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THE STRONG METRIC DIMENSION OF SOME GENERALIZED PETERSEN GRAPHS

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In this paper the strong metric dimension of generalized Petersen graphs $GP(n, 2)$ is considered. The exact value is determined for the cases $n = 4k$ and $n = 4k + 2$, while for $n = 4k + 1$ an upper bound of the strong metric dimension is presented.

1. INTRODUCTION

The strong metric dimension problem was introduced by Sebo and Tannier [**13**]. This problem is defined in the following way. Given a simple connected undirected graph $G = (V,E)$, where $V = \{1, 2, \ldots, n\}$, $|E| = m$ and $d(u, v)$ denotes the distance between vertices u and v, i.e. the length of a shortest $u - v$ path. A vertex w strongly resolves two vertices u and v if u belongs to a shortest $v - w$ path or v belongs to a shortest $u - w$ path. A vertex set S of G is a *strong resolving set* of G if every two distinct vertices of G are strongly resolved by some vertex of S. A *strong metric basis* of G is a strong resolving set of the minimum cardinality. The *strong metric dimension* of G, denoted by $sdim(G)$, is defined as the cardinality of a strong metric basis. Now, the strong metric dimension problem is defined as the problem of finding the strong metric dimension of a graph G.

Example 1. Consider the Petersen graph G given on Figure 1. It is easy to see that set $S = \{u_0, u_1, u_2, u_3, v_0, v_1, v_2, v_3\}$ is a strong resolving set, i.e. each pair of vertices in G is strongly resolved by a vertex from S. Since any pair with at least

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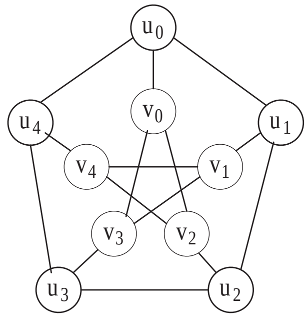

Figure 1: Petersen graph G

one vertex in S is strongly resolved by that vertex, the only interesting case is pair u_4, v_4. This pair is strongly resolved e.g. by $u_0 \in S$ since the shortest path u_0, u_4, v_4 contains u_4. In Section 2 it will be demonstrated that S is a strong resolving set with the minimum cardinality and, therefore, $sdim(G) = 8$.

The strong metric dimension has many interesting theoretical properties. If S is a strong resolving set of G, then the matrix of distances from all vertices from V to all vertices from S uniquely determines the graph G [**13**].

The strong metric dimension problem is NP-hard in general case [**10**]. Nevertheless, for some classes of graphs the strong metric dimension problem can be solved in polynomial time. For example, in [**8**] an algorithm for finding the strong metric dimension of distance hereditary graphs with $O(|V| \cdot |E|)$ complexity is presented.

In [**4**] an integer linear programming (ILP) formulation of the strong metric dimension problem was proposed. Let variable y_i determine whether vertex i belongs to a strong resolving set S or not, i.e. $y_i = \begin{cases} 1, & i \in S \\ 0, & i \notin S \end{cases}$. Now, the ILP model of the strong metric dimension problem is given by (1).

$$
\begin{aligned}
&\min \sum_{i=1}^{n} y_i && \text{subject to:} \\
(1) \qquad &\sum_{i=1}^{n} A_{(u,v),i} \cdot y_i \geq 1 && 1 \leq u < v \leq n \\
&y_i \in \{0,1\} && 1 \leq i \leq n
\end{aligned}
$$

where $A_{(u,v),i} = \begin{cases} 1, & d(u,i) = d(u,v) + d(v,i) \\ 1, & d(v,i) = d(v,u) + d(u,i) \\ 0, & otherwise \end{cases}$.

This ILP formulation will be used in Section 2 for finding the strong metric dimension of generalized Petersen graphs in individual cases, when dimensions are small. The following definition of mutually maximally distant vertices and two properties from the literature will also be used in Section 2.

Definition 1. ([**4**]) A pair of vertices $u, v \in V$, $u \neq v$, is *mutually maximally distant* if and only if

1. $d(w, v) \leq d(u, v)$ for each w such that $\{w, u\} \in E$ and
2. $d(u, w) \leq d(u, v)$ for each w such that $\{v, w\} \in E$.

Property 1. *([**4**]) If $S \subset V$ is a strong resolving set of graph G, then, for every two maximally distant vertices $u, v \in V$, it must be $u \in S$ or $v \in S$.*

Let $Diam(G)$ denote the diameter of the graph G, i.e. the maximal distance between two vertices in G.

Property 2. *([**4**]) If $S \subset V$ is a strong resolving set of the graph G, then, for every two vertices $u, v \in V$ such that $d(u, v) = Diam(G)$, it must be $u \in S$ or $v \in S$.*

A survey paper [**5**] contains results on the strong metric dimension of graphs up to mid 2013. Since then several interesting theoretical papers related to the properties of the strong metric dimension have been published (see e.g. [**6, 7, 12, 14**]).

This paper considers the strong metric dimension of a special class of graphs, so called generalized Petersen graphs. The generalized Petersen graph $GP(n, m)$ ($n \geq 3$; $1 \leq m < n/2$) has $2n$ vertices and $3n$ edges, with the vertex set $V = \{u_i, v_i \mid 0 \leq i \leq n-1\}$ and the edge set $E = \{\{u_i, u_{i+1}\}, \{u_i, v_i\}, \{v_i, v_{i+m}\} \mid 0 \leq i \leq n-1\}$, where vertex indices are taken modulo n. The Petersen graph from Figure 1 can be considered as $GP(5, 2)$.

The generalized Petersen graphs were first studied by Coxeter [**1**]. Various properties of $GP(n, m)$ have been recently theoretically investigated in the following areas: metric dimension [**9**], decycling number [**3**], component connectivity [**2**] and acyclic 3-coloring [**15**].

In the case when $k = 1$ it is easy to see that $GP(n, 1) \equiv C_n \Box P_2$, where $\Box$ is the Cartesian product of graphs, where C_n is the cycle on n vertices and P_2 is the path with 2 vertices. Using the following result from [**11**]: $sdim(C_n \Box P_r) = n$, for $r \geq 2$, it follows that $sdim(GP(n, 1)) = n$.

2. THE STRONG METRIC DIMENSION OF $GP(n, 2)$

In this section we consider the strong metric dimension of the generalized Petersen graph $GP(n, 2)$. The exact value is determined for the cases $n = 4k$ and $n = 4k + 2$, while for $n = 4k + 1$ an upper bound of the strong metric dimension is presented. In order to prove that the sets defined in Lemma 1, Lemma 3 and Lemma 5 are strong resolving we used shortest paths given in Tables 1-3, which are organized as follows:

- The first column named "*case*" contains the case number.
- The next two columns named "*vertices*" and "*res. by*" contain a pair of vertices and a vertex which strongly resolves them.
- The last two columns named "*condition*" and "*shortest path*" contain the condition under which the vertex in column three strongly resolves the pair in column two, and the corresponding shortest path, respectively.

Lemma 1. *The set* $S = \{u_{2i}, v_i | i = 0, 1, \ldots, 2k\}$ *is a strong resolving set of* $GP(4k+2, 2)$ *for* $k \geq 2$.

Proof. Let us consider pairs of vertices such that neither vertex is in S. There are three possible cases:

Case 1. (u_i, u_j), i, j are odd. Without loss of generality we may assume that $i < j$. If $j - i \leq 2k$, according to Table 1, vertices u_i and u_j are strongly resolved by vertex u_{i-1}. The shortest paths corresponding to the subcases $j - i = 2$ and $4 \leq j - i \leq 2k$ are also given in Table 1. As $i - 1$ is even, then $u_{i-1} \in S$. If $j - i \geq 2k + 2$, then the pair (u_i, u_j) can be represented as the pair $(u_{i'}, u_{j'})$, where $i' = j$ and $j' = i + 4k + 2$. Since $j' - i' \leq 2k$, the situation is reduced to the previous one.

Case 2. (v_i, v_j), $i, j \geq 2k + 1$. Without loss of generality we may assume that $i < j$. If $j - i$ is even, v_i and v_j are strongly resolved by u_j or u_{j+1}. Namely, when j is even $u_j \in S$, while $u_{j+1} \in S$ for odd j. If $j - i$ is odd, then v_i and v_j are strongly resolved by $v_{2k} \in S$ if i is even, and $v_{2k-1} \in S$ if i is odd, and hence u_i and u_j are strongly resolved by S.

Case 3. (u_i, v_j), i odd, $j \geq 2k + 1$. There are seven subcases, characterized by conditions presented in Table 1. Vertices which strongly resolve pair (u_i, v_j) listed in Table 1 belong to the set S. Indeed, vertices v_{2k-1} and v_{2k} belong to S by assumption, while $u_{i-1}, u_{i+1}, u_{i+5}$ belong to S since i is odd. Finally, $u_j \in S$ since j is even by condition.

□

Lemma 2. *If* S *is a strong resolving set of* $GP(4k+2, 2)$, *then* $|S| \geq 4k + 2$, *for any* $k \geq 2$.

Proof. Since $d(u_i, u_{i+2k+1}) = d(v_i, v_{i+2k+1}) = k + 3 = Diam(GP(4k+2, 2)), i = 0, 1, ..., 2k$, from Property 2, at least $2k + 1$ u-vertices and $2k + 1$ v-vertices belong to S. Therefore, $|S| \geq 4k + 2$. □

Table 1: Shortest paths in Lemma 1

Case	*vertices*	*res. by*	*condition*	*shortest path*
1	(u_i, u_j) ,	u_{i-1}	$j-i=2$	$u_{i-1}, u_i, u_{i+1}, u_{i+2} = u_j$
	i,j odd	u_{i-1}	$4 \leq j-i \leq 2k$	$u_{i-1}, u_i, v_i, v_{i+2}, ..., v_j, u_j$
2	(v_i, v_j),	u_j or u_{j+1}	$j-i$ even	$v_i, v_{i+2}, ..., v_j, u_j, u_{j+1}$
	$i, j \geq 2k+1$	v_{2k}	$j-i$ odd, i even	$v_{2k}, v_{2k+2}, ..., v_i, u_i, u_{i+1}, v_{i+1}, v_{i+3}, ..., v_j$
		v_{2k-1}	$j-i$ odd, i odd	$v_{2k-1}, v_{2k+1}, ..., v_i, u_i, u_{i+1}, v_{i+1}, v_{i+3}, ..., v_j$
3	(u_i, v_j),	u_{i+1}	$i \geq j$, j odd	$v_j, v_{j+2}, ..., v_j, v_{j+2}, ..., v_i, u_i, u_{i+1}$
	i odd, $j \geq 2k+1$	v_{2k}	$i > j$, j even	$v_{2k}, v_{2k+2}, ..., v_j, u_j, u_{j+1}, v_{j+1}, v_{j+3}, ..., v_i, u_i$
		u_{i-1}	j odd,$0 < j-i \leq 2k$	$u_{i-1}, u_i, v_i, v_{i+2}, ..., v_j$
		u_{i+1}	j odd,$j-i \geq 2k+2$	$v_j, v_{j+2}, ..., v_i, u_i, u_{i+1}$
		v_{2k-1}	$j-i=1$	$v_{2k-1}, v_{2k+1}, ..., v_i, u_i, u_{i+1}, v_{i+1} = v_j$
		$u_{j+2} = u_{i+5}$	$j-i=3$	$u_i, u_{i+1}, v_{i+1}, v_{i+3} = v_j, v_{j+2}, u_{j+2}$
		u_j	j even, $5 \leq j-i \leq 2k+1$	$u_i, u_{i+1}, v_{i+1}, v_{i+3}, ..., v_j, u_j$
		u_j	j even, $j-i \geq 2k+3$	$u_j, v_j, v_{j+2}, ..., v_{i-1}, u_{i-1}, u_i$

Table 2: Shortest paths in Lemma 3

Case	*vertices*	*res. by*	*condition*	*shortest path*
1	(u_i, u_j),	$u_{j+1} = u_{i+3}$	$j - i = 2$	$u_i, u_{i+1}, u_{i+2} = u_j, u_{j+1}$
	$i, j \geq 2k, i, j$ even	u_{j+1}	$j - i \geq 4$	$u_i, v_i, v_{i+2}, ..., v_j, u_j, u_{j+1}$
2	(v_i, v_j),	u_{j+1}	$j - i \leq 2k - 2$	$v_i, v_{i+2}, ..., v_j, u_j, u_{j+1}$
	i, j even	u_i and u_{i+2k}	$j - i = 2k$	$u_i, v_i, v_{i+2}, ..., v_{i+2k} = v_j, u_{i+2k} = u_j$
3	(u_i, v_j),	u_{i-1}	$0 \leq j - i \leq 2k - 2$	$u_{i-1}, u_i, v_i, v_{i+2}, ..., v_j$
	i, j even,	$u_{i-2k} = u_j$	$j - i = -2k$	$u_i, v_i, v_{i-2}, ..., v_{i-2k} = v_j, u_{i-2k} = u_j$
	$i \geq 2k$	u_{j-1}	$-2k < j - i < -2$	$u_{j-1}, u_j, v_j, v_{j+2}, ..., v_i, u_i$
	$i \geq 2k$	u_{i+1}	$j - i = -2$	$v_j, v_{j+2} = v_i, u_i, u_{i+1}$

The strong metric dimension of $GP(4k + 2, 2)$ is given in Theorem 1.

Theorem 1. *For all k it holds that $sdim(GP(4k + 2, 2)) = 4k + 2$.*

Proof. It follows directly from Lemmas 1 and 2 that $sdim(GP(4k+2, 2)) = 4k+2$ for $k \geq 2$. For $k = 1$, using CPLEX solver on ILP formulation (1), we have proved that the set S from Lemma 1 is also a strong metric basis of $GP(6, 2)$, i.e. $sdim(GP(6, 2)) = 6$. □

The strong metric dimension of $GP(4k, 2)$ will be determined using Lemmas 3 and 4.

Lemma 3. *The set $S = \{u_i | i = 0, 1, ..., 2k - 1\} \cup \{u_{2k+2i+1} | i = 0, 1, ...k - 1\} \cup \{v_{2i+1} | i = 0, 1, ...2k - 1\}$ is a strong resolving set of $GP(4k, 2)$ for $k \geq 2$.*

Proof. Since S contains u_i, $i = 1, ..., 2k - 1$ and all u_i and v_i for odd i, we need to consider only three possible cases:

Case 1. (u_i, u_j), $i, j \geq 2k$ are even. Without loss of generality we may assume that $i < j$. Vertices u_i and u_j are strongly resolved by vertex u_{j+1} (see Table 2). The shortest paths corresponding to subcases $j - i = 2$ and $j - i \geq 4$ are given in Table 2. As $j + 1$ is odd, then $u_{j+1} \in S$.

Case 2. (v_i, v_j), i, j are even. Without loss of generality we may assume that $i < j$. If $j - i \leq 2k - 2$, vertices v_i and v_j are strongly resolved by $u_{j+1} \in S$. If $j - i \geq 2k + 2$, then the pair (v_i, v_j) can be represented as the pair $(v_{i'}, v_{j'})$, where $i' = j$ and $j' = i + 4k$. Since $j' - i' \leq 2k - 2$, the situation is reduced to the previous one. If $j - i = 2k$, vertices v_i and $v_j = v_{i+2k}$ are strongly resolved by both u_i and $u_j = u_{i+2k}$. Since S contains $u_0, ..., u_{2k-1}$ it follows that u_i or u_{i+2k} belongs to S and hence v_i and v_j are strongly resolved by S.

Case 3. (u_i, v_j), i, j even, $i \geq 2k$. Let us assume first that $i \leq j$. If $j - i \leq 2k - 2$, vertices u_i and v_j are strongly resolved by vertex u_{i-1}. As $i - 1$ is odd it follows that $u_{i-1} \in S$ and u_i and v_j are strongly resolved by S. Assume now that $i > j$. If $j - i = -2k$ then vertices u_i and v_j=v_{i-2k} are strongly resolved by vertex u_j=u_{i-2k}. Since S contains $u_0, ..., u_{2k-1}$ and $i \geq 2k$, it follows that u_{i-2k} belongs to S and hence u_i and v_j are strongly resolved by S. If

$-2k < j - i < 0$ then the pair (u_i, v_j) is strongly resolved by u_{j-1}. Since $j - 1$ is odd, then $u_{j-1} \in S$. These three subcases cover all possible values for i and j having in mind that vertex indices are taken modulo n.

□

Lemma 4. *If $k \geq 10$ and S is a strong resolving set of $GP(4k, 2)$, then $|S| \geq 5k$.*

Proof. Let us note that $d(v_i, v_{i+2k-1}) = k+2 = Diam(GP(4k, 2)), i = 0, 1, ..., 4k-1$. If we suppose that S contains less than $2k$ v-vertices, since we have $4k$ pairs $(v_i, v_{i+2k-1}), i = 0, ..., 4k-1$, and each vertex appears exactly twice, there exists some pair $(v_i, v_{i+2k-1}), v_i \notin S, v_{i+2k-1} \notin S$. This is in contradiction with Property 2. Therefore, S contains at least $2k$ v-vertices.

Considering u-vertices, we have two cases:

Case 1. If there exist $u_{2i} \notin S$ and $u_{2j+1} \notin S$, then, as pairs $\{u_{2i}, u_{2i+2l-1}\}, l = 3, 4, ..., 2k-2$ and $\{u_{2j+1}, u_{2j+2l}\}, l = 3, 4, ..., 2k-2$ are mutually maximally distant, at most 8 additional vertices are not in S: $u_{2i-3}, u_{2i-1}, u_{2i+1}, u_{2i+3}, u_{2j-2}, u_{2j}, u_{2j+2}, u_{2j+4}$. Consequently, at most 10 vertices, where $10 \leq k$, are not in S.

Case 2. Indices of u-vertices which are not in S are all either even or odd. Without loss of generality we may assume that all these indices are even. Since $d(u_{2i}, u_{2i+2k}) = k+2 = Diam(GP(4k, 2)), i = 0, 1, ..., k-1$, according to Property 2, we have k pairs $(u_{2i}, u_{2i+2k}), i = 0, ..., k-1$, with at most one vertex not in S. Therefore, at most k u-vertices are not in S.

In both cases we have proved that at most k u-vertices are not in S, so at least $3k$ u-vertices are in S. Since we have already proved that at least $2k$ v-vertices should be in S, it follows that $|S| \geq 5k$. □

The strong metric dimension of $GP(4k, 2)$ is given in Theorem 2.

Theorem 2. *For all $k \geq 5$ it holds $sdim(GP(4k, 2)) = 5k$.*

Proof. Lemma 3 and Lemma 4 imply that $S = \{u_i | i = 0, 1, ..., 2k-1\} \cup \{u_{2k+2i+1} | i = 0, 1, ...k-1\} \cup \{v_{2i+1} | i = 0, 1, ...2k-1\}$ is a strong metric basis of $GP(4k, 2)$ for $k \geq 10$. Using CPLEX solver on ILP formulation (1), we have proved that the set S from Lemma 3 is a strong metric basis of $GP(4k, 2)$ for $k \in \{5, 6, 7, 8, 9\}$. □

In the case when $n = 4k + 1$, an upper bound of the strong metric dimension of $GP(4k + 1, 2)$ will be determined as a corollary of the following lemma.

Lemma 5. *The set $S = \{u_{2i+1} | i = 0, 1, ..., k-1\} \cup \{u_{2k+i} | i = 0, 1, ..., 2k\} \cup \{v_i | i = 0, 1, ...2k+3\}$ is a strong resolving set of $GP(4k + 1, 2)$ for $k \geq 3$.*

Proof. As in Lemma 1, there are three possible cases:

Table 3: Shortest paths in Lemma 5

Case	*vertices*	*res. by*	*condition*	*shortest path*
1	(u_{2i}, u_{2j}),	u_{2j+1}	$j - i \geq 2$	$u_{2i}, v_{2i}, v_{2i+2}, \ldots, v_{2j}, u_{2j}, u_{2j+1}$
	$0 \leq i, j \leq k-1$	u_{2j+1}	$j - i = 1$	$u_{2i}, u_{2i+1}, u_{2i+2} = u_{2j}, u_{2i+3} = u_{2j+1}$
2	(v_i, v_j),	v_1	i, j even	$v_i, v_{i+2}, \ldots, v_j, v_{j+2}, \ldots, v_1$
	$2k+4 \leq i, j \leq 4k$	v_0	i, j odd	$v_i, v_{i+2}, \ldots, v_j, v_{j+2}, \ldots, v_0$
		v_0	i even, j odd	$v_i, u_i, u_{i+1}, v_{i+1}, v_{i+3}, \ldots, v_j, v_{j+2}, \ldots, v_0$
		v_1	i odd, j even	$v_i, u_i, u_{i+1}, v_{i+1}, v_{i+3}, \ldots, v_j, v_{j+2}, \ldots, v_1$
3	(u_{2i}, v_j),	u_j	j even, $j - 2i \leq 2k$	$u_{2i}, v_{2i}, v_{2i+2}, \ldots, v_j, u_j$
	$0 \leq i \leq k-1$,	u_j	j even, $j - 2i \geq 2k+2$	$u_j, v_j, v_{j+2}, \ldots, v_{2i-1}, u_{2i-1}, u_{2i}$
	$2k+4 \leq j \leq 4k$	u_j	j odd, $j - 2i \leq 2k-1$	$u_{2i}, u_{2i+1}, v_{2i+1}, v_{2i+3}, \ldots, v_j, u_j$
		u_j	j odd, $j - 2i \geq 2k+1$	$u_j, v_j, v_{j+2}, \ldots, v_{2i}, u_{2i}$

Case 1. (u_{2i}, u_{2j}), $0 \leq i, j \leq k-1$. Without loss of generality we may assume that $i < j$. Vertices u_{2i} and u_{2j} are strongly resolved by vertex $u_{2j+1} \in S$. The shortest paths corresponding to subcases $j - i \geq 2$ and $j - i = 1$ are given in Table 3.

Case 2. (v_i, v_j), $2k+4 \leq i, j \leq 4k$. Without loss of generality we may assume that $i < j$. If j is even, vertices v_i and v_j are strongly resolved by $v_1 \in S$, while if j is odd, they are strongly resolved by $v_0 \in S$. The details about shortest paths can be seen in Table 3.

Case 3. (u_{2i}, v_j), $0 \leq i \leq k-1$, $2k+4 \leq j \leq 4k$. Vertices u_{2i} and v_j are strongly resolved by u_j, and the shortest paths corresponding to four subcases can be seen in Table 3. Since the set S contains u-vertices $u_{2k}, u_{2k+1}, ..., u_{4k}$ and $2k+4 \leq j \leq 4k$ it follows that $u_j \in S$.

□

Corollary 1. *If* $k \geq 3$ *then* $sdim(GP(4k+1, 2)) \leq 5k+5$.

The strong metric bases given in Lemma 1 and Lemma 3 and the strong resolving set given in Lemma 5 hold for $n \geq 20$. The strong metric bases for $n \leq 19$ have been obtained by CPLEX solver on ILP formulation (1). Computational results show that for $n \in \{6, 10, 14, 18\}$ the strong resolving set S from Lemma 1 is a strong metric basis. The strong metric bases for the remaining cases for $n \leq 19$ are given in Table 4.

3. CONCLUSIONS

In this paper we have studied the strong metric dimension of the generalized Petersen graphs $GP(n, 2)$. We have found closed formulas of the strong metric dimensions in the cases $n = 4k$ and $n = 4k+2$, and a tight upper bound of the strong metric dimension for $n = 4k+1$.

The experimental results for remaining cases of $GP(n, 2)$ indicate the following hypotheses: $sdim(GP(4k+1, 2)) = 5k+5$, for $k \geq 5$, $sdim(GP(4k+3, 2)) = 5k+6$ for $k = 5l-2$ and $sdim(GP(4k+3, 2)) = 5k+4$ for $k \neq 5l-2$, where $l \in \mathbb{N}$.

Table 4: Other strong metric bases of $GP(n, 2)$ for $n \leq 19$

n	$sdim(GP(n, 2))$	S
5	8	$\{u_0, u_1, u_2, u_3, v_0, v_1, v_2, v_3\}$
7	9	$\{u_0, u_1, u_2, u_3, v_0, v_1, v_2, v_3, v_4, v_6\}$
8	8	$\{u_4, u_5, u_6, u_7, v_1, v_3, v_5, v_7\}$
9	13	$\{u_2, u_4, u_5, u_6, u_7, u_8, v_0, v_2, v_3, v_4, v_5, v_6, v_7\}$
11	12	$\{u_0, u_1, u_2, u_3, u_4, u_5, v_0, v_1, v_2, v_5, v_6, v_7\}$
12	13	$\{u_0, u_1, u_2, u_3, u_4, u_5, u_6, v_0, v_2, v_4, v_6, v_8, v_{10}\}$
13	17	$\{u_0, u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ $\cup\{v_1, v_3, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{12}\}$
15	20	$\{u_0, u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9\}$ $\cup\{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$
16	19	$\{u_0, u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}\}$ $\cup\{v_0, v_2, v_4, v_6, v_8, v_{10}, v_{12}, v_{14}\}$
17	24	$\{u_0, u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}, u_{11}\}$ $\cup\{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}\}$
19	24	$\{u_0, u_4, u_6, u_7, u_8, u_9, u_{10}, u_{11}, u_{12}, u_{13}, u_{14}, u_{15}, u_{16}, u_{17}\}$ $\cup\{v_0, v_1, v_5, v_6, v_9, v_{10}, v_{11}, v_{14}, v_{15}, v_{16}\}$

Future work could be directed also towards obtaining the strong metric dimension of some other challenging classes of graphs.

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