Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **10** (2016), 332–344. doi:10.2298/AADM161005023L

ON THE SOLUTIONS TO A GENERALIZED FRACTIONAL CAUCHY PROBLEM

Barbara Łupińska, Tatiana Odzijewicz, Ewa Schmeidel

We show the existence and uniqueness of solutions to the fractional nonlinear Cauchy problem containing Katugampola derivative. Moreover, in the linear case, we give a formula for the solution.

1. INTRODUCTION

Fractional calculus studies problems with derivatives and integrals of real or complex order. As a purely mathematical field, the theory of fractional calculus was brought up for the first time in the XVIIth century and since then many renowned scientists worked on this topic, among them Euler, Laplace, Fourier, Abel, Liouville and Riemann [**13**]. When it comes to the practical applications, however, the notable development can only be observed during the last decades. Fractional operators have non–local character and consequently can be successfully applied in the study of non–local or time–dependent processes. The fields of applied sciences, where fractional calculus is found to be useful, include chaotic dynamics [**19**], material sciences [**14**], mechanics of fractal and complex media [**4, 18**], quantum mechanics [**6**], physical kinetics [**20**], and many others (see, e.g., [**5, 16, 17, 21, 22**]).

In the literature many different types of fractional operators have been proposed, with the choice of a relevant definition being linked to the considered system [**10, 11, 15**]. Here, in order to simplify the theory, we show that two important fractional problems with different notions of derivatives– the fractional Cauchy problem with Hadamard derivatives and the fractional Cauchy problem with Riemann–Liouville derivatives– can be joined together by considering Cauchy problem with derivative defined recently by Katugampola. The Katugampola operators depend

2010 Mathematics Subject Classification. 34A08, 26A33.
Keywords and Phrases. fractional calculus, fractional Cauchy problem.

on extra paramater $\rho > 0$, which by taking $\rho \to 0^+$ reduces to the Hadamard fractional operator and for parameter $\rho = 1$ becomes the Riemann–Liouville fractional operator. In this work, applying properties proved recently in [**12**], we show the existence and uniqueness of solutions to the Katugampola Cauchy problem. Moreover, we derive a Cauchy formula for the solution to the linear problem of such a type.

The text is organized as follows. Section 2 recalls definitions and main properties of the Katugampola fractional integrals and derivatives. Main results are then stated and proved in Section 3 and Section 4, where we show existence and uniqueness of solutions to the Katugampola Cauchy problem and give the exact solution in the linear case.

2. PRELIMINARIES

In this section we present some basic definitions and properties of fractional calculus used in the work. In the literature we can find several definitions for fractional integrals and derivatives, each one presenting its advantages and disadvantages [**10, 11, 15**]. In this paper we are focused on the Katugampola integral and differential operators, which first were introduced in [**8**]. The reader interested in the subject is referred to the recent works [**1, 2, 3**].

By $L^p([a,b],\mathbb{R})$, $1 \le p < \infty$, we denote the classical space of measurable functions $f : [a,b] \to \mathbb{R}$ which absolute value raised to the p-th power has a finite integral, or equivalently, that

$$\|f\|_{L^p([a,b],\mathbb{R})} := \left(\int_a^b |f(t)|^p \mathrm{d}t \right)^{\frac{1}{p}} < \infty.$$

Moreover, along the work, we assume that $0 < a < t < b < \infty$.

Definition 1. *Let $\alpha > 0,\ \rho > 0$ and $f \in L^p([a,b],\mathbb{R})$, $1 \le p < \infty$. The functions*

$$I^{\alpha,\rho}_{a+} f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} f(\tau)\mathrm{d}\tau,$$
$$I^{\alpha,\rho}_{b-} f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_t^b \frac{\tau^{\rho-1}}{(\tau^\rho - t^\rho)^{1-\alpha}} f(\tau)\mathrm{d}\tau,$$

for $t \in (a,b)$ are called the left-sided and right-sided Katugampola integrals of fractional order α, respectively.

Definition 2. *Let $\alpha > 0,\ \rho > 0,\ n = [\alpha] + 1$ and $f \in L^p([a,b],\mathbb{R})$, $1 \le p < \infty$. The functions*

$$D^{\alpha,\rho}_{a+} f(t) = \left(t^{1-\rho} \frac{\mathrm{d}}{\mathrm{d}t} \right)^n I^{n-\alpha,\rho}_{a+} f(t)$$
$$D^{\alpha,\rho}_{b-} f(t) = \left(-\, t^{1-\rho} \frac{\mathrm{d}}{\mathrm{d}t} \right)^n I^{n-\alpha,\rho}_{b-} f(t)$$

for $t \in (a,b)$ are called the left-sided and right-sided Katugampola derivatives of fractional order α, respectively.

These operators generalize two other operators, by introducing a new parameter $\rho > 0$ in the definition. The next theorem shows that the Riemann-Liouville and the Hadamard fractional derivatives are special cases of the generalized derivative in question.

Theorem 1 (Theorem 3.1, [**9**])**.** *Let $\alpha \in \mathbb{R},\ \alpha > 0,\ n = [\alpha] + 1$ and $\rho > 0$. Then, for $t > a$*

1. $\lim\limits_{\rho\to 1} I^{\alpha,\rho}_{a+} f(t) = \dfrac{1}{\Gamma(\alpha)} \int_a^t \big(t-\tau\big)^{\alpha-1} f(\tau)\mathrm{d}\tau,$

2. $\lim\limits_{\rho\to 0^+} I^{\alpha,\rho}_{a+} f(t) = \dfrac{1}{\Gamma(\alpha)} \int_a^t \Big(\log\dfrac{t}{\tau}\Big)^{\alpha-1} f(\tau)\dfrac{\mathrm{d}\tau}{\tau},$

3. $\lim\limits_{\rho\to 1} D^{\alpha,\rho}_{a+} f(t) = \Big(\dfrac{\mathrm{d}}{\mathrm{d}t}\Big)^n \dfrac{1}{\Gamma(n-\alpha)} \int_a^t \dfrac{f(\tau)}{(t-\tau)^{\alpha-n+1}}\mathrm{d}\tau,$

4. $\lim\limits_{\rho\to 0^+} D^{\alpha,\rho}_{a+} f(t) = \dfrac{1}{\Gamma(n-\alpha)}\Big(t\dfrac{\mathrm{d}}{\mathrm{d}t}\Big)^n \int_a^t \Big(\log\dfrac{t}{\tau}\Big)^{n-\alpha-1} f(\tau)\dfrac{\mathrm{d}\tau}{\tau}.$

Moreover, in [**8, 9**] the following properties of the Katugampola integrals and derivatives were proved. Let $f,\ g \in L^p([a,b],\mathbb{R})$ and $\alpha,\ \beta > 0$ then

$$I^{\alpha,\rho}_{a+} I^{\beta,\rho}_{a+} f(t) = I^{\alpha+\beta,\rho}_{a+} f(t), \tag{1}$$

$$I^{\alpha,\rho}_{a+}[f(t)+g(t)] = I^{\alpha,\rho}_{a+} f(t) + I^{\alpha,\rho}_{a+} g(t), \tag{2}$$

$$D^{\alpha,\rho}_{a+} I^{\alpha,\rho}_{a+} f(t) = f(t). \tag{3}$$

The similar results for the right-sided operators also exist.

Definition 3. *By $I^{\alpha,\rho}_{a+}(L^p([a,b],\mathbb{R}))$ or shortly $I^{\alpha,\rho}_{a+}(L^p)$ we denote the set*

$$I^{\alpha,\rho}_{a+}(L^p) = \left\{ f : [a,b] \to \mathbb{R} :\ \exists g \in L^p([a,b],\mathbb{R}) :\ f = I^{\alpha,\rho}_{a+} g(t) \text{ on } [a,b] \right\}.$$

Lemma 1 (Lemma 1, [**12**])**.** *Let $\alpha > 0,\ \rho > 0$ and $p \geq 1$. If $f \in I^{\alpha,\rho}_{a+}(L^p)$, then $I^{1-\alpha,\rho}_{a+} f(a) = 0$.*

Let us recall the following technical results.

Lemma 2 (Lemma 2, [**12**])**.** *If $f \in L^p([a,b],\mathbb{R}),\ 1 \leq p < \infty,\ \alpha > 0,\ \rho > 0$, then*

$$\left|I^{\alpha,\rho}_{a^+} f(t)\right|^p \leq c I^{\alpha,\rho}_{a^+} |f(t)|^p$$

with

$$c = \left(\Big(\frac{b^\rho - a^\rho}{\rho}\Big)^\alpha \frac{1}{\Gamma(\alpha+1)}\right)^{p-1}.$$

Consequently, $I^{\alpha,\rho}_{a^+} f \in L^p([a,b],\mathbb{R})$.

Lemma 3 (Lemma 3, [**12**])**.** *Let $\alpha \in (0,1)$, $\rho > 0$. Then we have*

$$I_{a+}^{1-\alpha,\rho}\left(\frac{\rho}{t^\rho - a^\rho}\right)^{1-\alpha} = \Gamma(\alpha).$$

Using the relation between the Katugampola fractional operators, we obtain that

$$D_{a+}^{\alpha,\rho}\left(\frac{\rho}{t^\rho - a^\rho}\right)^{1-\alpha} = 0, \tag{4}$$

for $\alpha \in (0,1)$ and $\rho > 0$.

The following rule of fractional integration by parts holds for Katugampola operators.

Theorem 2 (Theorem 2, [**12**])**.** *Let $\alpha > 0,\ \rho > 0,\ p \geq 1,\ q \geq 1$. If $f \in L^p([a,b],\mathbb{R})$ and $g \in L^q([a,b],\mathbb{R})$, then*

$$\int_a^b f(t) I_{a+}^{\alpha,\rho} g(t)\mathrm{d}t \leq B \int_a^b g(t) I_{b-}^{\alpha,\rho} f(t)\mathrm{d}t,$$

with $B = \max\left\{1, \left(\frac{b}{a}\right)^{1-\rho}\right\}$.

3. EXISTENCE AND UNIQUENESS OF SOLUTIONS TO THE GENERALIZED FRACTIONAL CAUCHY PROBLEM

In this section we show the existence and uniqueness of the solution to the problem

$$\begin{cases} D_{a+}^{\alpha,\rho} x(t) = h(t, x(t)) \\ I_{a+}^{1-\alpha,\rho} x(a) = 0. \end{cases} \tag{5}$$

By solution to problem (5) we mean function $x \in I_{a+}^{\alpha,\rho}(L^p)$, where $p \geq 1$, satysfying the equation (5) for $t \in [a,b]$. By Lemma 1, any function belonging to $I_{a+}^{\alpha,\rho}(L^p)$ satisfies initial condition (5).

Theorem 3. *Let $\alpha \in (0,1)$, $p \geq 1$. If*

- *the function $h(\cdot, x)$ is measurable for any $x \in \mathbb{R}$,*
- *there exists a constant $N > 0$ such that*

  $$|h(t,x_1) - h(t,x_2)| \leq N|x_1 - x_2|$$

  for all $x_1, x_2 \in \mathbb{R}$ and $t \in [a,b]$,
- *the function $[a,b] \ni t \longmapsto h(t,0) \in \mathbb{R}$ belongs to $L^p([a,b],\mathbb{R})$,*

then Cauchy problem (5) *possesses a unique solution* $x \in I^{\alpha;\rho}_{a+}(L^p)$.

Proof. Let $\alpha \in (0,1)$ and $p \geq 1$. The description of the set $I^{\alpha;\rho}_{a+}(L^p)$ implies that the problem of existence of a solution x to problem (5) in the set $I^{\alpha;\rho}_{a+}(L^p)$ is equivalent to the problem of existence of a solution φ to the equation

$$\varphi(t) = h\big(t, I^{\alpha;\rho}_{a+}\varphi(t)\big) \tag{6}$$

in the set $L^p([a,b],\mathbb{R})$. First, note that the above equation makes sense in any interval $[a,b_1] \subset [a,b]$, where b_1 is such that the inequality

$$\max\left\{1, \Big(\frac{b_1}{a}\Big)^{1-\rho}\right\} \cdot \left(\frac{N}{\Gamma(\alpha+1)}\Big(\frac{b_1-a}{\rho}\Big)^{\alpha}\right)^p < 1 \tag{7}$$

holds. In order to use the Banach fixed–point theorem for the space $L^p([a,b_1],\mathbb{R})$, we introduce the operator

$$S : L^p([a,b_1],\mathbb{R}) \ \ni \varphi \longmapsto h(t, I^{\alpha;\rho}_{a+}\varphi(t)) \in L^p([a,b_1],\mathbb{R}).$$

Let us observe that S is well defined. Indeed, using the Lipschitz condition, we have

$$\big|h(t, I^{\alpha;\rho}_{a+}\varphi(t))\big| \leq |h(t, I^{\alpha;\rho}_{a+}\varphi(t)) - h(t,0)| + |h(t,0)| \leq N|I^{\alpha;\rho}_{a+}\varphi(t)| + |h(t,0)|.$$

Now, from Minkowski's inequality we obtain

$$\begin{aligned}\left(\int_a^{b_1} \big|h(t, I^{\alpha;\rho}_{a+}\varphi(t))\big|^p \,\mathrm{d}t\right)^{\frac{1}{p}} &\leq \left(\int_a^{b_1} \big(N|I^{\alpha;\rho}_{a+}\varphi(t)| + |h(t,0)|\big)^p \,\mathrm{d}t\right)^{\frac{1}{p}} \\ &\leq \left(N\int_a^{b_1} |I^{\alpha;\rho}_{a+}\varphi(t)|^p \mathrm{d}t\right)^{\frac{1}{p}} + \left(\int_a^{b_1} |h(t,0)|^p \mathrm{d}t\right)^{\frac{1}{p}}.\end{aligned}$$

From Lemma 2 and our assumptions it follows that $S\varphi \in L_p([a,b_1],\mathbb{R})$. Using once again Lipschitz condition, Lemma 2 and Theorem 2, we obtain

$$\begin{aligned}&\|S(\varphi) - S(\psi)\|^p_{L_p([a,b_1],\mathbb{R})} = \int_a^{b_1} \big|h(t, I^{\alpha;\rho}_{a+}\varphi(t)) - h(t, I^{\alpha;\rho}_{a+}\psi(t))\big|^p \,\mathrm{d}t \\ &\leq N^p \int_a^{b_1} |I^{\alpha;\rho}_{a+}\varphi(t) - I^{\alpha;\rho}_{a+}\psi(t)|^p \mathrm{d}t \leq cN^p \int_a^{b_1} I^{\alpha;\rho}_{a+}|\varphi(t) - \psi(t)|^p \mathrm{d}t \\ &\leq cN^pB \int_a^{b_1} |\varphi(t) - \psi(t)|^p I^{\alpha;\rho}_{b_1-}1\mathrm{d}t.\end{aligned}$$

Note that

$$I^{\alpha;\rho}_{b_1-}1 = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_t^{b_1} \frac{\tau^{\rho-1}}{(\tau^\rho - t^\rho)^{1-\alpha}}\mathrm{d}\tau = \frac{(b_1^\rho - t^\rho)^\alpha}{\rho^\alpha\Gamma(\alpha+1)} \leq \frac{(b_1^\rho - a^\rho)^\alpha}{\rho^\alpha\Gamma(\alpha+1)}.$$

Therefore,

$$\left\|S(\varphi)-S(\psi)\right\|^p_{L_p([a,b_1],\mathbb{R})} \leq \frac{cN^pB(b_1^\rho-a^\rho)^\alpha}{\rho^\alpha\Gamma(\alpha+1)}\left\|\varphi-\psi\right\|^p_{L_p([a,b_1],\mathbb{R})},$$

where $c=\Big(\Big(\frac{b_1^\rho-a^\rho}{\rho}\Big)^\alpha\frac{1}{\Gamma(\alpha+1)}\Big)^{p-1}$ and $B=\max\Big\{1,\Big(\frac{b_1}{a}\Big)^{1-\rho}\Big\}$.
In accordance with (7),

$$\frac{cN^pB(b_1^\rho-a^\rho)^\alpha}{\rho^\alpha\Gamma(\alpha+1)}\in(0,1)$$

and consequently the operator S has a unique fixed point in $L^p([a,b_1],\mathbb{R})$. This means that problem (5) possesses a unique solution $x\in I^{\alpha;\rho}_{a+}(L^p)$ on the interval $[a,b_1]$.

Next, we consider the interval $[b_1,b_2]$ where $b_2=b_1+h_1$, $h_1>0$ is such that $b_2<b$ and

$$\max\left\{1,\Big(\frac{b_2}{b_1}\Big)^{1-\rho}\right\}\cdot\left(\frac{N}{\Gamma(\alpha+1)}\Big(\frac{b_2-b_1}{\rho}\Big)^\alpha\right)^p<1$$

Rewrite the equation (6) in the form

$$\varphi(t)=h\left(t,\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\left(\int_{b_1}^t\frac{\tau^{\rho-1}}{(t^\rho-\tau^\rho)^{1-\alpha}}\varphi(\tau)\mathrm{d}\tau+\int_a^{b_1}\frac{\tau^{\rho-1}}{(t^\rho-\tau^\rho)^{1-\alpha}}\varphi(\tau)\mathrm{d}\tau\right)\right).$$

Since the function $\varphi(t)$ is uniquely defined on the interval $[a,b_1]$, the second integral can be considered as the known function, and we rewrite the last equation as

$$\varphi(t)=h\left(t,\varphi_0(t)+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{b_1}^t\frac{\tau^{\rho-1}}{(t^\rho-\tau^\rho)^{1-\alpha}}\varphi(\tau)\mathrm{d}\tau\right).$$

By the same argument as before, we prove that there exists a unique solution $\varphi_2\in L^p([b_1,b_2],\mathbb{R})$ to equation (6) on $[b_1,b_2]$. Repeating the previous reasoning, choosing $b_k=b_{k-1}+h_{k-1}$ such that $h_{k-1}>0$, $b_k<b$ and

$$\max\left\{1,\Big(\frac{b_k}{b_{k-1}}\Big)^{1-\rho}\right\}\cdot\left(\frac{N}{\Gamma(\alpha+1)}\Big(\frac{b_k-b_{k-1}}{\rho}\Big)^\alpha\right)^p<1$$

we see that equation (6) possesses a solution $\varphi_k\in L^p([b_{k-1},b_k],\mathbb{R})$ on each interval $[b_{k-1},b_k]$, $(k=1,\dots,n)$, where $a=b_0<b_1<\dots<b_n=b$ and we conclude that for problem (6) there exists a unique solution $\varphi\in L^p([a,b],\mathbb{R})$. This means that problem (5) possesses a unique solution $x\in I^{\alpha;\rho}_{a+}(L^p)$ on the interval $[a,b]$. □

Now, let us consider the following Cauchy problem with non-homogeneous initial conditions

$$\begin{cases} D^{\alpha;\rho}_{a+}y(t)=g(t,y(t))\\ I^{1-\alpha,\rho}_{a+}y(a)=c,\end{cases}\tag{8}$$

where $g : [a,b] \times \mathbb{R} \to \mathbb{R}$, $c \in \mathbb{R}$. Solution to problem (8) is a function $y \in I^{\alpha,\rho}_{a+}(L^p) + \left\{ \frac{d}{(t^\rho - a^\rho)^{1-\alpha}};\ d \in \mathbb{R} \right\}$ satisfying the above equation on $[a,b]$ and the initial condition. Inspired by the method proposed in [**7**], we formulate and prove the following theorem.

Theorem 4. *Let* $\alpha \in (0,1)$, $p \geq 1$. *If*

- *the function* $g(\cdot, y)$ *is measurable for any* $y \in \mathbb{R}$,
- *there exists a constant* $N > 0$ *such that*

$$|g(t,y_1) - g(t,y_2)| \leq N|y_1 - y_2|$$

  for all $y_1, y_2 \in \mathbb{R}$ *and* $t \in [a,b]$,
- *the function* $[a,b] \ni t \longmapsto g(t,0) \in \mathbb{R}$ *belongs to* $L^p([a,b],\mathbb{R})$,

then Cauchy problem (8) *possesses a unique solution.*

Proof. Let $g : [a,b] \times \mathbb{R} \longmapsto \mathbb{R}$ be any function satisfying assumptions of Theorem 4 and we define

$$h(t,x(t)) := g\left(t, x(t) + \frac{c}{\Gamma(\alpha)}\left(\frac{\rho}{t^\rho - a^\rho}\right)^{1-\alpha}\right). \tag{9}$$

Now, we will show that this function satisfies the assumptions of Theorem 3. Let us observe that the function $t \mapsto \frac{c}{\Gamma(\alpha)}\left(\frac{\rho}{t^\rho - a^\rho}\right)^{1-\alpha}$ is measurable (in fact, it is continuous on (a,b)) and the function g is measurable in t and continuous (in fact, Lipschitzian) in y. Thus, we deduce that h is measurable in t. The fact that h is Lipschitzian in x follows from the fact that g is Lipschitzian in the second argument. Moreover,

$$\begin{aligned}
|h(t,0)| &= \left| g\left(t, \frac{c}{\Gamma(\alpha)}\left(\frac{\rho}{t^\rho - a^\rho}\right)^{1-\alpha}\right)\right| \\
&\leq \left| g\left(t, \frac{c}{\Gamma(\alpha)}\left(\frac{\rho}{t^\rho - a^\rho}\right)^{1-\alpha}\right) - g(t,0)\right| + |g(t,0)| \\
&\leq N\frac{c}{\Gamma(\alpha)}\left(\frac{\rho}{t^\rho - a^\rho}\right)^{1-\alpha} + |g(t,0)|
\end{aligned}$$

for $t \in [a,b]$ and the function $[a,b] \ni t \longmapsto h(t,0) \in \mathbb{R}$ belongs to $L^p([a,b];\mathbb{R})$. Therefore, by Theorem 3, we conclude that there exists $x \in I^{\alpha;\rho}_{a+}(L^p)$ such that $h(t,x(t)) = D^{\alpha;\rho}_{a+}x(t)$ for $t \in [a,b]$, which is equivalent to

$$g\left(t, x(t) + \frac{c}{\Gamma(\alpha)}\left(\frac{\rho}{t^\rho - a^\rho}\right)^{1-\alpha}\right) = D^{\alpha;\rho}_{a+}\left(x(t) + \frac{c}{\Gamma(\alpha)}\left(\frac{\rho}{t^\rho - a^\rho}\right)^{1-\alpha}\right),$$

by (4) and (9). Hence, there exists

$$y(t) = x(t) + \frac{c}{\Gamma(\alpha)} \left(\frac{\rho}{t^\rho - a^\rho} \right)^{1-\alpha}$$

which is a solution to the Cauchy problem (8).

Now, it is enough to prove the uniqueness of solution. Let us assume that $y_1(t),\ y_2(t)$ are solutions to (8). Let us put

$$x_1(t) = y_1(t) - \frac{c}{\Gamma(\alpha)} \left(\frac{\rho}{t^\rho - a^\rho} \right)^{1-\alpha},\ t \in (a,b),$$
$$x_2(t) = y_2(t) - \frac{c}{\Gamma(\alpha)} \left(\frac{\rho}{t^\rho - a^\rho} \right)^{1-\alpha},\ t \in (a,b).$$

Of course, by Lemma 3 and (4) we get $I_{a+}^{1-\alpha,\rho} x_1(a) = I_{a+}^{1-\alpha,\rho} x_2(a) = 0$ and

$$\begin{aligned} D_{a+}^{\alpha,\rho} x_1(t) = D_{a+}^{\alpha,\rho} y_1(t) = g(t, y_1(t)) &= g\left(t, x_1(t) + \frac{c}{\Gamma(\alpha)} \left(\frac{\rho}{t^\rho - a^\rho} \right)^{1-\alpha} \right) \\ &= h(t, x_1(t)),\ t \in [a,b] \end{aligned}$$

where h is given by (9).

In the same way, we obtain

$$D_{a+}^{\alpha,\rho} x_2(t) = h(t, x_2(t)),\ t \in [a,b].$$

Since h satisfies the assumptions of Theorem 3, we get that $x_1(t) = x_2(t)$ on $[a,b]$. Consequently, $y_1(t) = y_2(t)$.

4. LINEAR CAUCHY PROBLEM

Let us consider a linear Cauchy problem of the type (8) with $c = 0$ in the following form

$$(10) \qquad \begin{cases} D_{a^+}^{\alpha,\rho} x(t) = Ax(t) + B(t), \\ I_{a^+}^{1-\alpha,\rho} x(a) = 0 \end{cases}$$

where $B : [a,b] \longmapsto \mathbb{R}$, $B \in L^p([a,b], \mathbb{R})$, and $A \in \mathbb{R}$.

Theorem 5. *The unique solution to problem* (10) *is given by*

$$x(t) = \int_a^t \tau^{\rho-1} \phi_\alpha \left(\frac{t^\rho - \tau^\rho}{\rho} \right) B(\tau) \mathrm{d}\tau,$$

where $\phi_\alpha(t) = \sum_{k=0}^{\infty} \frac{A^k t^{\alpha(k+1)-1}}{\Gamma(\alpha(k+1))}$.

Proof. From Theorem 3 we know that problem (10) has a unique solution $x \in I^{\alpha,\rho}_{a+}(L^p)$ and that this solution is given by

$$x(t) = I^{\alpha,\rho}_{a+}\varphi_*(t),\ t \in [a,b],\ a.e.,$$

where $\varphi_* \in L^p([a,b],\mathbb{R})$ is a unique fixed point of the operator

$$S:\ L^p([a,b],\mathbb{R})\ \to\ L^p([a,b],\mathbb{R})$$

$$S\varphi_*(t) = AI^{\alpha,\rho}_{a+}\varphi_*(t) + B(t) = A\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t \frac{\tau^{\rho-1}}{(t^\rho-\tau^\rho)^{1-\alpha}}\varphi_*(\tau)\mathrm{d}\tau + B(t).$$

Therefore, $\varphi_* = S(\varphi_*)$ and

$$\begin{aligned}
x(t) &= I^{\alpha,\rho}_{a+}\varphi_*(t) = I^{\alpha,\rho}_{a+}S\varphi_*(t) = I^{\alpha,\rho}_{a+}\left(AI^{\alpha,\rho}_{a+}\varphi_*(t) + B(t)\right)\\
&= AI^{2\alpha,\rho}_{a+}\varphi_*(t) + I^{\alpha,\rho}_{a+}B(t) = AI^{2\alpha,\rho}_{a+}\left(AI^{\alpha,\rho}_{a+}\varphi_*(t) + B(t)\right) + I^{\alpha,\rho}_{a+}B(t)\\
&= A^2I^{3\alpha,\rho}_{a+}\varphi_*(t) + AI^{2\alpha,\rho}_{a+}B(t) + I^{\alpha,\rho}_{a+}B(t)\\
&= A^2I^{3\alpha,\rho}_{a+}\left(AI^{\alpha,\rho}_{a+}\varphi_*(t) + B(t)\right) + AI^{2\alpha,\rho}_{a+}B(t) + I^{\alpha,\rho}_{a+}B(t)\\
&= A^3I^{4\alpha,\rho}_{a+}\varphi_*(t) + A^2I^{3\alpha,\rho}_{a+}B(t) + AI^{2\alpha,\rho}_{a+}B(t) + I^{\alpha,\rho}_{a+}B(t),\ t \in [a,b].
\end{aligned}$$

One can easily check, applying mathematical induction principle, that

$$x(t) = R_m(t) + \sum_{i=0}^{m-1} A^i I^{(i+1)\alpha,\rho}_{a+}B(t),$$

for any $m \in \mathbb{N}$ and $t \in [a,b]$, where

$$R_m(t) = A^m I^{(m+1)\alpha,\rho}_{a+}\varphi_*(t).$$

Note that, for sufficiently large $m \in \mathbb{N}$, we have

$$\begin{aligned}
|R_m(t)| &\le |A|^m\,\frac{\rho^{1-(m+1)\alpha}}{\Gamma((m+1)\alpha)}\int_a^t \frac{\tau^{\rho-1}}{(t^\rho-\tau^\rho)^{1-(m+1)\alpha}}\,|\varphi_*(\tau)|\,\mathrm{d}\tau\\
&\le |A|^m\,\frac{\rho^{1-(m+1)\alpha}}{\Gamma((m+1)\alpha)}\int_a^b \frac{\tau^{\rho-1}}{(b^\rho-\tau^\rho)^{1-(m+1)\alpha}}\,|\varphi_*(\tau)|\,\mathrm{d}\tau\\
&\le |A|^m\,\frac{\rho^{1-(m+1)\alpha}}{\Gamma((m+1)\alpha)}(b^\rho-a^\rho)^{(m+1)\alpha-1}\int_a^b \tau^{\rho-1}\,|\varphi_*(\tau)|\,\mathrm{d}\tau.
\end{aligned}$$

Now, using the Hőlder inequality and the Gauss-Legendre formula

$$\Gamma(mz) = \frac{m^{mz-1/2}}{(2\pi)^{\frac{m-1}{2}}}\prod_{k=0}^{m-1}\Gamma\left(z+\frac{k}{m}\right),$$

we obtain

$$|R_m(t)| \leq \frac{|A|^m}{\Gamma((m+1)\alpha)} \left(\frac{b^\rho - a^\rho}{\rho}\right)^{(m+1)\alpha-1} \left(\frac{b^{(\rho-1)q+1} - a^{(\rho-1)q+1}}{(\rho-1)q+1}\right)^{\frac{1}{q}} \|\varphi_*\|_{L^p}$$

$$\leq \frac{|A|^m}{\Gamma((m+1)\alpha)} \max\left\{\left(\frac{b^\rho - a^\rho}{\rho}\right)^m, 1\right\} \left(\frac{b^{(\rho-1)q+1} - a^{(\rho-1)q+1}}{(\rho-1)q+1}\right)^{\frac{1}{q}} \|\varphi_*\|_{L^p}$$

$$\leq \frac{|A|^m (2\pi)^{\frac{m}{2}} \|\varphi_*\|_{L^p}}{(m+1)^{(m+1)\alpha-\frac{1}{2}} \prod_{k=0}^{m} \Gamma\left(\alpha + \frac{k}{m+1}\right)} \max\left\{\frac{b^\rho - a^\rho}{\rho}, 1\right\}^m \left(\frac{b^{(\rho-1)q+1} - a^{(\rho-1)q+1}}{(\rho-1)q+1}\right)^{\frac{1}{q}}$$

$$\leq \frac{1}{E}\left(\frac{|A| \cdot D\sqrt{2\pi}}{E}\right)^m \|\varphi_*\|_{L^p} \left(\frac{b^{(\rho-1)q+1} - a^{(\rho-1)q+1}}{(\rho-1)q+1}\right)^{\frac{1}{q}} \frac{1}{(m+1)^{(m+1)\alpha-\frac{1}{2}}},$$

where $D = \max\left\{\frac{b^\rho - a^\rho}{\rho}, 1\right\}$, E is the minimal value of the function Γ on the interval $(0, \infty)$. In an elementary way, we check that

$$\left(\frac{|A| \cdot D\sqrt{2\pi}}{E}\right)^m \frac{1}{(m+1)^{(m+1)\alpha-\frac{1}{2}}} \to 0, \text{ as } m \to \infty.$$

This means that

$$R_m(t) \to 0, \text{ as } m \to \infty, t \in [a,b].$$

Consequently,

$$x(t) = \sum_{k=0}^{\infty} A^k I_{a+}^{(k+1)\alpha,\rho} B(t) = \sum_{k=0}^{\infty} A^k \frac{\rho^{1-\alpha(k+1)}}{\Gamma(\alpha(k+1))} \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha(k+1)}} B(\tau) \mathrm{d}\tau$$

$$= \int_a^t \sum_{k=0}^{\infty} A^k \frac{\rho^{1-\alpha(k+1)}}{\Gamma(\alpha(k+1))} \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha(k+1)}} B(\tau) \mathrm{d}\tau$$

which completes the proof. □

Now, we consider the non–homogeneous linear Cauchy problem

$$\begin{cases} D_{a+}^{\alpha,\rho} y(t) = Ay(t) + B(t) \\ I_{a+}^{1-\alpha,\rho} y(a) = c, \end{cases} \tag{11}$$

where $c \in \mathbb{R}$, $B : [a,b] \longmapsto \mathbb{R}$, $B \in L^p([a,b], \mathbb{R})$, and $A \in \mathbb{R}$. The following theorem gives a formula for the solution to problem (11).

Theorem 6. *The unique solution to problem* (11) *is given by*

$$y(t) = \phi_\alpha\left(\frac{t^\rho - a^\rho}{\rho}\right) c + \int_a^t \tau^{\rho-1} \phi_\alpha\left(\frac{t^\rho - \tau^\rho}{\rho}\right) B(\tau) \mathrm{d}\tau,$$

where $\phi_\alpha(t) = \sum_{k=0}^{\infty} \frac{A^k t^{\alpha(k+1)-1}}{\Gamma(\alpha(k+1))}$.

Proof. Note that, by Theorem 4, there exists $y \in I^{\alpha,\rho}_{a+}(L^p)+\left\{\frac{d}{(t^\rho-a^\rho)^{1-\alpha}};\ d\in\mathbb{R}\right\}$ satisfying (11). Suppose that $x \in I^{\alpha;\rho}_{a+}(L^p)$ satisfies problem

$$\begin{cases} D^{\alpha,\rho}_{a+}x(t) = Ax(t) + \frac{Ac}{\Gamma(\alpha)}\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{\alpha-1} + B(t) \\ I^{1-\alpha,\rho}_{a+}x(a) = 0. \end{cases} \tag{12}$$

Then, the function

$$y(t) = \frac{c}{\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha-1} + x(t)$$

satisfies problem (11). Now, applying Theorem 5 to (12), we have

$$y(t) = \frac{c}{\Gamma(\alpha)}\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{\alpha-1} + \int_a^t \tau^{\rho-1}\phi_\alpha\Big(\frac{t^\rho-\tau^\rho}{\rho}\Big)\bigg(\frac{Ac}{\Gamma(\alpha)}\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{\alpha-1} + B(\tau)\bigg)\mathrm{d}\tau,$$

for $t \in [a,b]$. Note that, performing the substitution $\tau^\rho = a^\rho + u(t^\rho - a^\rho)$ and using the definition and properties of the beta function, we obtain

$$\begin{aligned}
&\int_a^t \tau^{\rho-1}\phi_\alpha\Big(\frac{t^\rho-\tau^\rho}{\rho}\Big)\frac{Ac}{\Gamma(\alpha)}\left(\frac{\tau^\rho - a^\rho}{\rho}\right)^{\alpha-1}\mathrm{d}\tau \\
&= \int_a^t \tau^{\rho-1}\sum_{k=0}^{\infty}\frac{A^k}{\Gamma(\alpha(k+1))}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\alpha(k+1)-1}\frac{Ac}{\Gamma(\alpha)}\left(\frac{\tau^\rho - a^\rho}{\rho}\right)^{\alpha-1}\mathrm{d}\tau \\
&= \sum_{k=1}^{\infty}\frac{A^k c}{\Gamma(\alpha)\Gamma(\alpha k)}\int_a^t \tau^{\rho-1}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\alpha k-1}\left(\frac{\tau^\rho - a^\rho}{\rho}\right)^{\alpha-1}\mathrm{d}\tau \\
&= \sum_{k=1}^{\infty}\frac{A^k c}{\Gamma(\alpha)\Gamma(\alpha k)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha(k+1)-1}\int_0^1 (1-u)^{\alpha k-1}u^{\alpha-1}\mathrm{d}u \\
&= \sum_{k=1}^{\infty}\frac{A^k c}{\Gamma(\alpha)\Gamma(\alpha k)}\Big(\frac{t^\rho-a^\rho}{\rho}\Big)^{\alpha(k+1)-1}B(\alpha,\alpha k) = \sum_{k=1}^{\infty}\frac{A^k c}{\Gamma(\alpha(k+1))}\Big(\frac{t^\rho-a^\rho}{\rho}\Big)^{\alpha(k+1)-1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
y(t) &= \sum_{k=0}^{\infty}\frac{A^k c}{\Gamma(\alpha(k+1))}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha(k+1)-1} + \int_a^t \tau^{\rho-1}\phi_\alpha\left(\frac{t^\rho-\tau^\rho}{\rho}\right)B(\tau)\mathrm{d}\tau \\
&= \phi_\alpha\left(\frac{t^\rho-a^\rho}{\rho}\right)c + \int_a^t \tau^{\rho-1}\phi_\alpha\left(\frac{t^\rho-\tau^\rho}{\rho}\right)B(\tau)\mathrm{d}\tau.
\end{aligned}$$

Acknowledgements. Research supported by the University of Białystok grant BST-137/2015 (B. Łupińska, E. Schmeidel), and by the Warsaw School of Economics grant KAE/S15/35/15 (T. Odzijewicz).

REFERENCES

1. R. Almeida: *Variational Problems Involving a Caputo-Type Fractional Derivative.* J. Optim. Theory Appl., (in press).
2. R. Almeida, N. R. O. Bastos: *An approximation formula for the Katugampola integral.* J. Math. Anal., **7** (2016), 23–30.
3. R. Almeida, A. B. Malinowska, T. Odzijewicz: *Fractional differential equations with dependence on the Caputo–Katugampola derivative.* J. Comput. Nonlinear Dynam., (in press).
4. A. Carpinteri, F. Mainardi: *Fractals and fractional calculus in continuum mechanics.* CISM Courses and Lectures, 378, Springer, Vienna, 1997.
5. E. Gerolymatou, I. Vardoulakis, R. Hilfer: *Modelling infiltration by means of a nonlinear fractional diffusion model.* J. Phys. D: Appl. Phys., **39** (2006), 4104–4110.
6. R. Hilfer: *Applications of fractional calculus in physics.* World Sci. Publishing, River Edge, NJ, 2000.
7. D. Idczak, R. Kamocki: *On the existence and uniqueness and formula for the solution of R-L fractional Cauchy problem in* $\mathbb{R}^n$. Fract. Calc. Appl. Anal., **14** (2011), 538–553.
8. U. N. Katugampola: *New approach to a generalized fractional integral.* Appl. Math. Comput., **218** (2011), 860–865.
9. U. N. Katugampola: *A new approach to generalized fractional derivatives.* Bull. Math. Anal. App., **6** (2014), 1–15.
10. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo: *Theory and applications of fractional differential equations.* North-Holland Mathematics Studies, 204, Elsevier, Amsterdam, 2006.
11. M. Klimek: *On solutions of linear fractional differential equations of a variational type.* The Publishing Office of Czestochowa University of Technology, Czestochowa, 2009.
12. B. Łupińska, T. Odzijewicz, E. Schmeidel: *Some properties of generalized fractional integrals and derivatives.* Proceedings of the International Conference of Numerical Analysis and Applied Mathematics 2016 (ICNAAM-2016) Book Series: AIP Conference Proceedings (accepted).
13. J. T. Machado, V. Kiryakova, F. Mainardi: *Recent history of fractional calculus.* Commun. Nonlinear Sci. Numer. Simul., **16** (2011), 1140–1153.
14. F. Mainardi: *Fractional calculus and waves in linear viscoelasticity.* Imp. Coll. Press, London, 2010.
15. A. B. Malinowska, D. F. M. Torres: *Introduction to the fractional calculus of variations.* Imp. Coll. Press, London, 2012.
16. R. Metzler, J. Klafter: *The random walk's guide to anomalous diffusion: A fractional dynamics approach.* Phys. Rep., **339** (2000), 1–77.
17. K. B. Oldham, J. Spanier: *The replacement of Fick's laws by a formulation involving semidifferentiation.* J. Electroanal. Chem., **26** (1970), 331–341.
18. J. Li, M. Ostoja-Starzewski: *Micropolar continuum mechanics of fractal media.* Internat. J. Engrg. Sci. **49** (2011), 1302–1310.

19. G. M. Zaslavsky: *Hamiltonian chaos and fractional dynamics*, reprint of the 2005 original, Oxford Univ. Press, Oxford, 2008.

20. G. M. Zaslavsky, M. A. Edelman: *Fractional kinetics: From pseudochaotic dynamics to Maxwell's demon.* Phys. D **193** (2007), 128–147.

21. A. Zoia, M. C. Néel, A. Cortis: *Continuous-time random-walk model of transport in variably saturated heterogeneous porous media.* Phys. Rev. E, **81** (2010).

22. A. Zoia, M. C. Néel, M. Joelson: *Mass transport subject to time-dependent flow with nonuniform sorption in porous media.* Phys. Rev. E, **80** (2009).

(Received March 19, 2016)
(Revised September 26, 2016)

Faculty of Mathematics and Computer Science,
University of Bialystok
Poland
E-mail: bpietruczuk@math.uwb.edu.pl

Department of Mathematics and Mathematical Economics,
Warsaw School of Economics
Poland
E-mail: tatiana.odzijewicz@sgh.waw.pl

Faculty of Mathematics and Computer Science,
University of Bialystok
Poland
E-mail: eschmeidel@math.uwb.edu.pl