The modern study of graph protection was initiated in the late twentieth century by the appearance of four publications in quick succession that referred to the military strategy of Emperor Constantine (Constantine The Great, 274-337 AD).

The seminal paper is Ian Stewart's "Defend the Roman Empire!" in *Scientific American*, December 1999 [**56**], which contains a reply to C. S. ReVelle's "Can you protect the Roman Empire?", *Johns Hopkins Magazine*, April 1997 [**54**], and which is based on ReVelle and K. E. Rosing's "Defendens Imperium Romanum: A Classical Problem in Military Strategy" in *American Mathematical Monthly*, August – September 2000 [**55**]. ReVelle's work [**54**] in turn is a response to the paper " Graphing' an Optimal Grand Strategy" by J. Arquilla and H. Fredricksen [**4**], which appeared in *Military Operations Research* in 1995

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and which is the oldest reference we could find that places the strategy of Emperor Constantine in a mathematical setting.

According to ancient history – some say mythology – Rome was founded by Romulus and Remus in 760 – 750 BC on the banks of the Tiber in central Italy. It was a country town whose power gradually grew until it was the centre of a large empire. In the third century AD Rome dominated not only Europe, but also North Africa and the Near East. The Roman army at that time was strong enough to use a *forward defense* strategy, deploying an adequate number of legions to secure on-site every region throughout the empire. However, the Roman Empire's power was greatly reduced over the following hundred years. By the fourth century AD only twenty-five legions of the Roman army were available, which made a forward defense strategy no longer feasible.

According to E. N. Luttwak, *The Grand Strategy of the Roman Empire*, as cited in [**55**], to cope with the reducing power of the Empire, Constantine devised a new strategy called a *defense in depth* strategy, which used local troops to disrupt invasion. He deployed mobile Field Armies (FAs), units of forces consisting of roughly six legions powerful enough to secure any one of the regions of the Roman Empire, to stop the intruding enemy, or to suppress insurrection. By the fourth century AD there were only four FAs available for deployment, whereas there were eight regions to be defended (Britain, Gaul, Iberia, Rome, North Africa, Constantinople, Egypt and Asia Minor) in the empire. An FA was considered capable of deploying to protect an adjacent region only if it moved from a region where there was at least one other FA to help launch it. The challenge that Constantine faced was to position four FAs in the eight regions of the empire. Consider a region to be *secured* if it has one or more FAs stationed in it already, and *securable* if an FA can reach it in one step. Constantine decided to place two FAs in Rome and another two FAs in Constantinople, making all regions either secured or securable – with the exception of Britain, which could only be secured after at least four movements of FAs.

It is mentioned in [**4, 55, 56**] that Constantine's "defense in depth" strategy was adopted during World War II by General Douglas MacArthur. When conducting military operations in the Pacific theatre he pursued a strategy of "island-hopping" – moving troops from one island to a nearby one, but only when he could leave behind a large enough garrison to keep the first island secure. The efficiency of Constantine's strategy under different criteria, and ways in which it can be improved, were also discussed in these three articles.

Constantine's strategy is now known in domination theory as **Roman domination**. A *Roman dominating function* on a graph $G = (V, E)$ is a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex u with $f(u) = 0$ is adjacent to at least one vertex v with $f(v) = 2$. The modern graph theoretical version of Roman domination was introduced in 2004 by Cockayne, Dreyer, Hedetniemi and Hedetniemi [**15**]. ***Weak Roman domination***, an alternative defense strategy that can be used if defense units can move without another unit being present, was introduced in 2003 by Henning and Hedetniemi in [**28**].

(Contrary to publication dates, the work in [**15**] preceded that in [**28**].) A function $f : V \to \{0, 1, 2\}$ is a *weak Roman dominating function of* G if each vertex u with $f(u) = 0$ is adjacent to a vertex v with $f(v) > 0$ such that the function $f' = (f - \{(v, f(v)), (u, 0)\}) \cup \{(v, f(v) - 1), (u, 1)\}$ also has the property that each vertex labelled 0 is adjacent to a vertex with positive label. **Secure domination** is a defense strategy that can be used when it is not possible or desirable to station two defense units at the same location. A *secure dominating function* is a weak Roman dominating function f such that $\{v \in V : f(v) = 2\} = \emptyset$. In this case the set $\{v \in V : f(v) = 1\}$ is a *secure dominating set* of G. Secure domination was introduced by Cockayne, Grobler, Gründlingh, Munganga and Van Vuuren [**17**]; again, the work in this paper preceded the work in papers with earlier publication dates, for example [**9, 16**].

A full discussion of Roman domination, weak Roman domination and secure domination is beyond the scope of this survey. A list of publications concerning these topics and their variations can be found in [**48**] on which this paper is based. Here we focus on securing the vertices and edges of graphs against infinite sequences of attacks, executed one at a time, by stationing defense units, henceforth called guards, at the vertices of the graph. At most one guard is stationed at each vertex, and guards that move in response to an attack do not return to their original positions before facing another attack. We refer to such models as *eternal*, as they can be thought of as protecting a graph for eternity. A number of different eternal protection models have been studied. We introduce them in the next section. A list of the parameters and symbols used can be found in Section 11.

2. GRAPH PROTECTION MODELS

A *dominating set* of a graph $G = (V, E)$ is a set $D \subseteq V$ such that each vertex in $V - D$ is adjacent to a vertex in D. The minimum cardinality amongst all dominating sets of G is the *domination number* $\gamma(G)$. By imposing conditions on the subgraph $G[D]$ of G induced by D we obtain several varieties of dominating sets and their associated parameters. For example, if $G[D]$ is connected, then D is a *connected dominating set* and the corresponding parameter is the *connected domination number* $\gamma_c(G)$, and if $G[D]$ has no isolated vertices, then D is a *total dominating set* and the minimum cardinality amongst all total dominating sets is the *total domination number* $\gamma_t(G)$. Only connected graphs have connected dominating sets, and only graphs without isolated vertices have total dominating sets. Domination theory can be considered the precursor to the study of graph protection: one may view a dominating set as an immobile set of guards protecting a graph. A thorough survey of domination theory can be found in the well-known book [**31**] by Haynes, Hedetniemi and Slater.

A *vertex cover* of G is a set $C \subseteq V$ such that each edge of G is incident with a vertex in C. The minimum cardinality of a vertex cover of G is the *vertex cover number* (also sometimes called the *vertex covering number*) $\tau(G)$ of G. An *independent set* of G is a set $I \subseteq V$ such that no two vertices in I are adjacent.

The maximum cardinality amongst all independent sets is the *independence number* $\alpha(G)$. The independence number of G equals the clique number $\omega(\overline{G})$ of the complement $\overline{G}$ of G. **Gallai's Theorem** [**22**] states that $\alpha(G) + \tau(G) = n$ for all graphs G of order n (also see e.g. [**13**, p. 241]). A *matching* in G is a set of edges, no two of which have a common vertex. The *matching number* $m(G)$ is the maximum cardinality of a matching of G. It is also well known that $\tau(G) \geq m(G)$ for all graphs, and that equality holds for bipartite graphs. The latter result is known as **König's Theorem** [**49**] (also see e.g. [**13**, Theorem 9.13]).

Let $\{D_i\}$, $D_i \subseteq V$, $i \geq 1$, be a collection of sets of vertices of the same cardinality, with one guard located on each vertex of D_i. Each protection strategy can be modeled as a two-player game between a *defender* and an *attacker*: the defender chooses D_1 as well as each D_i, $i > 1$, while the attacker chooses the locations of the attacks $r_1, r_2, \ldots$. Each attack is dealt with by the defender by choosing the next D_i subject to some constraints that depend on the particular game. The defender wins the game if they can successfully defend any sequence of attacks, subject to the constraints of the game described below; the attacker wins otherwise.

We say that a vertex (edge) is *protected* if there is a guard on the vertex or on an adjacent (incident) vertex. A vertex v is *occupied* if there is a guard on v, otherwise v is *unoccupied*. An attack is *defended* if a guard moves to the attacked vertex (across the attacked edge).

For the **eternal domination problem**, each D_i, $i \geq 1$, is required to be a dominating set, $r_i \in V$ (assume without loss of generality $r_i \notin D_i$), and D_{i+1} is obtained from D_i by moving one guard to r_i from an adjacent vertex $v \in D_i$. If the defender can win the game with the sets $\{D_i\}$, then each D_i is an *eternal dominating set*. The size of a smallest eternal dominating set of G is the *eternal domination number* $\gamma^\infty(G)$. This problem was first studied by Burger, Cockayne, Gründlingh, Mynhardt, Van Vuuren and Winterbach in [**10**] and will sometimes be referred to as the *one-guard moves* model.

For the **m-eternal dominating set problem**, each D_i, $i \geq 1$, is required to be a dominating set, $r_i \in V$ (assume without loss of generality $r_i \notin D_i$), and D_{i+1} is obtained from D_i by moving guards to neighboring vertices. That is, each guard in D_i may move to an adjacent vertex, as long as one guard moves to r_i. Thus it is required that $r_i \in D_{i+1}$. The size of a smallest *m-eternal dominating set* (defined as for an eternal dominating set) of G is the *m-eternal domination number* $\gamma_{\mathrm{m}}^\infty(G)$. This "multiple guards move" version of the problem was introduced by Goddard, Hedetniemi and Hedetniemi [**23**]. It is also called the "all-guards move" model. It is clear that $\gamma^\infty(G) \geq \gamma_{\mathrm{m}}^\infty(G) \geq \gamma(G)$ for all graphs G.

As for dominating sets, we obtain variations on the above-mentioned protection models by imposing conditions on $G[D_i]$. Thus we define the *eternal total* (*connected*, respectively) *domination number* $\gamma_t^\infty(G)$ ($\gamma_c^\infty(G)$, respectively) and the *m-eternal total* (*connected*, respectively) *domination number* $\gamma_{\mathrm{mt}}^\infty(G)$ ($\gamma_{\mathrm{mc}}^\infty(G)$, respectively) in the obvious way. Eternal total domination and eternal connected domination were introduced by the authors in [**43**].

For the **m-eternal vertex cover problem,** each D_i, $i \geq 1$, is required to be a vertex cover, $r_i \in E$, and D_{i+1} is obtained from D_i by moving guards to neighboring vertices; all guards in D_i may move to adjacent vertices provided that one of them moves across edge r_i (we assume without loss of generality that one of the two vertices incident with r_i is not in D_i, otherwise the two guards on the two vertices of r_i can simply interchange positions). If the defender can win the game with the sets $\{D_i\}$, then each D_i is an *eternal vertex cover.* The size of a smallest eternal vertex cover of G is the *eternal cover number* $\tau_{\mathrm{m}}^{\infty}(G)$. The m-eternal vertex cover problem (or just the eternal vertex cover problem, for simplicity) was introduced by the authors in [**42**] and was also studied by Fomin, Gaspers, Golovach, Kratsch and Saurabh in [**20, 21**] and Anderson, Brigham, Carrington, Dutton, Vitray and Yellen in [**2, 3**]. As in the case of domination, $\tau_{\mathrm{m}}^{\infty}(G) \geq \tau(G)$ for all graphs G. Also, for any graph G without isolated vertices, $\tau(G) \geq \gamma(G)$ and $\tau_{\mathrm{m}}^{\infty}(G) \geq \gamma_{\mathrm{m}}^{\infty}(G)$.

We discuss these and other related protection models in Sections 4–8, address complexity issues (a wide open field) in Section 9, and present a list of open problems in Section 10.

We conclude this section with some remarks about the nature of the attack sequence $\{r_i\}$. There are three main ways for the attacker to choose and reveal $\{r_i\}$. Following the notation used for the k-server problem (see Section 4.2), they are as follows.

1. **Offline problem:** the entire sequence $r_1, r_2, \ldots, r_m$ of attacks is chosen and revealed in advance.

2. **Adaptive online problem:** the sequence of attacks is chosen and revealed one by one by the attacker alternating with the guard movements by the defendant. The attacker is called an *adaptive adversary.*

3. **Oblivious online problem:** the sequence of attacks is constructed in advance by an adversary, but revealed one by one in response to each guard movement. The adversary in this case is called an *oblivious adversary.*

The offline problem, even if the finite sequence $r_1, r_2, \ldots, r_m$ is repeated indefinitely, is not the same as the eternal domination problem. The minimum number of guards required to defend such a predefined attack sequence could be strictly less than the eternal domination number. We only consider this type of attack sequence for the k-server problem in Section 4.2. The adaptive online problem is precisely the eternal domination problem as described above: the location of each attack is chosen by the attacker depending on the location of the guards at that time. At first glance, the oblivious online problem appears to be somewhat different from the adaptive online problem, and to be the same as the original eternal domination problem described in [**10**]. However, the defender is required to defend against *any* attack sequence and has no advance knowledge of the sequence. Furthermore, one can assume the attacker is aware of the defense strategy; and so the attacker can predict the moves of the defender unless the defender employs a randomized strategy. Because randomized strategies are not relevant for the types of results described in this paper, for our purposes, the two types of attack models are equivalent. Certainly, the associated parameters are equal. Randomized strategies are relevant when one asks questions that might concern the number of (expected) moves before some configuration is reached, for example.

3. DEFINITIONS

The *open* and *closed neighbourhoods* of $X \subseteq V$ are $N(X) = \{v \in V : v$ is adjacent to a vertex in $X\}$ and $N[X] = N(X) \cup X$, respectively, and $N(\{v\})$ and $N[\{v\}]$ are abbreviated, as usual, to $N(v)$ and $N[v]$. For any $v \in X$, the *private neighbourhood* $\mathrm{pn}(v, X)$ *of* v *with respect to* X is the set of all vertices in $N[v]$ that are not contained in the closed neighbourhood of any other vertex in X, i.e., $\mathrm{pn}(v, X) = N[v] - N[X - \{v\}]$. The elements of $\mathrm{pn}(v, X)$ are the *private neighbours of* v *relative to* X. The *external private neighbourhood of* v *with respect to* X is the set $\mathrm{epn}(v, X) = \mathrm{pn}(v, X) - \{v\} = N(v) - N[X - \{v\}]$.

The *clique covering number* $\theta(G)$ is the minimum number k of sets in a partition $V = V_1 \cup \cdots \cup V_k$ of V such that each $G[V_i]$ is complete. Hence $\theta(G)$ equals the chromatic number $\chi(\overline{G})$ of the complement $\overline{G}$ of G. Since $\chi(G) = \omega(G)$ if G is perfect, and G is perfect if and only if $\overline{G}$ is perfect [**13**, p. 203], $\alpha(G) = \theta(G)$ for all perfect graphs.

The *circulant graph* $C_n[a_1, \ldots, a_k]$, where $1 \le a_1 \le \cdots \le a_k \le \left\lfloor \frac{n}{2} \right\rfloor$, is the graph with vertex set $\{v_0, \ldots, v_{n-1}\}$ such that v_i and v_j are adjacent if and only if $i - j \equiv \pm a_\ell \pmod{n}$ for some $\ell \in \{1, \ldots, k\}$.

The Cartesian product of two graphs G and H is denoted $G \,\square\, H$; a definition can be found in [**31**].

4. ETERNAL DOMINATION

The eternal domination problem was first studied by Burger et al. [**10**] in 2004 where it was called infinite order domination. That paper, and this section, consider the one-guard moves model. Shortly thereafter, Goddard et al. published a second paper on the subject where they called it *eternal security* [**23**].

Consider an eternal dominating set D of a graph G. A necessary condition for a guard on D to defend a neighbouring vertex in a winning strategy is given below. As far as we know this result has not been stated explicitly before.

Proposition 1. *Let D be an eternal dominating set of a graph G. If a guard on $v \in D$ can move to a vertex $u \in V - D$ in a winning strategy, then* $\mathrm{pn}(v, D) \cup \{u\}$ *induces a clique.*

Proof. Suppose the guard g on v moves to u in a winning strategy. If the next attack is at $x \in \mathrm{pn}(v, D)$, g moves to x, as it is the only guard that protects x. Since this holds whether $u \in \mathrm{pn}(v, D)$ or not, $\mathrm{pn}(v, D) \cup \{u\}$ induces a clique. □

The converse of Proposition 1 is not true. Consider the graph G in Figure 1. The set $D = \{x, y, z\}$ is an eternal dominating set of G in which the guard on x (y, z) defends $\{x, u, r\}$ ($\{y, v, s\}$, $\{z, w\}$). Also, $\mathrm{pn}(y, D) = \{y, v\}$ and $G[\{y, v, r\}]$ is a clique. Suppose, however, the guard on y moves to r. If the next attack is at s, then only z has a guard adjacent to s. But moving this guard to s leaves w unprotected. In the graph H in Figure 1, $D = \{x, y\}$ is not an eternal dominating set, even though $\mathrm{pn}(x, D) \cup \{r\}$, $\mathrm{pn}(x, D) \cup \{w\}$, $\mathrm{pn}(y, D) \cup \{w\}$, $\mathrm{pn}(y, D) \cup \{s\}$ all induce cliques: first attack r; without loss of generality, the guard on x moves there. Now attack s. If the guard on y moves there, then w is not protected; if the guard on r moves there, then u is not protected.

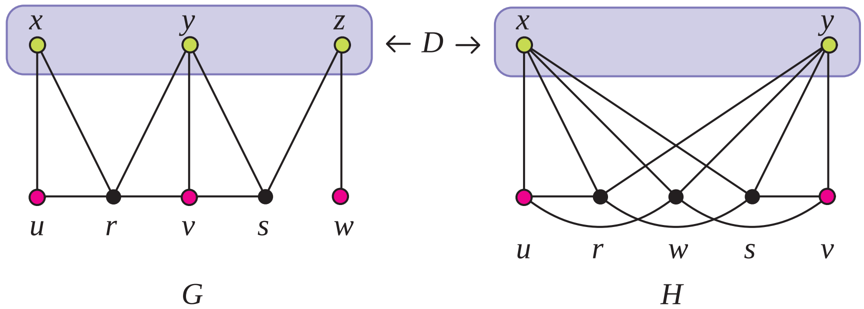

Figure 1. In G, y does not defend r, and D is not an eternal dominating set of H

4.1. Bounds for the eternal domination number

As first observed by Burger et al. [**10**], it does not take much imagination to see that γ^∞ lies between the independence and clique covering numbers.

Proposition 2. *For any graph G, $\alpha(G) \leq \gamma^\infty(G) \leq \theta(G)$.*

Proof. To see the lower bound, consider a sequence of consecutive attacks at the vertices of a maximum independent set. To see the upper bound, observe that a single guard can defend all vertices of a clique. □

Since $\alpha(G) = \theta(G)$ for perfect graphs, the bounds in Proposition 2 are tight for perfect graphs. A topic that has received much attention is finding classes of non-perfect graphs that satisfy one of the bounds in Proposition 2. Before proceeding, we should point out that the independence number, eternal domination number, and clique covering number can vary widely.

Theorem 3. [**37**] *For any positive integers c and d there exists a connected graph G such that $\alpha(G) + c \leq \gamma^\infty(G)$ and $\gamma^\infty(G) + d \leq \theta(G)$.*

Let C_n^k denote the k^{th} power (see [**13**, p. 105]) of the cycle of order n, where $2k + 1 < n$.

Theorem 4. *If G is a graph in one of the following classes, then $\gamma^\infty(G) = \theta(G)$.*

(a) [**10**] *Perfect graphs.*

(b) [**10**] *Any graph G such that $\theta(G) \leq 3$.*

(c) [**37**] C_n^k *and* $\overline{C_n^k}$*, for all* $k \geq 1$, $n \geq 3$.

(d) [**53**] *Circular-arc graphs (intersection graphs of a family of arcs of a circle).*

(e) [**1**] K_4*-minor-free graphs (a.k.a. series-parallel graphs, see e.g. [**58**, p. 336] for definition).*

(f) [**1**] $C_m \,\square\, C_n$; $P_m \,\square\, C_n$.

Goddard et al. [**23**] showed that

$$\text{if } \alpha(G) = 2, \text{ then } \gamma^\infty(G) \leq 3. \tag{1}$$

The Mycielski construction (see [**13**, p. 203]) yields triangle-free k-chromatic graphs for arbitrary k. The complements of these graphs have $\alpha = 2$ and $\theta = k$, and hence are examples of graphs with small eternal domination numbers and large clique covering numbers. The Grötzsch graph is the smallest 4-chromatic triangle-free graph, and its complement is the smallest known graph with $\gamma^\infty < \theta$. Goddard et al. [**23**] also gave the first example of a graph G with $\alpha(G) < \gamma^\infty(G) < \theta(G)$: the circulant graph $C_{18}[1,3,8]$, which satisfies $\alpha = 6$, $\gamma^\infty = 8$ and $\theta = 9$.

Klostermeyer and MacGillivray [**35**] proved the existence of graphs with $\gamma^\infty = \alpha$ and whose clique covering number is either equal to two (if $\alpha = 2$) or arbitrary otherwise. Their proof rests (i. a.) on the observation that if H is an induced subgraph of G and π is any of the parameters α, γ^∞, θ, then $\pi(H) \leq \pi(G)$. This is trivially true for α and θ. To see that it is true for γ^∞, note that a sequence of attacks on G but restricted to H requires $\gamma^\infty(H)$ guards, hence $\gamma^\infty(G) \geq \gamma^\infty(H)$.

Theorem 5. [**35**]

(a) *If* $\alpha(G) = \gamma^\infty(G) = 2$*, then* $\theta(G) = 2$.

(b) *For all integers* $k \geq a \geq 3$ *there exists a connected graph* G *such that* $\alpha(G) = \gamma^\infty(G) = a$ *and* $\theta(G) = k$.

Proof. (a) The statement is clearly true for graphs of order three and four. Assume it to be true for all graphs of order less than n, where $n \geq 5$, and let G be an n-vertex graph with $\alpha(G) = \gamma^\infty(G) = 2$. Let u and v be nonadjacent vertices of G. After consecutive attacks on u and v, both these vertices are occupied. By Proposition 1, $\mathrm{pn}(v, \{u,v\}) = V - N[u]$ and $\mathrm{pn}(u, \{u,v\}) = V - N[v]$ induce cliques. Let W and Y be the sets of all vertices in $N(u) \cap N(v)$ that are defended by the guard on v and the guard on u, respectively. By Proposition 1, each $w \in W$ ($y \in Y$, respectively) is adjacent to each vertex in $V - N[u]$ ($V - N[v]$, respectively).

Let $H' = G[N(u) \cap N(v)]$ and $H = G[V(H') \cup \{u,v\}]$. (Possibly $H = G$.) If H' is complete, then $(V - N[v]) \cup W$ and $(V - N[u]) \cup Y$ induce cliques that contain

$V(G)$. Hence suppose H' is not complete. Then $\alpha(H') \geq 2$ and so $\gamma^\infty(H') \geq 2$. Since H' is an induced subgraph of H, which is an induced subgraph of G, it follows that $\alpha(H) = \gamma^\infty(H) = 2$ and $\alpha(H') = \gamma^\infty(H') = 2$. By the induction hypothesis, $\theta(H') = 2$ and so $\theta(H) = 2$. Partition $V(H)$ into the cliques C_u, C_v, where $u \in V(C_u)$, $v \in V(C_v)$. Clearly, $V(C_u) - \{u\} \subseteq W$ and $V(C_v) - \{v\} \subseteq Y$. Therefore $(V - N[v]) \cup V(C_u)$ and $(V - N[u]) \cup V(C_v)$ induce a clique partition of G.

(b) Let H be the complement of a triangle-free k-chromatic graph, $k \geq 3$. Then $\alpha(H) = 2$ and $\theta(H) = k$. By (a), $\gamma^\infty(H) \geq 3$, and thus by (1), $\gamma^\infty(H) = 3$. Add a new vertices $v_1, \ldots, v_a$, joining each v_i to each vertex of H to form the graph G. Then $\alpha(G) = a$, and, since $a \leq k$, $\theta(G) = k$. Place a guard on each v_i, $i > 3$; these guards never move. The remaining three guards protect H and v_1, v_2, v_3 according to the strategy for H; when v_i is attacked, any guard moves there, and returns to H when required. □

Goddard et al. [**23**] asked whether the eternal domination number can be bounded by a constant times the independence number. That this is impossible in general follows from the next two theorems. One of the main results on eternal domination is the following upper bound, due to Klostermeyer and MacGillivray [**36**].

Theorem 6. [**36**] *For any graph G,*

$$\gamma^\infty(G) \leq \binom{\alpha(G)+1}{2}.$$

Proof. Assume $|V| > \binom{\alpha+1}{2}$, otherwise we are done. Consider pairwise disjoint sets $S_\alpha, S_{\alpha-1}, \ldots, S_1$, where S_α is a maximum independent set of G and, for $i = \alpha-1, \alpha-2, \ldots, 1$, the set S_i is either empty or an independent set of size i. Other than S_α, no S_i needs to be a maximal independent set. Among all collections of such sets, we choose one such that $|\bigcup_{i=1}^{\alpha} S_i|$ is maximum. Since $|V| > \binom{\alpha+1}{2}$, the set $S_1 \neq \emptyset$. Let $D = \bigcup_{i=1}^{\alpha} S_i$ and note that $|D| \leq \binom{\alpha(G)+1}{2}$. We describe a defense strategy ♣ which shows that D is an eternal dominating set of G.

♣ Whenever there is an attack at a vertex $v \notin D$, a guard on a vertex u from the set S_t with the smallest subscript among those with a vertex adjacent to v moves to v. Such a set S_t exists because S_α is a dominating set (as it is a maximum independent set).

The key technical part of the proof is to show that $(D - \{u\}) \cup \{v\}$ can be partitioned into disjoint independent sets with the same properties as the sets $\{S_i\}$. There are two cases.

If $S'_t = (S_t - \{u\}) \cup \{v\}$ is an independent set, then replacing S_t by S'_t yields another collection of disjoint independent sets as desired. Otherwise, v is adjacent to at least two vertices in S_t and $t > 1$. Let r be the greatest integer less than t such that $S_r \neq \emptyset$. We show that $r = t - 1$.

Suppose $r \leq t - 2$. Then $S_{r+1} = \emptyset$. By definition of t, no vertex in S_r is adjacent to v, hence $S_r \cup \{v\}$ is an independent set of cardinality $r + 1$. The collection of independent sets obtained by replacing S_{r+1} by $S_r \cup \{v\}$ and S_r by $\emptyset$ contradicts the maximality of $|\bigcup_{k=1}^{\alpha} S_k|$. Hence $r = t - 1$.

Replacing S_t by $S_{t-1} \cup \{v\}$ and S_{t-1} by $S_t - \{u\}$ gives another collection of independent sets with the desired properties. Thus we may repeat ♣ indefinitely to protect G against any sequence of attacks. □

Goldwasser and Klostermeyer [**24**] showed that this bound is sharp for certain graphs. Specifically, let $G(n, k)$ be the graph with vertex set consisting of the set of all k-subsets of an n-set and where two vertices are adjacent if and only if their intersection is nonempty (thus $G(n, k)$ is the complement of a Kneser graph).

Theorem 7. [**24**] *For each positive integer t, if k is sufficiently large, then the graph $G(kt + k - 1, k)$ has eternal domination number $\binom{t+1}{2}$.*

Regan [**53**] found another graph for which the bound is sharp: the circulant graph $C_{22}[1, 2, 4, 5, 9, 11]$. Theorems 6 and 7 show that it is impossible to find a constant c such that $\gamma^{\infty}(G) \leq c\alpha(G)$ for all graphs G. It would be of interest to find other graphs where the bound in Theorem 6 is sharp.

As shown by Klostermeyer and MacGillivray [**36**], the graph G obtained by joining a new vertex to m disjoint copies of C_5 satisfies $\alpha(G) = 2m$ and $\gamma^{\infty}(G) = 3m$, that is, $\gamma^{\infty}(G)/\alpha(G) = \frac{3}{2}$. This result and Theorem 5 can be placed in a more general setting (see Theorem 8), as explained in [**37**].

A triple (a, g, t) of positive integers is called *realizable* if there exists a connected graph G with $\alpha(G) = a$, $\gamma^{\infty}(G) = g$ and $\theta(G) = t$. Theorem 6 shows that no triple with $g > \binom{a+1}{2}$ is realizable. The following theorem, stated in [**37**], provides a partial solution to the question of which triples are realizable.

Theorem 8. *Let (a, g, t) be a triple of positive integers such that $a \leq g \leq t$.*

(a) *The only realizable triple with $a = 1$ is $(1, 1, 1)$.*

(b) [**10, 23, 35**] *The only realizable triples with $a = 2$ are $(2, 2, 2)$ and $(2, 3, t)$, $t \geq 3$.*

(c) [**10, 35, 37**] *For all integers a, g and t with $3 \leq a \leq g \leq \frac{3}{2}a$ and $g \leq t$, the triple (a, g, t) is realizable.*

The circulant $C_{21}[1, 3, 8]$, which satisfies $\gamma^{\infty}/\alpha = \frac{10}{6}$ (see [**23**]), shows that Theorem 8 does not characterize realizable triples.

4.2. The k-server problem

We briefly mention the k-server problem, which is related to the eternal domination problem. The k-server problem is an algorithmic problem set in the more general framework of metric spaces, but often focused on graphs. It was defined by Manasse, McGeoch and Sleator in [**51**] as follows. There are k mobile servers (or guards) located at vertices of a graph. In response to an attack on an unoccupied vertex r_i, a server must move to r_i. The objective is to minimize the total distance travelled by all the servers over the sequence of attacks. The three main variations of the problem are (1) the offline problem, (2) the adaptive online problem and (3) the oblivious online problem, as described in Section 2.

A simple polynomial time algorithm using dynamic programming can compute the optimal solution for the offline problem [**51**]. A faster algorithm is given by Chrobak, Karloff, Payne and Vishwanathan in [**14**].

Koutsoupias and Papadimitriou proved in [**50**] that a simple algorithm known as the work-function algorithm is $2k-1$ competitive. In other words, the distance the servers travel using the work function algorithm is at most $2k-1$ times the distance they would travel using any other algorithm, including an optimal algorithm that knew the entire attack sequence in advance, over all attack sequences. It is a famous conjecture in computer science that the work function algorithm is k-competitive and that this would be best possible.

A key difference between problems (2) and (3) is that a randomized algorithm can be effective in problem (3). Since an oblivious adversary cannot adapt the attack sequence to the moves of the algorithm, by using randomization an algorithm may be able to effectively prevent an adversary from constructing a costly attack sequence. A famous result by McGeoch and Sleator [**52**] is an H_k-competitive algorithm for the problem of k servers on a complete graph with $k+1$ vertices, where H_k is the k^{th} harmonic number. This result is known to be optimal.

5. m-ETERNAL DOMINATION

As mentioned in Section 2, m-eternal dominating sets are defined is the same way as eternal dominating sets, except that when an attack occurs, each guard is allowed to move to a neighbouring vertex to either defend the attacked vertex or to better position themselves for the future. As stated above, we refer to this as the "all-guards move" model of eternal domination.

Goddard et al. [**23**] determine $\gamma_{\mathrm{m}}^{\infty}(G)$ exactly for complete graphs, paths, cycles, and complete bipartite graphs. They also obtained the following fundamental bound.

Theorem 9. [**23**] *For all graphs G, $\gamma(G) \le \gamma_{\mathrm{m}}^{\infty}(G) \le \alpha(G)$.*

Outline of proof. The left inequality is obvious. The right inequality is proved by induction on the order of G, the result being easy to see for small graphs. If G has a vertex v that is not contained in any maximum independent set, then v is adjacent to at least two vertices of each maximum independent set of G. Therefore $\alpha(G - N[v]) \leq \alpha(G) - 2$. Hence (by induction) $G - N[v]$ can be protected by $\alpha(G) - 2$ guards. Since $K_{1,\deg(v)}$ is a spanning subgraph of $G[N[v]]$, $G[N[v]]$ can be protected by two guards. It follows that $\gamma_{\mathrm{m}}^{\infty}(G) \leq \alpha(G)$.

If each vertex of G is contained in a maximum independent set, place a guard on each vertex of a maximum independent set M. Defend an attack on $v \in V(G) - M$ by moving all guards to a maximum independent set M_v containing v. This is possible since Hall's Marriage Theorem ensures that there is a matching between M_v and M. □

Theorem 9 places $\gamma_{\mathrm{m}}^{\infty}$ nicely in the chain

$$\gamma(G) \leq \gamma_{\mathrm{m}}^{\infty}(G) \leq \alpha(G) \leq \gamma^{\infty}(G) \leq \theta(G). \tag{2}$$

Goddard et al. also claim that $\gamma_{\mathrm{m}}^{\infty}(G) = \gamma(G)$ for all Cayley graphs G. This claim, however, is false, as is shown in the recent paper [**7**] by Braga, de Souza and Lee. By computing $\gamma(G)$ and $\gamma_{\mathrm{m}}^{\infty}(G)$ for 7871 Cayley graphs of non-abelian groups, they found 61 connected Cayley graphs G such that $\gamma_{\mathrm{m}}^{\infty}(G) = \gamma(G) + 1$. For all other connected Cayley graphs they investigated, $\gamma_{\mathrm{m}}^{\infty}(G) = \gamma(G)$.

The upper bound in Theorem 9 is not always close to the actual value of γ^{∞}. For example, $K_{1,m}$ has independence number m and can be defended with just two guards in this model. But equality holds for many graphs, such as K_n, C_n, and $P_2 \,\square\, P_3$, just to name a few. By a careful analysis of the clique structure, it was shown by Braga, de Souza and Lee in [**8**] that $\gamma_{\mathrm{m}}^{\infty}(G) = \alpha(G)$ for all proper interval graphs (a subclass of perfect graphs). Characterizing graphs with m-eternal domination number equal to the bounds in Theorem 9 remains open, as mentioned in Section 10.2. However, trees for which equality holds in the upper bound, α, are characterized by Klostermeyer and MacGillivray [**39**].

Define a *neo-colonization* to be a partition $\{V_1, V_2, \ldots, V_t\}$ of G such that each V_i induces a connected graph. A part V_i is assigned weight one if it induces a clique, and $1 + \gamma_c(G[V_i])$ otherwise, where $\gamma_c(G[V_i])$ is the connected domination number of the subgraph induced by V_i. Then $\theta_c(G)$ is the minimum weight of any neo-colonization of G.

Goddard et al. [**23**] proved that $\gamma_{\mathrm{m}}^{\infty}(G) \leq \theta_c(G) \leq \gamma_c(G) + 1$. Klostermeyer and MacGillivray [**37**] proved that equality holds in the first inequality for trees.

Theorem 10. [**37**] *If T is a tree, then $\gamma_{\mathrm{m}}^{\infty}(T) = \theta_c(T)$.*

A different upper bound is given by Chambers, Kinnersly and Prince in [**12**]. A proof is given below. A *branch vertex* of a tree is a vertex of degree at least three.

Theorem 11. *If G is a connected graph of order n, then $\gamma_{\mathrm{m}}^{\infty}(G) \leq \left\lceil \frac{n}{2} \right\rceil$.*

Proof. The proof is by induction on n, the result being easy to see for paths and cycles. Let T be a spanning tree of G with $r \geq 1$ branch vertices.

If T has no vertex of degree two, then the subgraph of T induced by the branch vertices is connected and, by [**13**, Theorem 3.7], T has at least $r+2$ leaves. Hence $n \geq 2r+2$. Place a guard on each branch vertex and on one leaf. Whenever an unoccupied leaf u is attacked, guards move so that u and all branch vertices have guards. Hence $\gamma_{\mathrm{m}}^{\infty}(T) \leq r+1 \leq \left\lceil \frac{n}{2} \right\rceil$.

If T has a vertex v of degree two, and $N(v) = \{u_1, u_2\}$, then at least one of the graphs $T - \{vu_i\}$, $i = 1, 2$, has a component of even order. Let T_1 be this component and let T_2 be the other component. Say T_i has order n_i. By the induction hypothesis, $\gamma_{\mathrm{m}}^{\infty}(T_1) \leq \frac{n_1}{2}$ and $\gamma_{\mathrm{m}}^{\infty}(T_2) \leq \left\lceil \frac{n_2}{2} \right\rceil$. It follows that $\gamma_{\mathrm{m}}^{\infty}(T) \leq \left\lceil \frac{n}{2} \right\rceil$ and therefore $\gamma_{\mathrm{m}}^{\infty}(G) \leq \gamma_{\mathrm{m}}^{\infty}(T) \leq \left\lceil \frac{n}{2} \right\rceil$. □

The bound in Theorem 11 is exact for the corona of any graph with K_1 because they have domination numbers equal to half their order. It is also equal for odd length paths. It is not known which trees attain this bound [**39**], though partial results are given by Henning, Klostermeyer and MacGillivray in [**29**]. In that paper, an upper bound of $\frac{7n}{16}$ was obtained for cubic bipartite graphs. It remains open to improve this result. The Petersen graph is an example of a cubic graph with $\gamma_{\mathrm{m}}^{\infty}(G) = 2n/5$, but it is not known if there are infinitely many graphs with this property.

It is not hard to see that for many all-guards move models, the associated parameter is bounded above by 2γ. As far as we know this result has not appeared in the literature.

Proposition 12. *For any connected graph G, $\gamma_{\mathrm{m}}^{\infty}(G) \leq 2\gamma(G)$, and the bound is sharp for all values of $\gamma(G)$.*

Proof. The result is trivial for K_1, so assume $|V(G)| \geq 2$. As shown by Bollobas and Cockayne [**6**], every graph without isolated vertices has a minimum dominating set in which each vertex has an external private neighbour. Let D be such a minimum dominating set of G. For each $v \in D$, place a guard at v and at a private neighbour of v. This configuration is an m-eternal dominating set.

To see that the bound is sharp for $\gamma = 1$, consider any star with at least three vertices. For $\gamma = 2$, consider C_6 and let u and v be two vertices at distance three apart. Add two new internally disjoint $u - v$ paths of length three to form the graph G. Obviously, $\{u, v\}$ is a γ-set of G. Let D be any dominating set of G with $|D| = 3$. Suppose $u \notin D$. Since $N(u)$ is independent with $|N(u)| = 4$, and no two vertices in $N(u)$ have a common neighbour other than u, D does not dominate $N(u)$, a contradiction. Thus $u \in D$ and similarly $v \in D$. Without loss of generality say $D = \{u, v, w\}$, where $w \in N(u)$. Then D cannot repel an attack at a vertex in $N(v) - N(w)$. It follows that $\gamma_{\mathrm{m}}^{\infty}(G) = 4 = 2\gamma(G)$.

For $\gamma = k \geq 3$, consider the cycle $C_{3k} = u_0, u_1, \ldots, u_{3k-1}, u_0$ and the γ-set $\{u_0, u_3, \ldots, u_{3k-3}\}$ of C_{3k}. For each $i = 0, \ldots, k-1$, add a new $u_{3i} - u_{3(i+1) (\mathrm{mod}\ 3k)}$ path of length three to form G. Then $\gamma(G) = k$, but it can be shown as in the previous case that no set of $2k - 1$ vertices eternally protects the vertices of G. □

Klostermeyer and MacGillivray [**39**] characterized trees for which equality holds in the following bounds: $\gamma_{\rm m}^{\infty}(T) \le \gamma_c(T)+1$, $\gamma(T) \le \gamma_{\rm m}^{\infty}(T)$, $\gamma_{\rm m}^{\infty}(T) \le 2\gamma(T)$, and $\gamma_{\rm m}^{\infty}(T) \le \alpha(T)$.

Grid graphs, i.e. $P_n \,\square\, P_m$, are a well-studied class of graphs in domination theory; see [**31**]. We sometimes refer to $P_n \,\square\, P_m$ as the $n \times m$ grid graph. As shown by Goldwasser, Klostermeyer and Mynhardt [**25**], $\gamma_{\rm m}^{\infty}(P_2 \,\square\, P_n) = \left\lceil \frac{2n}{3} \right\rceil$ for any $n \ge 2$, while $\gamma_{\rm m}^{\infty}(P_3 \,\square\, P_n) = n$ for $2 \le n \le 8$. Based on these results, the next two theorems by Goldwasser et al. and Finbow, Messinger and Van Bommel [**19**], respectively, are quite surprising.

Theorem 13. [**25**] *For* $n \ge 9$, $\gamma_{\rm m}^{\infty}(P_3 \,\square\, P_n) \le \left\lceil \frac{8n}{9} \right\rceil$.

Theorem 14. [**19**] *For* $n > 11$, $1 + \left\lceil \frac{4n}{5} \right\rceil \le \gamma_{\rm m}^{\infty}(P_3 \,\square\, P_n) \le 2 + \left\lceil \frac{4n}{5} \right\rceil$.

Theorem 14 shows that the bound in Theorem 13 is not sharp in general, although it is sharp for $n = 9, 10$ for example. It is conjectured in [**25**] that the lower bound in Theorem 14 gives the exact value of $\gamma_{\rm m}^{\infty}(P_3 \,\square\, P_n)$ for $n \ge 10$.

Beaton, Finbow and MacDonald [**5**] continued the study of m-eternal domination in grid graphs and obtained the following results.

Theorem 15. [**5**]

(a) *For any* $n \in \mathbb{Z}^+$, $\gamma_{\rm m}^{\infty}(P_4 \,\square\, P_n) = 2\left\lceil \frac{n+1}{2} \right\rceil$, *with the exceptions* $\gamma_{\rm m}^{\infty}(P_4 \,\square\, P_2) = 3$ *and* $\gamma_{\rm m}^{\infty}(P_4 \,\square\, P_6) = 7$.

(b) *For any* $n \in \mathbb{Z}^+$, $\left\lfloor \frac{10(n+1)}{7} \right\rfloor \le \gamma_{\rm m}^{\infty}(P_6 \,\square\, P_n) \le \left\lceil \frac{8n}{5} \right\rceil + 8$.

(c) $\gamma_{\rm m}^{\infty}(P_5 \,\square\, P_5) = 7$, $\gamma_{\rm m}^{\infty}(P_6 \,\square\, P_6) = 10$, *and* $13 \le \gamma_{\rm m}^{\infty}(P_7 \,\square\, P_7) \le 14$.

Van Bommel and Van Bommel [**57**] studied protection in $5 \times n$ grids, obtaining exact values of $\gamma_{\rm m}^{\infty}(P_5 \,\square\, P_n)$ for $n \le 12$ and the following bounds.

Theorem 16. [**57**] $\left\lfloor \frac{6n+9}{5} \right\rfloor \le \gamma_{\rm m}^{\infty}(P_5 \,\square\, P_n) \le \left\lfloor \frac{4n+3}{3} \right\rfloor$.

Note that $\gamma_{\rm m}^{\infty}(P_n \square P_n) = \gamma(P_n \square P_n)$ for $n \in \{1,2,3,5,6\}$, while $\gamma_{\rm m}^{\infty}(P_4 \square P_4) = 6$ while $\gamma(P_4 \,\square\, P_4) = 4$. This raises the problem of determining the largest value of n for which $\gamma_{\rm m}^{\infty}(P_n \,\square\, P_n) = \gamma(P_n \,\square\, P_n)$ (see Problem 13).

We now compare the m-eternal domination number and the vertex cover number. This may seem like an unusual pair of parameters to compare, but the comparison turns out to be interesting.

Theorem 17. (a) [**44**] *If* G *is connected, then* $\gamma_{\rm m}^{\infty}(G) \le 2\gamma(G) \le 2\tau(G)$.

(b) [**44**] *If, in addition,* $\delta(G) \ge 2$, *then* $\gamma_{\rm m}^{\infty}(G) \le \tau(G)$.

(c) [**45**] *If, in addition to (a) and (b), G has girth seven or at least nine, then* $\gamma_{\mathrm{m}}^{\infty}(G) < \tau(G)$.

(d) [**45**] *For any nontrivial tree T,* $\tau(T) \leq \gamma_{\mathrm{m}}^{\infty}(T) \leq 2\tau(T)$.

It is not possible to relax the girth condition in Theorem 17(c) to girth less than five. Examples of graphs with girth less than five for which $\gamma_{\mathrm{m}}^{\infty}(G) = \tau(G)$ are given by the authors in [**45**]. The problem remains open for girths five, six, and eight, though it is believed that $\gamma_{\mathrm{m}}^{\infty}(G) < \tau(G)$ for such graphs. The trees where the bounds in Theorem 17(d) are sharp are characterized in [**45**].

The results stated up until now in this paper apply to the case when only one guard is allowed to occupy each vertex. A question stated in [**23**] is whether there is any advantage in allowing two guards to occupy the same vertex in the m-eternal domination problem. There is no advantage allowing multiple guards to occupy a single vertex in the "one guard moves" model [**10**]. However, Finbow, Gaspers, Messinger and Ottoway [**18**] have shown that there exist graphs for which it is an advantage in the all-guards move model to allow more than one guard on a vertex at a time. We sketch a proof of this, using $\gamma_{\mathrm{m}}^{*\infty}(G)$ to denote the number of guards needed if more than one guard is allowed on a vertex at a time (and all guards are allowed to move in response to an attack).

Theorem 18. *For each* $k \geq 4$ *there exists a graph* G_k *such that* $\gamma_{\mathrm{m}}^{*\infty}(G) = k + 4$ *and* $\gamma_{\mathrm{m}}^{\infty}(G) = 2k + 1$.

Proof. (Sketch) Let $K_4 - e$ be the graph formed from K_4 by deleting an edge. Define a *widget* to be the graph formed by taking two copies of $K_4 - e$ and combining one degree three vertex from each into a single vertex (so a widget has seven vertices, two of them of degree three). Form the graph G_k by taking k widgets and an additional vertex x, adding an edge between x and the vertices of degree three in each widget.

To see that $\gamma_{\mathrm{m}}^{\infty}(G_k) = 2k + 1$, observe that if there are $2k$ guards on G_k and one of them is on x, then one widget contains only one guard, which is on the vertex of degree six to dominate the four independent degree two vertices. If one of these vertices is attacked, no possible guard movement results in a configuration in which all four degree two vertices are dominated. However, 2 guards on each widget and one on x, the latter never moving, easily defend G_k.

On the other hand, by maintaining one guard on each of $k - 1$ widgets, three on one widget and two on x, one can protect G_k with $k + 4$ guards. This is done by moving both guards from x if a widget with one guard is attacked, hence that widget contains three guards immediately after an attack, and two guards from the previously attacked widget to x, whilst the remaining guard in that widget moves to the degree six vertex in the widget. If $k \geq 4$, then certainly $k + 3$ guards cannot defend G_k if there is at most one guard on x, and if there are two guards on x, then one widget has two guards and all the others only one, and no strategy suffices to protect G_k. □

It remains open to prove the existence or otherwise of a constant $c > 1$ such that G $c\gamma_{\mathrm{m}}^{*\infty}(G) \geq \gamma_{\mathrm{m}}^{\infty}(G)$ for all graphs.

If any number of guards per vertex is allowed, then the bound in Theorem 11 can be improved to $\left\lceil \frac{n}{2} \right\rceil - 1$ when $\delta(G) \geq 2$ (with four small exceptions) [**12**]. It is not known whether their result holds if each vertex contains at most one guard. Under these conditions Nordhaus-Gaddum results were also shown in [**12**], for example the following bound; they also characterize the graphs for which equality holds.

Theorem 19. [**12**] $\gamma_{\mathrm{m}}^{\infty}(G) + \gamma_{\mathrm{m}}^{\infty}(\overline{G}) \leq n + 1$.

6. ETERNAL TOTAL DOMINATION

Some results on eternal total domination are reviewed in this section. The first result applies to the "one-guard moves" model.

Theorem 20. [**43**] *For all graphs $G = (V, E)$ without isolated vertices,*

(a) $\gamma_t^{\infty}(G) > \gamma^{\infty}(G)$

(b) $\gamma_t^{\infty}(G) \leq \gamma^{\infty}(G) + \gamma(G) \leq 2\gamma^{\infty}(G) \leq 2\theta(G)$.

The authors in [**43**] give a number of results on eternal total domination in the all-guards move model, such as the following.

Theorem 21. [**43**] *For all graphs $G = (V, E)$ without isolated vertices, $\gamma_{\mathrm{mt}}^{\infty}(G) \leq 2\gamma(G)$.*

Results from [**25**] focus on grid graphs and include the following.

Theorem 22. [**25**]

(a) *For any $n \geq 3$, $\gamma_{\mathrm{mt}}^{\infty}(P_2 \square P_n) = \left\lfloor \frac{2n}{3} \right\rfloor + 2$.*

(b) *For all $n \geq 1$, $\gamma_{\mathrm{mt}}^{\infty}(P_3 \square P_n) = n + 1$.*

(c) *For any $n \geq 1$, $\gamma_{\mathrm{mt}}^{\infty}(P_4 \square P_n) \leq \left\lfloor \frac{4n}{3} \right\rfloor + 2$.*

Achieving good bounds for larger grid graphs seems quite difficult; by "good" bounds we mean better than simply partitioning the grid into disjoint, say $3 \times n$, grids.

7. ETERNAL VERTEX COVER

The eternal vertex cover problem was introduced by the authors in [**42**]. We emphasize that eternal vertex cover is non-trivial only for the all-guards move model and thus our attention is limited to that model. Some simple examples are as follows: $\tau_{\mathrm{m}}^{\infty}(C_4) = 2$, $\tau_{\mathrm{m}}^{\infty}(C_5) = 3$ and $\tau_{\mathrm{m}}^{\infty}(P_n) = 2\tau(P_n)$ if n is odd [**42**].

As mentioned in the introduction, for any graph G without isolated vertices, $\tau(G) \geq \gamma(G)$ and $\tau_{\mathrm{m}}^{\infty}(G) \geq \gamma_{\mathrm{m}}^{\infty}(G)$. A fundamental bound for $\tau_{\mathrm{m}}^{\infty}$ is given next.

Theorem 23. [**42**] *For any nontrivial connected graph G, $\tau(G) \leq \tau_{\mathrm{m}}^{\infty}(G) \leq 2\tau(G)$.*

Graphs satisfying the upper bound in Theorem 23 are characterized in [**42**]. Some graphs where the lower bound is sharp are described next.

Proposition 24. *Each graph in the following classes satisfies $\tau_{\mathrm{m}}^{\infty}(G) = \tau(G)$.*

(a) K_n

(b) *Petersen graph*

(c) $K_m \,\square\, K_n$

(d) $C_m \,\square\, C_n$

(e) *Circulant graphs* (*to repel an attack along the edge uv, move* (*say*) *the guard on u to v and move each other guard along its incident edge that corresponds to uv in the same orientation of the cycle*).

We next give some exact bounds for trees and grid graphs. Let L denote the number of leaves of a tree T.

Theorem 25. [**42**] *For any nontrivial tree T, $\tau_{\mathrm{m}}^{\infty}(T) = |V - L| + 1$.*

Theorem 26. [**42**]

(a) $\tau_{\mathrm{m}}^{\infty}(P_1 \,\square\, P_n) = n - 1$.
(b) *If n is even, then $\tau_{\mathrm{m}}^{\infty}(P_n \,\square\, P_m) = \dfrac{nm}{2} = \tau(P_n \,\square\, P_m)$.*
(c) *If $n, m > 1$ are odd, $n \geq m$, then $\tau_{\mathrm{m}}^{\infty}(P_n \,\square\, P_m) = \left\lceil \dfrac{nm}{2} \right\rceil = \tau(P_n \,\square\, P_m) + 1$.*

We next compare $\tau_{\mathrm{m}}^{\infty}$ with some of the other graph protection parameters.

Theorem 27. [**33**] *If G is connected, then $\tau_{\mathrm{m}}^{\infty}(G) = \gamma(G)$ if and only if $G \in \{C_4, K_2\}$.*

Theorem 28. [**42**] *If $G \neq C_4$ is a connected graph of order $n \geq 3$ with $\delta(G) \geq 2$, then $\gamma_{\mathrm{m}}^{\infty}(G) < \tau_{\mathrm{m}}^{\infty}(G)$.*

It seems a challenging problem to describe graphs with pendant vertices and $\gamma_{\mathrm{m}}^{\infty}(G) = \tau_{\mathrm{m}}^{\infty}(G)$. Some examples are given next. The proof of Proposition 29 is by the authors [**44**] and we thank Michael Fisher for pointing out an example in the proof of part (b).

Proposition 29. *Let G be a 2-connected graph of order n. Let G' be a graph obtained from G by attaching a pendant vertex to each vertex of G except the two vertices u, v.*

(a) *If* $uv \in E$ *then* $\tau_{\mathrm{m}}^{\infty}(G') = n$ *and* $\gamma_{\mathrm{m}}^{\infty}(G') = n - 1$.

(b) *If* $uv \notin E$ *then* $\tau_{\mathrm{m}}^{\infty}(G') \geq n - 1 = \gamma_{\mathrm{m}}^{\infty}(G')$.

Proof. (a) Suppose we could eternally defend the edges of G' with $n-1$ guards. Let $x \in V(G) - \{u, v\}$ and let y be the pendant vertex attached to x. We can force guards onto both vertices x, y. Since each end-vertex is dominated, the edge uv is not protected. To see that n guards suffice to defend the edges, initially place guards on the vertices of G and then maintain at most one guard on a pendant vertex at any time. It is easy to see that the vertices of G' can be protected by $n-1$ guards by using a clique cover of $n-1$ K_2's.
(b) As in (a), $n-2$ guards do not protect the edges of G'. Letting $G = C_5$ is an example where $\tau_{\mathrm{m}}^{\infty}(G') = n$ and $G = K_4 - e$ is an example where $\tau_{\mathrm{m}}^{\infty}(G') = n - 1$. For $\gamma_{\mathrm{m}}^{\infty}(G')$, it is clear that $n-2$ guards are not enough to protect the vertices of G'. Place $n-1$ guards on G with a guard on each vertex except u. While vertices of G' except u are attacked, move guards to and fro along the pendant edges, maintaining at most one guard on a pendant vertex at all times. Suppose u is attacked. Since G is 2-connected, G has a $u-v$ path P with a guard on each vertex except u. Each such guard moves along P until each vertex of P except u has a guard.

Proposition 30. [**42**] *Let* G *be a* 2*-connected graph with* $n \geq 3$ *vertices. Add one pendant vertex to* $n-1$ *vertices of* G *and call the resulting graph* G'*. Then* $\tau_{\mathrm{m}}^{\infty}(G') = \gamma_{\mathrm{m}}^{\infty}(G') = n$.

It is an open question whether the condition of G being 2-connected in Proposition 30 can be replaced by minimum degree two.

An analog of realizable triples can be defined for edge protection. Results on graphs G having realizable triples $(\tau(G), \tau_{\mathrm{m}}^{\infty}(G), \tau_{\mathrm{mt}}^{\infty}(G))$, where $\tau_{\mathrm{mt}}^{\infty}(G)$ is the *total eternal vertex cover*, are given in [**3, 33**]. Any such realizable triple must satisfy the basic bound that for a connected graph G with more than two vertices, $\tau_{\mathrm{mt}}^{\infty}(G) < 2\tau_{\mathrm{m}}^{\infty}(G)$ [**33**]. In [**3**] it is shown that $\tau_{\mathrm{mt}}^{\infty}(G) \leq \tau_c(G) + 1 \leq 2\tau(G)$, where τ_c is the size of a smallest connected vertex cover of G.

It is shown in [**21**] that there exist graphs for which allowing multiple guards to reside on a vertex at the same time reduces the number of guards needed to defend the edges of the graph, in comparison to the eternal vertex cover number. These authors leave obtaining good bounds on k in the following statement as an open problem:

If one can defend any sequence of k attacks on edges,
then one can defend any infinite sequence of attacks on edges.

Partial results on this question are given in [**3**]. For instance:

Theorem 31. [**3**] *If* T *is a tree with* $n - L$ *guards, then there exists a strategy to defend* $V(T)$ *attacks on the edges of* T*. That is, an adversary can be forced to make* $V(T)$ *attacks before winning the eternal vertex cover game.*

An alternate type of eternal vertex cover problem in which attacks are at vertices while a vertex cover must be maintained at all times is explored in [**32**].

8. OTHER MODELS

Eternal independent dominating sets were studied by Hartnell and Mynhardt [**27**] and secure independent sets (analogous to secure dominating sets) were studied by Regan [**53**].

8.1. Eviction Model

In the eviction model, each configuration D_i, $i \geq 1$, of guards is required to be a dominating set. An attack occurs at a vertex $r_i \in D_i$ such that there exists at least one $v \in N(r_i)$ with $v \notin D_i$. The next guard configuration D_{i+1} is obtained from D_i by moving the guard from r_i to a vertex $v \in N(r_i)$, $v \notin D_i$ (i.e., this is the "one-guard moves" model). The size of a smallest eternal dominating set in the eviction model for G is denoted $e^\infty(G)$. Simply put, attacks occur at vertices with guards and we must move that guard to an unoccupied neighboring vertex. An attacked vertex is required to have at least one neighboring vertex with no guard, otherwise there would be no place for the guard to go.

This problem models a problem in computer networks where copies of a file are stored throughout the network and files must sometimes be moved, or migrated, due to maintenance at the server at which they are located. The goal is to ensure a copy of the file is close to every vertex in the network. That is, the locations of the files induce a dominating set at all times. The eviction problem was introduced by Klostermeyer, Lawrence and MacGillivray [**34**] and "one-guard moves" and "all-guards move" versions were defined. Most of the results in that paper are for the one-guard moves model and we focus our attention to that model here.

Some easy examples to illustrate the concept are $e^\infty(K_{1,m}) = m$, $e^\infty(C_5) = 2$, and $e^\infty(P_5) = 3$.

Theorem 32. [**34**] *If G is connected, then $e^\infty(G) \leq \theta(G)$.*

Theorem 33. [**34**] *If G is bipartite, then $e^\infty(G) = \alpha(G)$.*

Unlike in the traditional eternal domination model, there are graphs G for which $e^\infty(G) < \alpha(G)$: take a copy of K_2 and a large independent set I and join every vertex of the K_2 to every vertex of I. This graph has $e^\infty(G) = 1$.

Theorem 34. [**34**] *There exists a graph G such that $e^\infty(G) > \alpha(G)$. In fact, for $k \geq 3$, $e^\infty(C_{2k+1}) = k + 1$ (and $\alpha(C_{2k+1}) = k$).*

Theorem 35. [**34**] *Let G be a graph with $\alpha(G) = 2$. If G has two dominating vertices, then $e^\infty(G) = 1$. Otherwise, $e^\infty(G) = 2$.*

Proof. If G has dominating vertices x and y, then a single guard can relocate back and forth between them and maintain a dominating set.

Finally, suppose G has at most one dominating vertex. Then G is the complement of a triangle-free graph with at most one isolated vertex. Initially locate the guards on any dominating set of size two, say $\{u, v\}$. Suppose the guard on u is attacked. If v has a non-neighbor $w \neq u$, then whether or not u and v are adjacent, the guard at u guard can relocate to v and the resulting configuration is a dominating set. If no such vertex w exists, the guard at u can relocate to any vertex z and the resulting configuration of guards is a dominating set. □

The following result is much more difficult to prove.

Theorem 36. [**34**] *Let graph G have $\alpha(G) = 3$. Then $e^\infty(G) \leq 5$.*

It is not known whether or not $e^\infty(G) \leq \gamma^\infty(G)$ for all graphs G, see [**34**].

Less is known about the eviction model when all guards are allowed to move in response to an attack, though some elementary results are given in [**34**] and by the authors in [**46**]. The all-guards move eviction problem in grid graphs was studied by Klostermeyer and Messinger in [**41**] where the exact number of guards is determined for $m \times n$ grids with $m \leq 4$ and upper bounds are shown for large grids.

The eviction model for eternal independent sets was investigated by Caro and Klostermeyer in [**11**].

8.2. Eternal Connected Domination

Let $\gamma_c^\infty(G)$ denote the size of a smallest *eternal connected dominating set* (*ECDS*) in which the vertices containing guards induce a connected graph. Denote the all-guards move version of this parameter (the cardinality of a minimum *m-eternal connected dominating set*, or *m-ECDS*) by $\gamma_{\mathrm{mc}}^\infty(G)$. The ordinary connected domination number of G is denoted $\gamma_c(G)$ [**31**]. Obviously, these parameters are only defined for connected graphs. They were initially studied by the authors in [**43**].

Theorem 37. [**43**] *If G is connected and $\theta(G) \geq 2$, then $\gamma_{\mathrm{mc}}^\infty(G) \leq 2\theta(G) - 1$. This bound is sharp for all $\theta \geq 2$.*

Theorem 38. [**43**] *For all graphs $G = (V, E)$ without isolated vertices,*

(a) $\gamma_c^\infty(G) > \gamma^\infty(G)$

(b) $\gamma_c^\infty(G) \leq \gamma^\infty(G) + \gamma(G) \leq 2\gamma^\infty(G) \leq 2\theta(G)$.

Klostermeyer and Mynhardt [**43**] also give a number of results on eternal connected domination in the all-guards move model, such as the following bound.

Theorem 39. [**43**] *For all graphs $G = (V, E)$ without isolated vertices, $\gamma_{\mathrm{mc}}^\infty(G) \leq 2\gamma(G)$.*

8.3. Foolproof Eternal Domination

In the definition of eternal domination, the decision of which guard to send to defend an attack may require knowledge of the locations of future attacks. The definition states "there exists" a guard to send to defend the attack such that all subsequent attacks can be defended by the resulting guard configuration. It may be difficult in practice to decide which guard to send to defend an attack.

Burger et al. [**10**] defined a "foolproof" variation on eternal domination in which the resulting configuration of guards must be able to defend all subsequent attacks if a guard from **any** vertex adjacent to the attacked vertex is sent to defend an attack at an unoccupied vertex. That is, no matter which guard is sent, the resulting configuration can defend all future attacks. They proved that $n - \delta(G)$ guards are necessary and sufficient for all graphs G, where $\delta(G)$ is the minimum vertex degree in the graph. To see this, note that any set of $n-\delta(G)$ vertices form a dominating set. On the other hand, if we have fewer than $n-\delta(G)$ guards in G, then by a series of attacks, an adversary can force the closed neighborhood of a vertex to contain no guards. For example, consider C_6, and observe that $\gamma^\infty(C_6) = 3$. Now suppose we could defend the graph with three guards in the foolproof model. Since any neighboring guard can move to defend an attack, an adversary can force the three guards to migrate to three consecutive vertices, thereby leaving a vertex undominated.

The foolproof variety has been studied in the all-guards move model by Klostermeyer and MacGillivray [**38**]. The problem is the same as the m-eternal dominating set problem in that attacks are at (unoccupied) vertices and all guards can move in response to an attack on a vertex v, but the attacker chooses which guard moves to v. One can also imagine there being a victim of the attack at v and allowing the victim to choose which guard to send to its defense. For example, when a site is attacked, it may want to choose which of the nearby defenders it calls in, perhaps because of particular expertise in defending certain types of attacks. The size of a smallest m-eternal dominating set for G in the foolproof model is denoted $\rho^\infty_{\mathrm{m}}(G)$.

Proposition 40. [**38**] *For any graph G, $\gamma^\infty_{\mathrm{m}}(G) \le \rho^\infty_{\mathrm{m}}(G) \le \tau^\infty_{\mathrm{m}}(G)$.*

Proof. The first inequality is obvious. For the second inequality, observe that in the m-eternal vertex cover problem, when an attack occurs on an edge with guards on either end, the two guards can swap places and no other guards need to move; hence there is no net change in the guard configuration. If there is only one guard incident to attacked edge uv, that guard must move across the edge, say from u to v, to defend the attack. This is equivalent to the attacker choosing the guard to defend the attack. Now rather than having attacks at edges, imagine the attack is at v and the attacker chooses the guard at u to defend it. It follows that $\rho^\infty_{\mathrm{m}}(G) \le \tau^\infty_{\mathrm{m}}(G)$.

Theorem 41. [**38**]

(a) *If G is a connected bipartite graph, then $\rho^\infty_{\mathrm{m}}(G) \le \alpha(G)$.*

(b) *For any graph G, $\rho_{\mathrm{m}}^{\infty}(G) \leq 2\theta(G)$.*

It is not known if the bound in Theorem 41 (b) is sharp. There does exist a graph G with $\rho_{\mathrm{m}}^{\infty}(G) \geq \frac{3}{2}\theta(G)$ [**38**].

9. COMPLEXITY

The complexity of deciding whether a given set of vertices is an eternal dominating set, or another of the variations discussed, as well as the complexity of determining the protection parameters themselves, are generally difficult problems. The precise complexity remains unknown in most cases. For example, it is unknown whether deciding whether a given set of vertices is an eternal dominating set lies in the class PSPACE (though it is not too difficult to see that it can be decided in exponential time, based on the "configuration graph" idea from [**34**]). One problem in assessing in which complexity class the eternal domination problem lies is to determine how many attacks one must evaluate to determine whether a set of guards can defend any infinite sequence of attacks in the graph. That is, is there a polynomial function $f(n)$, where n is the number of vertices in G, such that if one can defend any sequence of $f(n)$ attacks, then one can defend any infinite sequence of attacks? If there is no such polynomial function, then what bounds can be placed on such a function?

We mention some results, besides the obvious cases like for perfect graphs. From the results in [**37**], the m-eternal domination number for a tree can be computed in polynomial time. In addition, we can determine in polynomial time whether each of these protection parameters is at most k for a fixed constant k, based the configuration graph method of [**34**]. On a related note, the parameterized complexity of the eternal vertex cover problem was studied in [**20**].

10. OPEN PROBLEMS

We present a number of conjectures and open problems on some of the models discussed above. The first problem is of a general type and for this reason we state it here and not in the dedicated subsections.

Theorem 19 presents the only known Nordhaus-Gaddum type result for graph protection, namely for the m-eternal domination number $\gamma_{\mathrm{m}}^{\infty}$.

Problem 1. *Determine Nordhaus-Gaddum type results for other graph protection parameters.*

10.1. Eternal domination

Problem 2. *Study classes of graphs G such that (i) $\gamma^{\infty}(G) = \alpha(G)$, (ii) $\gamma^{\infty}(G) = \theta(G)$.*

As mentioned above, $\gamma^{\infty}(G) = \theta(G)$ if G is series-parallel, so it makes sense to pose the following question.

Problem 3. *Is it true that* $\gamma^\infty(G) = \theta(G)$ *if* G *is planar?*

Problem 4. *Does there exist a constant* c *such that* $\gamma^\infty(G) \le c\tau(G)$ *for all graphs* G?

The following is motivated by an error discovered in [**37**], where it is claimed that no such graph exists.

Problem 5. *Does there exist a graph* G *with* $\gamma(G) = \gamma^\infty(G)$ *and* $\gamma(G) < \theta(G)$?

In [**47**], it was shown that (i) every triangle-free G with $\gamma(G) = \gamma^\infty(G)$ has $\gamma(G) = \theta(G)$ and (ii) every graph G with $\Delta(G) \le 3$ with $\gamma(G) = \gamma^\infty(G)$ has $\gamma(G) = \theta(G)$.

It would also be of interest to determine if the graph with 11 vertices given in [**23**] having $\gamma^\infty < \theta$ is the smallest such graph.

Problem 6. [**37**] *Characterize graphs* G *with* $\gamma(G) = \gamma^\infty(G) = \theta(G)$.

It is not hard to argue that any graph G satisfying $\gamma(G) = \gamma^\infty(G) < \theta(G)$ contains a triangle.

Problem 7. (a) *Describe classes of graphs with* $\gamma^\infty/\alpha > \frac{3}{2}$.

(b) *Characterize realizable triples with* $\gamma^\infty/\alpha > \frac{3}{2}$.

A Vizing-like question was asked by the authors in [**47**].

Problem 8. *It is true for all graphs* G *and* H *that* $\gamma^\infty(G \,\square\, H) \ge \gamma^\infty(G) * \gamma^\infty(H)$?

Interestingly, such a Vizing-like condition was shown not to hold for all graphs G in the all-guards-move model in [**47**].

The following conjecture was stated by Klostermeyer and MacGillivray [**40**] and shown to hold in some special cases.

Conjecture 1. *Let* G *be a graph with no isolated vertices. Let* D *be a minimum eternal dominating set of* G. *For every vertex* $u \in D$ *with an unoccupied neighbor, there exists an eternal dominating set* D' *with* $|D| = |D'|$ *such that* $D' = (D - \{u\}) \cup \{v\}$, *where* $v \notin D$ *and* $v \in N(u)$.

The following weaker analog of Conjecture 1 was shown to be true in [**40**].

Theorem 42. *Let* G *be a graph with no isolated vertices and* D *a minimum eternal dominating set of* G. *For every vertex* $u \in D$, *there exists a minimum eternal dominating set* D' *of* G *such that* (i) $u \notin D'$ *and* (ii) D' *is reachable from* D *by a sequence of guard moves.*

10.2. m-Eternal Domination

Recall the inequality chain (2) $\gamma(G) \le \gamma^\infty_{\mathrm{m}}(G) \le \alpha(G) \le \gamma^\infty(G) \le \theta(G)$ from Section 5.

Problem 9. *Describe classes of graphs having* $\gamma(G) = \gamma^\infty_{\mathrm{m}}(G)$, $\gamma^\infty(G) = \gamma^\infty_{\mathrm{m}}(G)$, $\gamma^\infty_{\mathrm{m}}(G) = \tau(G)$, *or* $\gamma^\infty_{\mathrm{m}}(G) = \alpha(G)$.

Although this survey does not cover Roman, weak Roman and secure domination, comparing these parameters with eternal protection parameters could be interesting.

Problem 10. *Where do the Roman, weak Roman and secure domination numbers lie in the chain* (2)?

As shown in [**7**], there exist connected Cayley graphs, necessarily of non-Abelian groups, whose m-eternal domination numbers exceed their domination numbers by one. This implies that there exist disconnected Cayley graphs G such that $\gamma_{\mathrm{m}}^{\infty}(G) - \gamma(G)$ is an arbitrary positive integer. The picture for connected Cayley graphs is not so clear.

Problem 11. *Does there exist a connected Cayley graph G such that $\gamma_{\mathrm{m}}^{\infty}(G) > \gamma(G) + 1$?*

Problem 12. *Find conditions under which the bound $\gamma_{\mathrm{m}}^{\infty}(G) \leq \left\lceil \frac{n}{2} \right\rceil$ in Theorem* 11 *can be improved, and conditions under which equality holds.*

10.2.1. Grids

Determining the domination number of grid graphs was a long and arduous process which was concluded by Gonçalves, Pinlou, Rao and Thomasse [**26**] about thirty years after the first results on the domination number of grids were obtained by Jacobson and Kinch [**30**]. By contrast, the study of eternal domination in grids still has a long way to go. A solution to the next problem will provide a nice link between the two parameters for these graphs.

Problem 13. *Determine the largest value of n for which $\gamma_{\mathrm{m}}^{\infty}(P_n \Box P_n) = \gamma(P_n \Box P_n)$.*

Problem 14. *Determine the value of $\gamma_{\mathrm{m}}^{\infty}(P_n \Box P_m)$. In particular, is $\gamma_{\mathrm{m}}^{\infty}(P_n \Box P_n) \leq \gamma(P_n \Box P_n) + c$, for some constant c?* (*The latter is conjectured to be true by* S. Finbow *and* W. Klostermeyer, *personal communication*).

Conjecture 2. [**25**] *If $\gamma_{\mathrm{m}}^{\infty}(P_3 \Box P_n) \leq r$, then $\gamma_{\mathrm{m}}^{\infty}(P_3 \Box P_{n+1}) \leq r + 1$.*

Conjecture 3. [**25**] For $n > 9$, $\gamma_{\mathrm{m}}^{\infty}(P_3 \Box P_n) = 1 + \left\lceil \frac{4n}{5} \right\rceil$.

The latter conjecture has been nearly resolved by Finbow et al. [**19**] in Theorem 14, as discussed above.

10.3. Eternal Vertex Cover

Problem 15. [**42**] *For which* (*bipartite*) *graphs is $\tau_{\mathrm{m}}^{\infty}(G) = \tau(G)$?*

Problem 16. [**42**] *Do all vertex transitive graphs G satisfy $\tau_{\mathrm{m}}^{\infty}(G) = \tau(G)$?*

Conjecture 4. [**42**] *Let G and H be graphs such that $\tau_{\mathrm{m}}^{\infty}(G) = \tau(G)$ and $\tau_{\mathrm{m}}^{\infty}(H) = \tau(H)$. Then $\tau_{\mathrm{m}}^{\infty}(G \Box H) = \tau(G \Box H)$.*

Conjecture 5. [**42**] *Let* $G = (V, E)$ *be a connected graph with subgraph* H *such that* $\delta(H) \geq 2$ *and* $\delta(G[V - V(H)]) \geq 2$*. Then* $\tau_{\rm m}^{\infty}(G) \geq \tau_{\rm m}^{\infty}(H) + \tau_{\rm m}^{\infty}(G[V - V(H)])$.

Problem 17. [**42**] *Characterize graphs that are edge-critical for eternal vertex cover.*

If $e \in E(\overline{G})$, then possibly $\tau_{\rm m}^{\infty}(G+e) > \tau_{\rm m}^{\infty}(G)$ (such as $G = C_4$) or possibly $\tau_{\rm m}^{\infty}(G+e) < \tau_{\rm m}^{\infty}(G)$. An example of the latter is to let $G+e$ be the 2×4 grid graph laid out in the usual manner (this graph has eternal vertex cover number four) and choose e be the middle edge on the upper P_4.

In general, vertex and edge criticality has not been studied for any of the eternal protection parameters.

10.4 Other models

We mention some open problems in some of the other models discussed.

Conjecture 6. [**43**] *For all connected graphs* G *with* $\Delta(G) < n-1$, $\gamma_c^{\infty}(G) > \theta(G)$.

Problem 18. [**34**] *Is* $e^{\infty}(G) \leq \gamma^{\infty}(G)$ *for all graphs* G*?*

The analogous problem in the all-guards-move model of eviction (in which an attacked vertex may be occupied by another guard after the attack) is also open. In the model in all-guards-move model of eviction in which an attacked vertex must remain unoccupied until the next attack, it is sometimes the case that more guards are needed in the eviction model than the traditional all-guards-move model: $\gamma_{\rm m}^{\infty}(K_{1,m}) = 2$ when $m \geq 2$, but this graph requires m guards in the eviction model in which an attacked vertex must remain unoccupied until the next attack.

Conjecture 7. [**38**] *For a graph* $G = (V, E)$ *with* n *vertices and no isolated vertices,* $\rho_{\rm m}^{\infty}(G) \leq \left\lceil \frac{n}{2} \right\rceil$.

It was shown in [**38**] that $\rho_{\rm m}^{\infty}(G) \leq \left\lceil \frac{5n}{6} \right\rceil$, for all graphs G.

11. LIST OF PARAMETERS

$\alpha(G)$:	independence number
$\gamma(G)$:	domination number
$\gamma_c(G)$:	connected domination number
$\gamma_t(G)$:	total domination number
$\gamma^{\infty}(G)$:	eternal domination number
$\gamma_{\rm m}^{\infty}(G)$:	m-eternal domination number
$\gamma_{\rm m}^{*\infty}(G)$:	m-eternal domination number with multiple guards per vertex
$\gamma_{\rm mc}^{\infty}(G)$:	m-eternal connected domination number
$\gamma_{\rm mt}^{\infty}(G)$:	m-eternal total domination number

$\gamma_c^\infty(G)$: eternal connected domination number
$\rho_{\mathrm{m}}^\infty(G)$: foolproof m-eternal domination number
$\tau(G)$: vertex covering number
$\tau_{\mathrm{m}}^\infty(G)$: m-eternal covering number
$e^\infty(G)$: eviction model eternal domination number
$\theta(G)$: clique covering number
$\theta_c(G)$: neo-colonization number

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