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ALMOST AUTOMORPHIC SOLUTIONS OF DELAYED NEUTRAL DYNAMIC SYSTEMS ON HYBRID DOMAINS

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We study the existence of almost automorphic solutions of the delayed neutral dynamic system on hybrid domains that are additively periodic. We use exponential dichotomy and prove the uniqueness of projector of exponential dichotomy to obtain some limit results leading to sufficient conditions for existence of almost automorphic solutions to neutral system. Unlike the existing literature we prove our existence results without assuming boundedness of the coefficient matrices in the system. Hence, we significantly improve the results in the existing literature. Finally, we also provide an existence result for almost periodic solutions of the system.

1. INTRODUCTION

The theory of neutral type equations has attracted a prominent attention due to the potential of its application in variety of fields in the natural sciences dealing with models that analyze and control real life processes. In particular, investigation of periodic solutions of neutral dynamic systems has a particular importance for scientists interested in biological models of certain type of populations having periodical structures (see [**10**], [**20**] and [**23**]). There is a vast literature on neutral type equations on continuous and discrete domains which focus on the stability, oscillation and existence results (see [**30**], [**32**], [**15**], [**28**] and references therein). Theory of time scales enables researchers to combine differential and difference equations under one theory called dynamic equations on time scales. For brevity we assume reader is familiar with time scale calculus. For an excellent review on time scale calculus, we refer to the pioneering work [**8**]. In order to study existence of periodic

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solutions of dynamic equations on time scales, one has to assume that the time scale is periodic.

D1. A time scale $\mathbb{T}$ is periodic if there exists $p > 0$ such that if $t \in \mathbb{T}$ then $t \pm p \in \mathbb{T}$. For $\mathbb{T} \neq \mathbb{R}$, the smallest positive p is called the period of the time scale (see [**16**]).

Since there is another periodicity notion based on shifts on time scales (see [**2**]), it is more suitable to call time scales satisfying (D1) additively periodic and to observe that additively periodic time scales must be unbounded from above and below. In addition, for any additively periodic time scale, the set

$$\mathcal{T} := \{p \in \mathbb{R} : t \pm p \in \mathbb{T}, \forall t \in \mathbb{T}\} \tag{1}$$

is nonempty. For example, the time scales $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, $\mathbb{T} = h\mathbb{Z}$, and $\mathbb{T} = \bigcup_{k \in \mathbb{Z}} [2k, 2k+1]$ are additively periodic. An additive periodic time scale may have discrete, continuous or piecewise continuous structure. Hence, it may be more perceptible to use the phrase "hybrid domain" instead of "additive periodic time scale". We refer the reader to [**15**], [**16**], [**17**], [**28**], [**29**] and [**31**] for studies handling existence of periodic solutions or related topics of neutral dynamic equations on hybrid domains. A more general approach to periodicity notion on time scales has been introduced in [**2**] and applied to neutral dynamic systems in [**3**].

Periodicity may be a strong restriction in some specific real life models including functions that are not strictly periodic but having values close enough to each other at every different period. Many mathematical models (see e.g. [**26**], and [**27**]) in signal processing require the use of almost periodic functions. Informally, a nearly periodic function means that any one period is virtually identical to its adjacent periods but not necessarily similar to periods much further away in time. The theory of almost periodic functions was first introduced by H. Bohr and generalized by A. S. Besicovitch, W. Stepanov, S. Bochner, and J. von Neumann at the beginning of 20^{th} century. We have the following definitions.
A continuous function $f : \mathbb{R} \to \mathbb{R}$ is said to be almost periodic if the following characteristic property holds:

D2. *For any* $\varepsilon > 0$, *the set*

$$E(\varepsilon, f(x)) := \{\tau \in \mathbb{R} : |f(x+\tau) - f(x)| < \varepsilon \text{ for all } x \in \mathbb{R}\}$$

is relatively dense in the real line $\mathbb{R}$. *That is, for any* $\varepsilon > 0$, *there exists a number* $\ell(\varepsilon) > 0$ *such that any interval of length* $\ell(\varepsilon)$ *contains a number in* $E(\varepsilon, f(x))$.

Afterwards, S. Bochner showed that almost periodicity is equivalent to the following characteristic property which is also called *the normality condition*:

D3. *From any sequence of the form* $\{f(x+h_n)\}$, *where* h_n *are real numbers, one can extract a subsequence converging uniformly on the real line* (*see* [**6**]).

Theory of almost automorphic functions was first studied by S. Bochner [**7**]. It is a property of a function which can be obtained by replacing convergence with uniform convergence in normality definition D3. More explicitly, a continuous function $f:\mathbb{R}\to\mathbb{R}$ is said to be almost automorphic if for every sequence $\{h'_n\}_{n\in\mathbb{Z}_+}$ of real numbers there exists a subsequence $\{h_n\}$ such that $\lim_{m\to\infty}\lim_{n\to\infty} f(t+h_n-h_m)=f(t)$ for each $t\in\mathbb{R}$. For more reading on almost automorphic functions, we refer to [**14**].

Almost automorphic solutions of difference equations have been investigated in [**5**], [**11**] and [**22**]. In addition, C. Lizama and J. G. Mesquita generalized the notion of almost automorphy in [**21**] by studying almost automorphic solutions of dynamic equations on time scales that are invariant under translation. In [**25**] Mishra et. al. investigated almost automorphic solutions to functional differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(x(t)-F_1(t,x(t-g(t)))\big)=A(t)x(t)+F_2\big(t,x(t),x(t-g(t))\big) \tag{2}$$

using the theory of evolution semigroup. Note that almost periodic solutions of Eq. (2) have also been studied in [**1**] by means of the theory of evolution semigroup.

In this study, we propose existence results for almost automorphic solutions of the delayed neutral dynamic system

$$x^{\Delta}(t)=A(t)x(t)+Q^{\Delta}(t,x(t-g(t)))+G(t,x(t),x(t-g(t)))$$

by using fixed point theory. The highlights of the paper can be summarized as follows:

- In the present paper, we use exponential dichotomy instead of theory of evolution semigroup since the conditions required by theory of evolution semigroup are strict and not easy to check (for related discussion see [**12**]).
- We prove the uniqueness of projector of exponential dichotomy on hybrid domains satisfying (D1) (see Theorem 3).
- In [**21**], the authors obtain the limiting properties of exponential dichotomy by using the product integral on time scales (see [**33**]). This method requires boundedness of inverse matrices $A(t)^{-1}$ and $\big(I+\mu(t)A(t)\big)^{-1}$ as compulsory conditions. We obtain our limit results by using a different technique, including uniqueness of projector of exponential dichotomy, which does not require boundedness of inverse matrices $A(t)^{-1}$ and $\big(I+\mu(t)A(t)\big)^{-1}$ (see Theorem 4).
- Using a different approach we improve the existence results of [**21**]. Furthermore, our results also extend the results of [**11**] and [**22**] in the particular time scale $\mathbb{T}=\mathbb{Z}$.

2. ALMOST AUTOMORPHY NOTION ON TIME SCALES

This part of the paper is devoted to the discussion of almost automorphic functions and their properties on time scales. Throughout the paper, we assume that the reader is familiar with the theory of time scale calculus.

Definition 1 ([**21**]). *Let $\mathcal{X}$ be a (real or complex) Banach space and $\mathbb{T}$ an additively periodic time scale. Then, an rd-continuous function $f : \mathbb{T} \to \mathcal{X}$ is called almost automorphic in $\mathbb{T}$ if for every sequence $(\alpha'_n) \in \mathcal{T}$, there exists a subsequence (α_n) such that*

$$\lim_{n\to\infty} f(t+\alpha_n) = \bar{f}(t)$$

is well defined for each $t \in \mathbb{T}$ and

$$\lim_{n\to\infty} \bar{f}(t-\alpha_n) = f(t)$$

for every $t \in \mathbb{T}$, where $\mathcal{T}$ is defined in (1).

Remark 1. In particular cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, it is well known that every continuous periodic function is almost periodic and every almost periodic function is almost automorphic. This relationship is preserved on an additively periodic time scale $\mathbb{T}$.

Hereafter, we denote by $\mathcal{A}(\mathcal{X})$ the set of all almost automorphic functions on an additively periodic time scale. Obviously, $\mathcal{A}(\mathcal{X})$ is a Banach space when it is endowed by the norm

$$\|f\|_{\mathcal{A}(\mathcal{X})} = \sup_{t\in\mathbb{T}} \|f(t)\|_{\mathcal{X}},$$

where $\|.\|_{\mathcal{X}}$ is the norm defined on $\mathcal{X}$.

The following result summarizes the main properties of almost automorphic functions on time scales:

Theorem 1 ([**21**]). *Let $\mathbb{T}$ be an additively periodic time scale and suppose rd-continuous functions $f, g : \mathbb{T} \to \mathcal{X}$ are almost automorphic. Then*

i. *$f + g$ is almost automorphic function on time scales,*

ii. *cf is almost automorphic function on time scales for every scalar c,*

iii. *For each $\ell \in \mathbb{T}$, the function $f_\ell : \mathbb{T} \to \mathcal{X}$ defined by $f_\ell(t) := f(\ell + t)$ is almost automorphic on time scales,*

iv. *The function $\hat{f} : \mathbb{T} \to \mathcal{X}$ defined by $\hat{f}(t) := f(-t)$ is almost automorphic on time scales,*

v. *Every almost automorphic function on a time scale is bounded, that is*

$$\|f\|_{\mathcal{A}(\mathcal{X})} < \infty,$$

vi. $\|\bar{f}\|_{\mathcal{A}(\mathcal{X})} \leq \|f\|_{\mathcal{A}(\mathcal{X})}$, *where*

$$\lim_{n\to\infty} f(t+\alpha_n) = \bar{f}(t) \text{ and } \lim_{n\to\infty} \bar{f}(t-\alpha_n) = f(t).$$

The following definition is also necessary for our further analysis.

Definition 2 ([**21**])**.** *Let $\mathcal{X}$ be a (real or complex) Banach space and $\mathbb{T}$ an additively periodic time scale. Then, an rd-continuous function $f : \mathbb{T} \times \mathcal{X} \to \mathcal{X}$ is called almost automorphic for $t \in \mathbb{T}$ for each $x \in \mathcal{X}$, if for every sequence $(\alpha'_n) \in \mathcal{T}$, there exists a subsequence (α_n) such that*

$$\lim_{n\to\infty} f(t+\alpha_n, x) = \bar{f}(t,x) \tag{3}$$

is well defined for each $t \in \mathbb{T}$, $x \in \mathcal{X}$ and

$$\lim_{n\to\infty} \bar{f}(t-\alpha_n, x) = f(t,x)$$

for every $t \in \mathbb{T}$ and $x \in \mathcal{X}$.

Remark 2. Almost automorphic functions of two or more variables have the similar properties of almost automorphic functions of one variable. If $f, g : \mathbb{T} \times \mathcal{X} \to \mathcal{X}$ are almost automorphic, then $f+g$ and cf (c is a constant) are almost automorphic. Furthermore,

$$\left\|f(.,x)\right\|_{\mathcal{A}(\mathcal{X})} = \sup_{t\in\mathbb{T}} \left\|f(t,x)\right\|_{\mathcal{X}} < \infty \text{ for each } x \in \mathcal{X}$$

and

$$\left\|\bar{f}(.,x)\right\|_{\mathcal{A}(\mathcal{X})} = \sup_{t\in\mathbb{T}} \left\|\bar{f}(t,x)\right\|_{\mathcal{X}} < \infty \text{ for each } x \in \mathcal{X},$$

where $\bar{f}$ is as in (3).

Theorem 2 ([**21**])**.** *Let $\mathbb{T}$ be an additively periodic time scale. Suppose that $f : \mathbb{T} \times \mathcal{X} \to \mathcal{X}$ is an almost automorphic function in t for each $x \in \mathcal{X}$ and $f(t,x)$ satisfies Lipschitz condition in x uniformly in t, that is*

$$\left\|f(t,x) - f(t,y)\right\|_{\mathcal{A}(\mathcal{X})} \leq L\|x-y\|_{\mathcal{X}},$$

for all $x, y \in \mathcal{X}$. Suppose that $\phi : \mathbb{T} \to \mathcal{X}$ is almost automorphic, then the function $U : \mathbb{T} \to \mathcal{X}$ defined by $U(t) := f\big(t, \phi(t)\big)$ is almost automorphic.

3. EXPONENTIAL DICHOTOMY AND LIMITING RESULTS

In this section, we use exponential dichotomy to obtain some convergence results for principal fundamental matrix solution of the regressive linear nonautonomous system

$$x^{\Delta}(t) = A(t)x(t), \; x(t_0) = x_0, \; t \in \mathbb{T} \tag{4}$$

on an additively periodic time scale $\mathbb{T}$.

Let $\mathcal{B}(\mathcal{X})$ be a Banach space of all bounded linear operators from $\mathcal{X}$ to itself with the norm $\|.\|_{\mathcal{B}(\mathcal{X})}$ given by

$$\|L\|_{\mathcal{B}(\mathcal{X})} := \sup\big\{\|Lx\|_{\mathcal{X}} : x \in \mathcal{X} \text{ and } \|x\|_{\mathcal{X}} \leq 1\big\}.$$

Then, the definition of exponential dichotomy is as follows:

Definition 3 (Exponential dichotomy, [**21**])**.** *Let $A(t)$ be an $n \times n$ rd-continuous matrix valued function on $\mathbb{T}$. We say that the linear system* (4) *has an exponential dichotomy on $\mathbb{T}$ if there exist positive constants $K_{1,2}$ and $\alpha_{1,2}$, and an invertible projection $\mathcal{P}$ commuting with $X(t)$, where $X(t)$ is principal fundamental matrix solution of* (4) *satisfying*

$$\big\|X(t)\mathcal{P}X^{-1}(s)\big\|_{\mathcal{B}(\mathcal{X})} \leq K_1 e_{\ominus\alpha_1}(t,s),\ \ s,t \in \mathbb{T},\ t \geq s, \tag{5}$$

and

$$\big\|X(t)(I-\mathcal{P})X^{-1}(s)\big\|_{\mathcal{B}(\mathcal{X})} \leq K_2 e_{\ominus\alpha_2}(s,t),\ \ s,t \in \mathbb{T},\ t \leq s, \tag{6}$$

where $\ominus\alpha_i := -\alpha_i\big(1+(\sigma(t)-t)\alpha_i\big)^{-1}$, $i = 1,2$.

The following lemma can be proven in a similar way to [**9**, Lemma 1], hence we omit its proof.

Lemma 1. *The following two statements hold.*

i. *Suppose that $-p \in \mathcal{R}^+$, i.e. positively regressive, and $t_0 \in \mathbb{T}$. If*

$$y^{\Delta}(t) + p(t)y(t) \leq 0 \text{ for all } t_0 \geq t,$$

then

$$y(t_0) \leq e_{-p}(t,t_0)y(t) \text{ for all } t_0 \geq t.$$

ii. *Suppose that p is a positive valued function and $t_0 \in \mathbb{T}$. If*

$$y^{\Delta}(t) - p(t)^{-1}y(t) \geq 0 \text{ for all } t \geq t_0,$$

then

$$y(t) \geq e_{p^{-1}}(t,t_0)y(t_0) \text{ for all } t \geq t_0.$$

Lemma 2. *Suppose that the time scale $\mathbb{T}$ is unbounded from above and below. Let $\Upsilon : \mathbb{T} \to (0,\infty)$ and $\Psi : \mathbb{T} \to (0,\infty)$ be rd-continuous functions satisfying*

$$\Upsilon(t)\int_{-\infty}^{t} \big(\Upsilon(s)\big)^{-1}\Delta s \leq \beta,\ \ t \in \mathbb{T} \tag{7}$$

and

$$\Psi(t)\int_{t}^{\infty} \big(\Psi(s)\big)^{-1}\Delta s \leq \nu,\ \ t \in \mathbb{T} \tag{8}$$

where β, ν are positive constants. Then

$$\Upsilon(t) \leq c e_{\ominus \beta^{-1}}(t, t_0)$$

and

$$\Psi(t) \leq \tilde{c} e_{\ominus \nu^{-1}}(t_0, t).$$

Proof. Define $u(t) := \int\limits_{-\infty}^{t} \big(\Upsilon(s)\big)^{-1} \Delta s$. Then by (7) we get $u^{\Delta}(t) = \big(\Upsilon(t)\big)^{-1} \geq \beta^{-1} u(t)$, and hence,

$$u(t) \geq e_{\beta^{-1}}(t, t_0) u(t_0) \text{ for all } t \geq t_0$$

by (ii) of Lemma 1. This implies

$$\begin{aligned} \Upsilon(t) &\leq \beta \big(u(t)\big)^{-1} \\ &\leq e_{\ominus \beta^{-1}}(t, t_0) \big(u(t_0)\big)^{-1} \beta \\ &= c e_{\ominus \beta^{-1}}(t, t_0) \end{aligned}$$

for $c = \big(u(t_0)\big)^{-1} \beta$. Similarly if we let $z(t) := \int\limits_{t}^{\infty} \big(\Psi(s)\big)^{-1} \Delta s$, then (8) implies

$$z^{\Delta}(t) \leq -\nu^{-1} z(t) \leq \nu^{-1} z(t)$$

which along with (i) of Lemma 1 yields

$$z(t_0) \leq e_{\nu^{-1}}(t, t_0) z(t) \text{ for all } t_0 \geq t.$$

Then, we have

$$\begin{aligned} \Psi(t) &\leq \nu \big(z(t)\big)^{-1} \\ &\leq \nu \big(z(t_0)\big)^{-1} e_{\nu^{-1}}(t, t_0) \\ &= \tilde{c} e_{\ominus \nu^{-1}}(t_0, t) \end{aligned}$$

for $\tilde{c} = \nu \big(z(t_0)\big)^{-1}$. The proof is complete.

Lemma 3 ([**9**, Lemma 2])**.** *If p is nonnegative and $-p$ is positively regressive, then*

$$1 - \int_s^t p(u) \Delta u \leq e_{-p}(t, s) \leq \exp\left\{ - \int_s^t p(u) \Delta u \right\} \text{ for all } t \geq s.$$

Lemma 4. *Assume that $\mathbb{T}$ is unbounded above and below. If the homogeneous system (4) admits an exponential dichotomy, then $x = 0$ is the unique bounded solution of the system.*

Proof. Let B_0 be the set of initial conditions ϑ belonging to bounded solutions of the system (4). Assume $(I-\mathcal{P})\vartheta \neq 0$ and define $\big(\phi(t)\big)^{-1} := \big\|X(t)(I-\mathcal{P})\vartheta\big\|_{\mathcal{X}}$. Using the equality $(I-\mathcal{P})^2 = I-\mathcal{P}$, we get

$$\int_t^\infty X(t)(I-\mathcal{P})\vartheta\phi(s)\Delta s = \int_t^\infty X(t)(I-\mathcal{P})X^{-1}(s)X(s)(I-\mathcal{P})\vartheta\phi(s)\Delta s.$$

Taking the norm on both sides, we obtain

$$\begin{aligned}\big(\phi(t)\big)^{-1}\int_t^\infty \phi(s)\Delta s &\leq \int_t^\infty \big\|X(t)(I-\mathcal{P})X^{-1}(s)\big\|_{\mathcal{B}(\mathcal{X})}\Delta s\\ &\leq K_2\int_t^\infty e_{\ominus\alpha_2}(s,t)\Delta s\end{aligned}$$

uniformly in $t\in\mathbb{T}$. Setting $p = -\ominus\alpha_2 = \dfrac{\alpha_2}{1+(\sigma(t)-t)\,\alpha_2}$ in Lemma 3 we can conclude boundedness of the last integral. This means

$$\lim \inf_{s\in[t,\infty)\cap\mathbb{T}} \phi(s) = 0,$$

and hence, $\|X(t)\,(I-\mathcal{P})\,\vartheta\|_{\mathcal{X}}$ is unbounded.
Using a similar procedure with $\mathcal{P}\vartheta \neq 0$ and

$$\int_{-\infty}^t X(t)\mathcal{P}\vartheta\phi(s)\Delta s = \int_{-\infty}^t X(t)\mathcal{P}X^{-1}(s)X(s)\mathcal{P}\vartheta\phi(s)\Delta s,$$

where $\big(\phi(t)\big)^{-1} := \big\|X(t)\mathcal{P}\vartheta\big\|_{\mathcal{X}}$, we get

$$\int_{-\infty}^t \phi(s)\Delta s < \infty$$

and hence

$$\lim \inf_{s\in(-\infty,t]\cap\mathbb{T}} \phi(s) = 0.$$

This shows that $\|X(t)\mathcal{P}\vartheta\|_{\mathcal{X}}$ is unbounded. Thus, the system (4) has a bounded solution if $B_0 = \{0\}$. Equivalently, if x is a bounded solution of the system (4), then $x = 0$. The proof is complete.

Theorem 3. *If the homogeneous system* (4) *admits an exponential dichotomy, then the projection* $\mathcal{P}$ *of the exponential dichotomy is unique on additively periodic time scales.*

Proof. Assume that system (4) admits an exponential dichotomy. Define $\Upsilon(t) := \big\|X(t)\mathcal{P}\big\|_{\mathcal{B}(\mathcal{X})}$ and consider

$$\int_{-\infty}^t X(t)\mathcal{P}\Upsilon(s)^{-1}\Delta s = \int_{-\infty}^t X(t)\mathcal{P}X^{-1}(s)X(s)\mathcal{P}\Upsilon(s)^{-1}\Delta s.$$

Taking the norm of both sides and employing Lemma 3, we obtain

$$\Upsilon(t)\int_{-\infty}^{t}\big(\Upsilon(s)\big)^{-1}\Delta s \le \Gamma,$$

where Γ is a positive constant. By Lemma 2 we arrive at the following inequality

$$\Upsilon(t) \le ce_{\ominus\Gamma^{-1}}(t,t_0) \text{ for } t \ge t_0.$$

This shows that $\|X(t)\mathcal{P}\|_{\mathcal{B}(\mathcal{X})}$ is bounded. By repeating the same procedure, we conclude that $\|X(t)(I-\mathcal{P})\|_{\mathcal{B}(\mathcal{X})}$ is bounded for $t \le t_0$.

Suppose that there exists another projection $\tilde{\mathcal{P}} \ne \mathcal{P}$ of exponential dichotomy of the homogeneous system (4). Then by using the similar arguments we may find positive constants N and $\hat{N}$ such that

$$\|X(t)\mathcal{P}\|_{\mathcal{B}(\mathcal{X})} < N \text{ for } t \ge t_0,$$

and

$$\|X(t)(I-\mathcal{P})\|_{\mathcal{B}(\mathcal{X})} < \hat{N} \text{ for } t \le t_0.$$

Using (5) - (6), for any nonzero vector ϑ we get

$$\begin{aligned}\|X(t)\mathcal{P}(I-\tilde{\mathcal{P}})\vartheta\|_{\mathcal{X}} &= \|X(t)\mathcal{P}X^{-1}(t_0)X(t_0)(I-\tilde{\mathcal{P}})\vartheta\|_{\mathcal{X}}\\ &\le K_1\|(I-\tilde{\mathcal{P}})\vartheta\|_{\mathcal{X}} \text{ for } t \ge t_0,\end{aligned}$$

and

$$\begin{aligned}\|X(t)\mathcal{P}(I-\tilde{\mathcal{P}})\vartheta\|_{\mathcal{X}} &= \|X(t)\mathcal{P}X^{-1}(t)X(t)(I-\tilde{\mathcal{P}})X^{-1}(t_0)X(t_0)(I-\tilde{\mathcal{P}})\vartheta\|_{\mathcal{X}}\\ &\le K_1K_2\|(I-\tilde{\mathcal{P}})\vartheta\|_{\mathcal{X}} \text{ for } t \le t_0.\end{aligned}$$

Thus, $x(t) = X(t)\mathcal{P}(I-\tilde{\mathcal{P}})\vartheta$ is a bounded solution of (4). Similarly, $x(t) = X(t)(I-\mathcal{P})\tilde{\mathcal{P}}\vartheta$ is also a bounded solution of (4). It follows from Lemma 4 that $x = 0$, and therefore, $\tilde{\mathcal{P}} = \mathcal{P}\tilde{\mathcal{P}} = \mathcal{P}$. This completes the proof.

Theorem 4. *Let $\mathbb{T}$ be an additively periodic time scale and assume that the system* (4) *admits exponential dichotomy with projection $\mathcal{P}$ and positive constants $K_{1,2}$ and $\alpha_{1,2}$. Let the matrix valued function A in* (4) *be almost automorphic. That is, for any sequence $\{\theta'_k\}$ in $\mathcal{T}$ there exists a subsequence $\{\theta_k\}$ such that $\lim_{k\to\infty} A(t+\theta_k) := \bar{A}(t)$ is well defined and $\lim_{k\to\infty}\bar{A}(t-\theta_k) = A(t)$ for each $t \in \mathbb{T}$ and $\mathcal{T}$ is as in* (1). *Then*

$$\lim_{k\to\infty} X(t+\theta_k)\mathcal{P}X^{-1}\big(\sigma(s+\theta_k)\big) := \bar{X}(t)\bar{\mathcal{P}}\bar{X}^{-1}\big(\sigma(s)\big),\ t \ge \sigma(s) \tag{9}$$

and

$$\lim_{k\to\infty} X(t+\theta_k)(I-\mathcal{P})X^{-1}\big(\sigma(s+\theta_k)\big) := \bar{X}(t)(I-\bar{\mathcal{P}})\bar{X}^{-1}(\sigma(s)),\ t \le \sigma(s) \tag{10}$$

are well defined for each $t \in \mathbb{T}$, *and the limiting system*

$$x^{\Delta}(t) = \bar{A}(t)x(t),\ x(t_0) = x_0 \tag{11}$$

admits an exponential dichotomy with the projection $\bar{\mathcal{P}}$ *and the same constants. In addition, for each* $t \in \mathbb{T}$ *we obtain*

$$\lim_{k\to\infty} \bar{X}(t-\theta_k)\bar{P}\bar{X}^{-1}\big(\sigma(s-\theta_k)\big) = X(t)PX^{-1}\big(\sigma(s)\big),\ t \geq \sigma(s) \tag{12}$$

and

$$\lim_{k\to\infty} \bar{X}(t-\theta_k)(I-\bar{P})\bar{X}^{-1}\big(\sigma(s-\theta_k)\big) = X(t)(I-P)X^{-1}\big(\sigma(s)\big),\ t \leq \sigma(s). \tag{13}$$

Proof. First we show that $X(t_0+\theta_k)\mathcal{P}X^{-1}\big(\sigma(t_0+\theta_k)\big)$ is convergent. Suppose the contrary that it is not convergent. Thus, there exist two subsequences $X(t_0+\theta_{k_m})\mathcal{P}X^{-1}\big(\sigma(t_0+\theta_{k_m})\big)$ and $X(t_0+\theta_{k'_m})\mathcal{P}X^{-1}\big(\sigma(t_0+\theta_{k'_m})\big)$ converging to two different numbers $\bar{\mathcal{P}}$ and $\underline{\mathcal{P}}$, respectively. From (5), we can write

$$\big\|X(t+\theta_{k_m})\mathcal{P}X^{-1}(\sigma(s+\theta_{k_m}))\big\|_{\mathcal{B}(\mathcal{X})} \leq K_1 e_{\ominus\alpha_1}\big(t,\sigma(s)\big) \text{ for } t \geq \sigma(s), \tag{14}$$

$$\big\|X(t+\theta_{k'_m})\mathcal{P}X^{-1}(\sigma(s+\theta_{k'_m}))\big\|_{\mathcal{B}(\mathcal{X})} \leq K_1 e_{\ominus\alpha_1}\big(t,\sigma(s)\big) \text{ for } t \geq \sigma(s). \tag{15}$$

Let $X_{k_m}(t)$ and $X_{k'_m}(t)$ denote the principal fundamental matrix solutions of the following systems:

$$x^{\Delta}(t) = A(t+\theta_{k_m})x(t),\ x(t_0) = x_0, \tag{16}$$

and

$$x^{\Delta}(t) = A(t+\theta_{k'_m})x(t),\ x(t_0) = x_0,$$

respectively. Then

$$X(t+\theta_{k_m}) = X_{k_m}(t)X(t_0+\theta_{k_m}). \tag{17}$$

To see this, consider

$$\begin{aligned}
\big[X_{k_m}^{-1}(t)X(t+\theta_{k_m})\big]^{\Delta} &= \big[X_{k_m}^{-1}(t)\big]^{\Delta}X^{\sigma}(t+\theta_{k_m}) + X_{k_m}^{-1}(t)\big[X(t+\theta_{k_m})\big]^{\Delta} \\
&= \big[X_{k_m}^{-1}(t)\big]^{\Delta}\big[X(t+\theta_{k_m}) + \mu(t)X^{\Delta}(t+\theta_{k_m})\big] \\
&\quad + X_{k_m}^{-1}(t)A(t+\theta_{k_m})X(t+\theta_{k_m}) \\
&= \big[X_{k_m}^{-1}(t)\big]^{\Delta}\big[I + \mu(t)A(t+\theta_{k_m})\big]X(t+\theta_{k_m}) \\
&\quad + X_{k_m}^{-1}(t)A(t+\theta_{k_m})X(t+\theta_{k_m})
\end{aligned}$$

which along with

$$\big[X_{k_m}^{-1}(t)\big]^{\Delta} = -X_{k_m}^{-1}(t)A(t+\theta_{k_m})\big[I+\mu(t)A(t+\theta_{k_m})\big]^{-1}$$

implies

$$\left[X_{k_m}^{-1}(t)X(t+\theta_{k_m})\right]^{\Delta}=0.$$

This means $X_{k_m}^{-1}(t)X(t+\theta_{k_m})$ is constant. Letting $t=t_0$, we obtain (17). Similarly, we have

$$X(t+\theta_{k'_m})=X_{k'_m}(t)X(\theta_{k'_m}).$$

Since $\{A(t+\theta_{k_m})\}$ converges to $\bar{A}(t)$ and $\{A(t+\theta_{k_m})x(t)\}$ converges to $\bar{A}(t)x(t)$, we have

$$A(t+\theta_{k_m})x(t)\to\bar{A}(t)x(t),$$

and

$$A(t+\theta_{k'_m})x(t)\to\bar{A}(t)x(t).$$

Thus the sequences $X_{k_m}(t)$ and $X_{k'_m}(t)$ converge to $\bar{X}(t)$ as $m\to\infty$ for each $t\in\mathbb{T}$. Now, the exponential dichotomy of the linear homogeneous system (4) plays an important role. Using (17) along with (14) and (15), we get

$$\left\|X_{k_m}(t)X(t_0+\theta_{k_m})\mathcal{P}X^{-1}(t_0+\theta_{k_m})X_{k_m}^{-1}(\sigma(s))\right\|_{\mathcal{B}(\mathcal{X})}\le K_1e_{\ominus\alpha_1}\big(t,\sigma(s)\big), \tag{18}$$

for $t\ge\sigma(s)$ and

$$\left\|X_{k'_m}(t)X(t_0+\theta_{k'_m})\mathcal{P}X^{-1}(t_0+\theta_{k'_m})X_{k'_m}^{-1}(\sigma(s))\right\|_{\mathcal{B}(\mathcal{X})}\le K_1e_{\ominus\alpha_1}\big(t,\sigma(s)\big) \tag{19}$$

for $t\ge\sigma(s)$. Taking the limit as $m\to\infty$, we obtain

$$\left\|\bar{X}(t)\bar{\mathcal{P}}\bar{X}^{-1}\big(\sigma(s)\big)\right\|_{\mathcal{B}(\mathcal{X})}\le K_1e_{\ominus\alpha_1}(t,\sigma(s)) \text{ for } t\ge\sigma(s), \tag{20}$$

$$\left\|\bar{X}(t)\underline{\mathcal{P}}\bar{X}^{-1}(\sigma(s))\right\|_{\mathcal{B}(\mathcal{X})}\le K_1e_{\ominus\alpha_1}\big(t,\sigma(s)\big) \text{ for } t\ge\sigma(s). \tag{21}$$

Similarly, we have

$$\left\|\bar{X}(t)(I-\bar{\mathcal{P}})\bar{X}^{-1}\big(\sigma(s)\big)\right\|_{\mathcal{B}(\mathcal{X})}\le K_2e_{\ominus\alpha_2}(\sigma(s),t) \text{ for } \sigma(s)\ge t, \tag{22}$$

$$\left\|\bar{X}(t)(I-\underline{\mathcal{P}})\bar{X}^{-1}(\sigma(s))\right\|_{\mathcal{B}(\mathcal{X})}\le K_2e_{\ominus\alpha_2}(\sigma(s),t) \text{ for } \sigma(s)\ge t. \tag{23}$$

Inequalities (20)-(23) show that the limiting system of (16) admits an exponential dichotomy. From uniqueness of projector of exponential dichotomy (Theorem 3), we get that $\bar{\mathcal{P}}=\underline{\mathcal{P}}$. This leads to a contradiction. Assume that

$$X(t_0+\theta_k)\mathcal{P}X^{-1}\big(\sigma(t_0+\theta_k)\big)\to\bar{\mathcal{P}}.$$

Let $X_k(t)$ denote the principal fundamental matrix solution of the system

$$x^{\Delta}(t)=A(t+\theta_k)x(t),\ x(t_0)=x_0.$$

Then $X_k(t)\to\bar{X}(t)$ and $X_k^{-1}\big(\sigma(s)\big)\to\bar{X}^{-1}\big(\sigma(s)\big)$ as $k\to\infty$ for each $t,s\in\mathbb{T}$. This means

$$X(t+\theta_k)\mathcal{P}X^{-1}\big(\sigma(s+\theta_k)\big)\to\bar{X}(t)\bar{\mathcal{P}}\bar{X}^{-1}\big(\sigma(s)\big) \text{ for } t\ge\sigma(s),$$

and similarly,

$$X(t+\theta_k)(I-\mathcal{P})X^{-1}\big(\sigma(s+\theta_k)\big) \to \bar{X}(t)(I-\bar{\mathcal{P}})\bar{X}^{-1}\big(\sigma(s)\big) \text{ for } \sigma(s) \geq t.$$

Furthermore, from (14) and (15), we arrive at the following inequalities:

$$\big\|X(t+\theta_k)\mathcal{P}X^{-1}(\sigma(s+\theta_k))\big\|_{\mathcal{B}(\mathcal{X})} \leq K_1 e_{\ominus\alpha_1}(t,\sigma(s)) \text{ for } t \geq \sigma(s),$$

and

$$\big\|X(t+\theta_k)(I-\mathcal{P})X^{-1}(\sigma(s+\theta_k))\big\|_{\mathcal{B}(\mathcal{X})} \leq K_2 e_{\ominus\alpha_2}(\sigma(s),t) \text{ for } \sigma(s) \geq t.$$

Taking the limit as $k \to \infty$, we show that the limiting system (11) admits exponential dichotomy with the projection $\bar{\mathcal{P}}$ and positive constants $\alpha_{1,2}$, $K_{1,2}$. To prove (12) and (13), we can follow the similar procedure that we used to get (9) and (10). This completes the proof.

4. EXISTENCE RESULTS

In this section, we propose some sufficient conditions guaranteeing existence of almost automorphic solutions of the following nonlinear neutral delay dynamic system

$$x^{\Delta}(t) = A(t)x(t) + Q^{\Delta}\big(t, x(t-g(t))\big) + G\big(t, x(t), x(t-g(t))\big), \tag{24}$$

where $g(t)$ is a scalar delay function, $A(t)$ is regressive, rd-continuous $n \times n$ matrix valued function, $Q \in C_{rd}(\mathbb{T} \times \mathcal{X}, \mathcal{X})$, and $G \in C_{rd}(\mathbb{T} \times \mathcal{X} \times \mathcal{X}, \mathcal{X})$. Henceforth, we assume that the time scale $\mathbb{T}$ is additively periodic and we use the following fixed point theorem in our further analysis.

Theorem 5 (Krasnoselskii). *Let $\mathbb{M}$ be a closed, convex and nonempty subset of a Banach space $\big(\mathbb{B}, \|.\|\big)$. Suppose that H_1 and H_2 maps $\mathbb{M}$ into $\mathbb{B}$ such that*

i. *$x, y \in \mathbb{M}$ implies $H_1 x + H_2 y \in \mathbb{M}$,*

ii. *H_2 is continuous and $H_2\mathbb{M}$ contained in a compact set,*

iii. *H_1 is a contraction mapping.*

Then there exists $z \in \mathbb{M}$ with $z = H_1 z + H_2 z$.

Let $\zeta, \psi \in A(\mathcal{X})$. We have the following list of assumptions:

A1 There exists a constant $E_1 > 0$ such that

$$\big\|Q(t,\zeta) - Q(t,\psi)\big\|_{\mathcal{X}} \leq E_1 \|\zeta - \psi\|_{A(\mathcal{X})} \text{ for all } t \in \mathbb{T}.$$

A2 There exists a constant $E_2 > 0$ such that

$$\left\|G(t,u,\zeta) - G(t,u,\psi)\right\|_{\mathcal{X}} \le E_2\big(\|u-v\|_{\mathcal{X}} + \|\zeta - \psi\|_{A(\mathcal{X})}\big) \text{ for all } t \in \mathbb{T}$$

and for any vector valued functions u and v defined on $\mathbb{T}$.

A3 Functions $g(t)$, $A(t)$, $Q(t,u)$ and $G(t,u,v)$ are almost automorphic in t.

A4 The linear homogeneous system (4) admits an exponential dichotomy with the positive constants $K_{1,2}$ and $\alpha_{1,2}$ and invertible projection $\mathcal{P}$ commuting with $X(t)$, where $X(t)$ is principal fundamental matrix solution of (4).

The following result can be proven similar to [**12**, Lemma 2.4], hence we omit it.

Lemma 5. *If u, $v : \mathbb{T} \to \mathcal{X}$ are almost automorphic functions, then $u\big(t - v(t)\big)$ is also almost automorphic on time scale $\mathbb{T}$.*

Theorem 6. *If* (A4) *holds, then the nonhomogeneous system*

$$x^{\Delta}(t) = A(t)x(t) + f(t) \tag{25}$$

has a solution $x(t)$ of the form

$$x(t) = \int_{-\infty}^{t} X(t)\mathcal{P}X^{-1}\big(\sigma(s)\big)f(s)\Delta s - \int_{t}^{\infty} X(t)(I-\mathcal{P})X^{-1}\big(\sigma(s)\big)f(s)\Delta s, \tag{26}$$

where $X(t)$ is the principal fundamental matrix solution of the system (4)*. Moreover, we have*

$$\|x\|_{\mathcal{X}} \le \left(\frac{K_1}{\alpha_1} + \frac{K_2}{\alpha_2}\right)\|f\|_{\mathcal{X}}.$$

Proof. It is trivial to show that $x(t)$ given by (26) satisfies the equation (25). The boundedness of x follows from the inequality

$$\begin{aligned}
\|x\|_{\mathcal{X}} &= \left\|\int_{-\infty}^{t} X(t)\mathcal{P}X^{-1}\big(\sigma(s)\big)f(s)\Delta s - \int_{t}^{\infty} X(t)(I-\mathcal{P})X^{-1}\big(\sigma(s)\big)f(s)\Delta s\right\|_{\mathcal{X}} \\
&\le \left(\int_{-\infty}^{t} \left\|X(t)\mathcal{P}X^{-1}(\sigma(s))\right\|_{\mathcal{B}(\mathcal{X})}\Delta s + \int_{t}^{\infty} \left\|X(t)(I-\mathcal{P})X^{-1}(\sigma(s))\right\|_{\mathcal{B}(\mathcal{X})}\Delta s\right)\|f\|_{\mathcal{X}} \\
&\le \left(\int_{-\infty}^{t} K_1 e_{\ominus\alpha_1}\big(t,\sigma(s)\big)\Delta s + \int_{t}^{\infty} K_2 e_{\ominus\alpha_2}\big(\sigma(s),t\big)\Delta s\right)\|f\|_{\mathcal{X}} \\
&\le \left(\frac{K_1}{\alpha_1} + \frac{K_2}{\alpha_2}\right)\|f\|_{\mathcal{X}},
\end{aligned}$$

where we used [**8**, Theorem 2.36] and [**9**, Remark 2] to get the inequalities

$$\begin{aligned}(27)\quad \int_{-\infty}^{t} K_1 e_{\ominus\alpha_1}(t,\sigma(s))\Delta s &= K_1\int_{-\infty}^{t}\frac{1}{1+\mu(s)\alpha_1}\left(\frac{1}{\alpha_1}e_{\ominus\alpha_1}(t,s)\right)^{\Delta_s}\Delta s\\ &\le \frac{K_1}{\alpha_1}+\lim_{s\to-\infty}e_{\ominus\alpha_1}(t,s)=\frac{K_1}{\alpha_1}+\lim_{s\to-\infty}\frac{1}{e_{\alpha_1}(t,s)}\\ &\le \frac{K_1}{\alpha_1}+\lim_{s\to-\infty}\frac{1}{1+\alpha_1(t-s)}=\frac{K_1}{\alpha_1},\end{aligned}$$

and

$$\begin{aligned}(28)\quad \int_{t}^{\infty} K_2 e_{\ominus\alpha_2}\big(\sigma(s),t\big)\Delta s &= -\frac{K_2}{\alpha_2}\int_{t}^{\infty}\left(\frac{1}{e_{\alpha_2}(s,t)}\right)^{\Delta_s}\Delta s\\ &= \frac{K_2}{\alpha_2}-\frac{K_2}{\alpha_2}\lim_{s\to\infty}\frac{1}{e_{\alpha_2}(s,t)}\\ &\le \frac{K_2}{\alpha_2}-\frac{K_2}{\alpha_2}\lim_{s\to\infty}\frac{1}{1+\alpha_2(s-t)}=\frac{K_2}{\alpha_2}.\end{aligned}$$

The proof is complete. □

Now, define the mapping H by

$$(Hx)(t) := (H_1x)(t) + (H_2x)(t),$$

where

$$(29)\quad (H_1x)(t) := Q\big(t, x(t-g(t))\big),$$

$$\begin{aligned}(30)\quad (H_2x)(t) := &\int_{-\infty}^{t} X(t)\mathcal{P}X^{-1}\big(\sigma(s)\big)\Lambda(s,x)\Delta s\\ &-\int_{t}^{\infty} X(t)(I-\mathcal{P})X^{-1}\big(\sigma(s)\big)\Lambda(s,x)\Delta s,\end{aligned}$$

and

$$(31)\quad \Lambda(s,x) := \big[A(s)Q(s,x(s-g(s))) + G(s,x(s),x(s-g(s)))\big].$$

Then the next result follows.

Lemma 6. *The mapping H maps $\mathcal{A}(\mathcal{X})$ into $\mathcal{A}(\mathcal{X})$.*

Proof. Suppose that $x \in \mathcal{A}(\mathcal{X})$. First, we deduce by using (A1)-(A3) along with Theorem 2 that the functions Q and G are almost automorphic. That is,

$$\lim_{n\to\infty} Q\big(t+k_n, x(t+k_n-g(t+k_n))\big) := \overline{Q}\big(t,\overline{x}(t-\overline{g}(t))\big),$$

and

$$\lim_{n\to\infty} G\big(t+k_n, x\,(t+k_n)\,, x(t+k_n-g(t+k_n))\big) := \overline{G}\big(t,\overline{x}(t),\overline{x}(t-\overline{g}(t))\big),$$

are well defined for each $t \in \mathbb{T}$. Moreover,

$$\lim_{n\to\infty} \overline{Q}\big(t-k_n, \overline{x}(t-k_n-\bar{g}(t-k_n))\big) = Q\big(t, x(t-g(t))\big),$$

and

$$\lim_{n\to\infty} \overline{G}\big(t-k_n, \overline{x}(t-k_n), \overline{x}(t-k_n-\overline{g}(t-k_n))\big) = G\big(t, x(t), x(t-g(t))\big),$$

for each $t \in \mathbb{T}$.
As for the mapping H, we have

$$\begin{aligned}(Hx)(t+k_n) &= Q\big(t+k_n, x(t+k_n-g(t+k_n))\big)\\ &\quad+\int_{-\infty}^{t} X(t+k_n)\mathcal{P}X^{-1}\big(\sigma(s+k_n)\big)\Lambda(s+k_n,x)\Delta s\\ &\quad-\int_{t}^{\infty} X(t+k_n)(I-\mathcal{P})X^{-1}\big(\sigma(s+k_n)\big)\Lambda(s+k_n,x)\Delta s.\end{aligned}$$

Taking the limit $n \to \infty$ and employing Lebesgue convergence theorem, we conclude that the mapping

$$\begin{aligned}\overline{(Hx)}(t) := \lim_{n\to\infty}(Hx)(t+k_n) &= \overline{Q}\big(t, \overline{x}(t-\overline{g}(t))\big)\\ &\quad+\int_{-\infty}^{t} \overline{X}(t)\overline{\mathcal{P}X}^{-1}\big(\sigma(s+k_n)\big)\overline{\Lambda}(s,x)\Delta s\\ &\quad-\int_{t}^{\infty} \overline{X}(t)(I-\overline{\mathcal{P}})\overline{X}^{-1}\big(\sigma(s+k_n)\big)\overline{\Lambda}(s,x)\Delta s\end{aligned}$$

is well defined for each $t \in \mathbb{T}$, where

$$\overline{\Lambda}(s,x) := \overline{A}(s)\overline{Q}\big(s, \overline{x}(s-\overline{g}(s))\big) + \overline{G}\big(s, \overline{x}(s), \overline{x}(s-\overline{g}(s))\big).$$

Applying similar procedure to the following

$$\begin{aligned}\overline{(Hx)}(t-k_n) &= \overline{Q}\big(t-k_n, \overline{x}(t-k_n-\overline{g}(t-k_n))\big)\\ &\quad+\int_{-\infty}^{t} \overline{X}(t-k_n)\overline{\mathcal{P}X}^{-1}\big(\sigma(s-k_n)\big)\overline{\Lambda}(s-k_n,x)\Delta s\\ &\quad-\int_{t}^{\infty} \overline{X}(t-k_n)(I-\overline{\mathcal{P}})\overline{X}^{-1}\big(\sigma(s-k_n)\big)\overline{\Lambda}(s-k_n,x)\Delta s,\end{aligned}$$

we get

$$\lim_{n\to\infty} \overline{(Hx)}(t-k_n) = (Hx)(t),$$

for each $t \in \mathbb{T}$. This means $Hx \in \mathcal{A}(\mathcal{X})$. This completes the proof. □

The following lemma follows from (29) and (A1).

Lemma 7. *Assume* (A1). *If* $E_1 < 1$, *then the mapping* H_1 *given by* (29) *is a contraction.*

Lemma 8. *Assume* (A1)-(A4). *Define the set*

$$\Pi_M := \left\{x \in \mathcal{A}(\mathcal{X}), \|x\|_{\mathcal{A}(\mathcal{X})} \leq M\right\}$$

where M *is a fixed constant. The mapping* H_2 *given by* (30) *is continuous and the image* $H_2(\Pi_M)$ *is contained in a compact set.*

Proof. By (A4), (30), and Theorem 6, we have the following:

$$\big\|(H_2x)(t)\big\|_{\mathcal{A}(\mathcal{X})} \leq \big\|\Lambda(.,x(.)\big\|_{\mathcal{A}(\mathcal{X})}\left(\frac{K_1}{\alpha_1}+\frac{K_2}{\alpha_2}\right), \tag{32}$$

where Λ is defined by (31). To see that H_2 is continuous, suppose ζ, $\psi \in \mathcal{A}(\mathcal{X})$ and for any given $\varepsilon > 0$ define the number $\delta(\varepsilon) > 0$ by

$$\delta := \frac{\varepsilon}{\big[\|A\|E_1\|\zeta-\psi\|_{\mathcal{A}(\mathcal{X})}+2E_2\|\zeta-\psi\|_{\mathcal{A}(\mathcal{X})}\big]\Big(\frac{K_1}{\alpha_1}+\frac{K_2}{\alpha_2}\Big)},$$

where the norm of matrix function A is defined by

$$\|A\| = \sup_{t\in\mathbb{T}}|A(t)|, \tag{33}$$

and

$$\big|A(t)\big| := \max_{1\leq i\leq n}\sum_{j=1}^{n}|a_{ij}(t)|.$$

If $\|\zeta-\psi\|_{\mathcal{A}(\mathcal{X})} < \delta$, then we have

$$\begin{aligned}
\big\|H_2(\zeta)(t)-H_2(\psi)(t)\big\|_{\mathcal{X}} \leq & \int_{-\infty}^{t}\Big(\big\|X(t)\mathcal{P}X^{-1}(\sigma(s))\big\|_{\mathcal{B}(\mathcal{X})} \\
& \times\big[\|A\|\;\|Q(s,\zeta(s-g(s)))-Q(s,\psi(s-g(s)))\big\|_{\mathcal{X}} \\
& +\|G(s,\zeta(s),\zeta(s-g(s)))-G(s,\psi(s),\psi(s-g(s)))\|_{\mathcal{X}}\big]\Big)\Delta s \\
& +\int_{t}^{\infty}\Big(\big\|X(t)(I-\mathcal{P})X^{-1}(\sigma(s))\big\|_{\mathcal{B}(\mathcal{X})} \\
& \times\big[\,\|A\|\;\|Q(s,\zeta(s-g(s)))-Q(s,\psi(s-g(s)))\|_{\mathcal{X}} \\
& +\|G(s,\zeta(s),\zeta(s-g(s)))-G(s,\psi(s),\psi(s-g(s)))\|_{\mathcal{X}}\,\big]\Big)\Delta s.
\end{aligned}$$

By (A2)-(A4), we get

$$\begin{aligned}\left\|H_2(\zeta)(t) - H_2(\psi)(t)\right\|_{\mathcal{X}} \leq &\int_{-\infty}^{t} K_1 e_{\ominus\alpha_1}(t, \sigma(s)) \\ &\times \left[\|A\| E_1 \|\zeta - \psi\|_{\mathcal{A}(\mathcal{X})} + 2E_2 \|\zeta - \psi\|_{\mathcal{A}(\mathcal{X})}\right] \Delta s \\ &+ \int_{t}^{\infty} K_2 e_{\ominus\alpha_2}\big(\sigma(s), t\big) \left[\, \|A\|\, E_1\, \|\zeta - \psi\|_{\mathcal{A}(\mathcal{X})} + 2E_2\, \|\zeta - \psi\|_{\mathcal{A}(\mathcal{X})} \,\right] \Delta s \\ &\leq \left[\|A\| E_1\, \|\zeta - \psi\|_{\mathcal{A}(\mathcal{X})} + 2E_2\, \|\zeta - \psi\|_{\mathcal{A}(\mathcal{X})}\,\right] \left(\frac{K_1}{\alpha_1} + \frac{K_2}{\alpha_2}\right) < \varepsilon,\end{aligned}$$

which shows that H_2 is continuous.

Now, we show that $H_2(\Pi_M)$ is contained in a compact set. For any $\zeta, \psi \in \Pi_M$ we have

$$\begin{aligned}\left\|G(t, \zeta(t), \psi(t - g(t)))\right\|_{\mathcal{X}} &\leq \left\|G(t, \zeta(t), \psi(t - g(t))) - G(t, 0, 0)\right\|_{\mathcal{X}} + \left\|G(t, 0, 0)\right\|_{\mathcal{X}} \\ &\leq E_2\big(\|\zeta\|_{\mathcal{A}(\mathcal{X})} + \|\psi\|_{\mathcal{A}(\mathcal{X})}\big) + a \\ &\leq 2ME_2 + a,\end{aligned}$$

and

$$\begin{aligned}\left\|Q(t, \zeta(t - g(t)))\right\|_{\mathcal{X}} &\leq \left\|Q(t, \zeta(t - g(t))) - Q(t, 0)\right\|_{\mathcal{X}} + \|Q(t, 0)\|_{\mathcal{X}} \\ &\leq E_1 \|\zeta\|_{\mathcal{A}(\mathcal{X})} + b \\ &\leq E_1 M + b\end{aligned}$$

where $a := \left\|G(t, 0, 0)\right\|_{\mathcal{X}}$ and $b := \left\|Q(t, 0)\right\|_{\mathcal{X}}$. This implies

$$\left\|H_2(\zeta_n)(t)\right\|_{\mathcal{A}(\mathcal{X})} \leq \left[\|A\|(E_1 M + b) + 2E_2 M + a\right] \left(\frac{K_1}{\alpha_1} + \frac{K_2}{\alpha_2}\right)$$

for any sequence $\{\zeta_n\}$ in Π_M. Moreover, from (A1),(A4) and (32), we deduce that

$$\big(H_2(\zeta_n(t))\big)^{\Delta} = A(t)\big(H_2(\zeta_n(t))\big) + A(t) Q\big(t, \zeta_n(\delta_-(\tau, t))\big) + G\big(t, \zeta_n(t), \zeta_n(\delta_-(\tau, t))\big)$$

is bounded. That is, $H_2(\zeta_n)$ is uniformly bounded an equicontinuous. Employing the Arzela-Ascoli theorem, we conclude that $H_2(\Pi_M)$ is contained in a compact set.

Theorem 7. *Assume* (A1)-(A4). *Let M_0 be a constant satisfying the following inequality*

$$E_1 M_0 + b + \left[\, \|A\|(E_1 M_0 + b) + 2E_2 M_0 + a\right] \left(\frac{K_1}{\alpha_1} + \frac{K_2}{\alpha_2}\right) \leq M_0,$$

where $E_1 \in (0, 1)$ and

$$a := \left\|G(t, 0, 0)\right\|_{\mathcal{X}}, b := \left\|Q(t, 0)\right\|_{\mathcal{X}}.$$

Then the equation (24) *has an almost automorphic solution in Π_{M_0}.*

Proof. For $\psi \in \Pi_{M_0}$, we have

$$\begin{aligned}\left\|H_1(\psi(t)) + H_2(\psi(t))\right\|_{\mathcal{A}(\mathcal{X})} &\leq \left\|Q(t,\psi(t-g(t))) - Q(t,0)\right\|_{\mathcal{X}} + \left\|Q(t,0)\right\|_{\mathcal{X}} \\ &+ \int_{-\infty}^{t} \Big(\left\|X(t)\mathcal{P}X^{-1}(\sigma(s))\right\|_{\mathcal{B}(\mathcal{X})} \\ &\quad \times \left\|A(s)Q(s,x(s-g(s))) + G(s,\psi(s),\psi(s-g(s)))\right\|_{\mathcal{X}} \Delta s\Big) \\ &+ \int_{t}^{\infty} \Big(\left\|X(t)(I-\mathcal{P})X^{-1}(\sigma(s))\right\|_{\mathcal{B}(\mathcal{X})} \\ &\quad \times \left\|A(s)Q(s,x(s-g(s))) + G(s,\psi(s),\psi(s-g(s)))\right\|_{\mathcal{X}} \Delta s\Big) \\ &\leq E_1 M_0 + b + \left[\ \|A\| \left(E_1 M_0 + b\right) + 2E_2 M_0 + a\right] \left(\frac{K_1}{\alpha_1} + \frac{K_2}{\alpha_2}\right) \leq M_0\end{aligned}$$

which means $H_1(\psi) + H_2(\psi) \in \Pi_{M_0}$. Consequently, all conditions of Theorem 5 are satisfied. Thus, there exists a $x \in \Pi_{M_0}$ such that $x(t) = H_1(x(t)) + H_2(x(t))$.

Example 1. Let $\mathbb{T} = \mathbb{Z}$ and consider the discrete system given by

$$\tag{34} \begin{aligned}\Delta x(t) &= \begin{bmatrix} \frac{1}{3}\operatorname{sgn}(\cos 2\pi t\theta) - 1 & 0 \\ 0 & \frac{1}{3}\operatorname{sgn}(\cos 2\pi t\theta) - 1 \end{bmatrix} x(t) \\ &\quad + \frac{1}{10}\Delta x(t-\tau) + \begin{bmatrix} \sin\frac{\pi}{2}t + \sin\frac{\pi}{2}t\sqrt{2} \\ \cos\pi t + \cos\pi t\sqrt{2} \end{bmatrix} + \frac{1}{20}x(t-\tau),\end{aligned}$$

where θ is an irrational number, τ is a positive integer with $t > \tau$ and Banach space $\mathcal{X} = \mathbb{R}$. In this case,

$$A(t) = \begin{bmatrix} \frac{1}{3}\operatorname{sgn}(\cos 2\pi t\theta) - 1 & 0 \\ 0 & \frac{1}{3}\operatorname{sgn}(\cos 2\pi t\theta) - 1 \end{bmatrix},$$

$$Q(t,x(t-g(t))) = \begin{bmatrix} \frac{1}{10}x_1(t-\tau) \\ \frac{1}{10}x_2(t-\tau) \end{bmatrix},$$

and

$$G(t,x(t),x(t-g(t))) = \begin{bmatrix} \sin\frac{\pi}{2}t + \sin\frac{\pi}{2}t\sqrt{2} + \frac{1}{20}x_1\left(t-\tau\right) \\ \cos\pi t + \cos\pi t\sqrt{2} + \frac{1}{20}x_2(t-\tau) \end{bmatrix}.$$

In [**34**], it is shown that $\operatorname{sgn}(\cos 2\pi t\theta)$ is an almost automorphic function for $t \in \mathbb{Z}$ and θ is irrational. Hence, the matrix function $A(t)$ is discrete almost automorphic. Furthermore, the functions Q and G are discrete almost automorphic in t for $x \in \mathcal{A}(\mathbb{R})$. Then assumption (A3) is satisfied. For any ς, $\psi \in \mathbb{M}$, we have

$$\left|Q(t,\varsigma(t-g(t))) - Q(t,\psi(t-g(t)))\right| \leq \frac{1}{10}\|\varsigma - \psi\|_{A(\mathbb{R})}$$

and

$$\left|G(t,\varsigma(t),\varsigma(t-g(t))) - G(t,\psi(t),\psi(t-g(t)))\right| \leq \frac{1}{20}\left\|\varsigma - \psi\right\|_{A(\mathbb{R})}.$$

Then (A1-A2) hold with $E_1 = \frac{1}{10}$, $E_2 = \frac{1}{20}$, $a = 2$ and $b = 0$.
Then the principal fundamental matrix solution of the homogeneous system

$$x(t+1) = \frac{1}{3}\operatorname{sgn}(\cos 2\pi t\theta)x(t)I$$

can be written as

$$X(t) = \begin{bmatrix} 3^{-t}\left(\prod_{j=0}^{t-1}\operatorname{sgn}(\cos 2\pi j\theta)\right) & 0 \\ 0 & 3^{-t}\left(\prod_{j=0}^{t-1}\operatorname{sgn}(\cos 2\pi j\theta)\right) \end{bmatrix},$$

and with projections $P_0 = I_{2\times 2}$ and $P_1 = 0_{2\times 2}$ the following inequality

$$\left|3^{s-t}\left(\prod_{j=s}^{t-1}\operatorname{sgn}(\cos 2\pi j\theta)\right)\right| = 3^{s-t} \leq K_1(1+\alpha_1)^{s-t} \text{ for } t \geq s$$

is satisfied for $K_1 = 1$ and $\alpha_1 = 1$, the homogeneous system admits exponential dichotomy, as desired in (A4). Moreover, we may assume $\alpha_1 = \alpha_2$ and $K_1 = K_2$ since $P_1 = 0_{2\times 2}$. That is, all assumptions of Theorem 7 hold. Hence, we conclude that the system (34) has an almost automorphic solution in Π_{M_0} whenever M_0 satisfies the inequality

$$\frac{1}{10}M_0 + \frac{4}{10}M_0 + \frac{3}{10}M_0 + 6 \leq M_0$$

or equivalently

$$30 \leq M_0.$$

The following theorem is useful for our next examples:

Theorem 8 ([**36**, Theorem 5.1])**.** *If $A(t)$ is uniformly bounded, rd-continuous $n\times n$ matrix valued function on $\mathbb{T}$, and there exists $\delta > 0$ such that*

$$|a_{ii}(t)| - \sum_{j\neq i}|a_{ij}(t)| - \frac{1}{2}\mu(t)\left(\sum_{j=1}^{n}|a_{ij}(t)|\right)^2 \geq 2\delta + \delta^2\mu(t),\ t\in\mathbb{T},\ i = 1,2,\dots,n, \tag{35}$$

then the regressive homogeneous system

$$x^{\Delta}(t) = A(t)x(t),\ t\in\mathbb{T}$$

admits an exponential dichotomy.

Example 2. The time scale $\hat{\mathbb{T}} = \bigcup_{k\in\mathbb{Z}}[2k, 2k+1]$ is an important time scale for mathematical models in biomathematics. In particular, a prey-predator model can be constructed on the time scale $\hat{\mathbb{T}}$. For example, two insect prey-predator populations living on nonoverlapping intervals and dying between two consecutive intervals while their eggs are incubating can be an applicable scenario on $\hat{\mathbb{T}}$. Now, consider

$$x^{\Delta}(t) = \begin{bmatrix} 1+\frac{\sin \pi t}{2} & 0 \\ 0 & 1 \end{bmatrix} x(t) + c_1 x^{\Delta}(t-\tau) + c_2 x(t-\tau),\ t\in\hat{\mathbb{T}}, \tag{36}$$

where $c_1 < 1$ and c_2 are real constants, τ is constant delay and the abstract Banach space $\mathcal{X}=\mathbb{R}$. If we compare system (36) with (24), we have

$$A(t) = \begin{bmatrix} 1 + \frac{\sin \pi t}{2} & 0 \\ 0 & 1 \end{bmatrix}, \qquad Q\big(t, x(t-g(t))\big) = c_1 x\,(t-\tau)$$

and

$$G(t, x(t), x(t-g(t))) = c_2 x\,(t-\tau)\,.$$

The matrix function A is periodic (consequently almost automorphic) and satisfies condition (35) for all $t \in \hat{\mathbb{T}}$. Then, by Theorem 8, the regressive homogeneous system

$$x^{\Delta}(t) = \begin{bmatrix} 1 + \frac{\sin \pi t}{2} & 0 \\ 0 & 1 \end{bmatrix} x(t)$$

admits an exponential dichotomy and suppose that dichotomy constants are denoted by $\hat{K}_{1,2}$ and $\hat{\alpha}_{1,2}$. Furthermore, for $x \in \mathcal{A}(\mathbb{R})$ assumptions (A1-A3) hold with $E_1 = c_1$, $E_2 = c_2$ and $a = b = 0$. Then Theorem 7 implies that system (36) has an almost automorphic solution in $\Pi_{\hat{M}_0} := \big\{x \in \mathcal{A}(\mathbb{R}), \|x\|_{\mathcal{A}(\mathbb{R})} \leq \hat{M}_0\big\}$ whenever the inequality

$$c_1\hat{M}_0 + \big[c_1\hat{M}_0 + 2c_2\hat{M}_0\big]\left(\frac{K_1}{\alpha_1} + \frac{K_2}{\alpha_2}\right) \leq \hat{M}_0$$

holds.

The following existence result is given in [**22**].

Theorem 9 ([**22**, Theorem 4.3])**.** *Suppose that $A(k)$ is discrete almost automorphic and a non-singular matrix and the set $\big\{A^{-1}(k)\big\}_{k\in\mathbb{Z}}$ is bounded. Also, assume that the homogeneous system $U(k+1) = A(k)U(k)$, $k \in \mathbb{Z}$, admits an exponential dichotomy on $\mathbb{Z}$ with positive constants η, ν, β, α and the function $f : \mathbb{Z} \times E^n \to E^n$ is discrete almost automorphic in k for each u in E^n, satisfying the following condition:*

1. *There exists a constant $0 < L < \dfrac{(1-e^{-\alpha})(e^{\beta}-1)}{\eta(e^{\beta}-1)+\nu(1-e^{-\alpha})}$ such that*

$$\|f(k,u) - f(k,v)\| \leq L\,\|u - v\|$$

for every $u, v \in E^n$ and $k \in \mathbb{Z}$. Then the system

$$U(k+1) = A(k)U(k) + f\big(k, u(k)\big), k \in \mathbb{Z}$$

has a unique almost automorphic solution.

In the following example we use Theorem 7 to obtain existence of almost periodic solutions of a discrete system for which [**22**, Theorem 4.3] is invalid.

Example 3. Consider the function $\varphi : \mathbb{Z} \to \mathbb{R}$ defined by

$$\varphi(0) = 2, \varphi(1) = 1, \varphi(-2) = \frac{3}{2}, \varphi(-1) = \frac{1}{2},$$

$$\vdots$$

$$\varphi(t) := \varphi(t + 2^{2i-1}) - \frac{1}{2^{2i-1}} \text{ if } t \in \left[-\sum_{k=1}^{i} 2^{2k-1}, -\left(1 + \sum_{k=1}^{i-1} 2^{2k-1}\right)\right] \cap \mathbb{Z} \text{ for } i \in \mathbb{N}$$

$$\varphi(t) := \varphi(t - 2^{2i}) - \frac{1}{2^{2i}} \text{ if } t \in \left[2 + \sum_{k=1}^{i-1} 2^{2k}, 1 + \sum_{k=1}^{i} 2^{2k}\right] \cap \mathbb{Z} \text{ for } i \in \mathbb{N}$$

see [**35**, Example 2.19]. Obviously, the function φ never vanishes on $\mathbb{Z}$. It is proven in [**35**, Theorem 1.16] that the function φ is discrete almost periodic (and hence, discrete almost automorphic) and that

$$\lim_{i\to\infty} \varphi\big(2^0 - 2^1 + 2^2 - 2^3 + \cdots + (-2)^i\big) = 0.$$

We construct the following discrete system

$$x(t+1) = \begin{bmatrix} \varphi(t) & 0 \\ 0 & \varphi(t) \end{bmatrix} x(t) + f(t,x),\ t \in \mathbb{Z} \tag{37}$$

for any discrete almost automorphic function f satisfying

$$\|f(t,\xi) - f(t,\psi)\|_{\mathcal{X}} \le E \|\xi - \psi\|_{\mathcal{A}(\mathcal{X})}$$

for a sufficiently small positive constant E and $\xi, \psi \in \mathcal{A}(\mathcal{X})$. In this case the discrete almost automorphic matrix function $A(t) = \varphi(t) I_{2\times 2}$ is invertible and has unbounded inverse. Furthermore, we can say that the homogeneous part of (37) admits an discrete exponential dichotomy by using discrete counterpart of Theorem 8. Then the assumptions of Theorem 7 are satisfied and the sysem (37) has a discrete almost automorphic solution. However, [**22**, Theorem 4.3] is invalid for the system (37) since the matrix $A(t)$ has no bounded inverse.

One may repeat the same procedure in the last section by replacing $\mathcal{A}(\mathcal{X})$ with $\mathcal{AP}(\mathcal{X})$, the space of all almost periodic functions on $\mathcal{X}$, and the assumption (A3) with the following

A3′ Functions $A(t)$, $g(t)$, $Q(t,u)$ and $G(t,u,v)$ are almost periodic in t

to arrive at the following result:

Theorem 10 (Almost periodic solutions of the system (24))**.** *Assume* (A3′) *and* (A2)-(A4). *Let* M_0 *be a constant satisfying the following inequality*

$$E_1 M_0 + b + \left[\, \|A\| (E_1 M_0 + b) + 2E_2 M_0 + a \right] \left(\frac{K_1}{\alpha_1} + \frac{K_2}{\alpha_2} \right) \le M_0,$$

where $E_1 \in (0,1)$ *and*

$$a := \|G(t,0,0)\|_{\mathcal{X}}, b := \|Q(t,0)\|_{\mathcal{X}}.$$

Then the equation (24) *has an almost periodic solution in*

$$\widetilde{\Pi}_{M_0} := \big\{x \in \mathcal{AP}(\mathcal{X}), \|x\|_{\mathcal{AP}(\mathcal{X})} \leq M_0\big\}.$$

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