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FIXED POINT AND COINCIDENCE POINT THEOREMS IN b-METRIC SPACES WITH APPLICATIONS

Adrian Petruşel, Gabriela Petruşel, Jen-Chih Yao*

In this paper, we will consider the coincidence point problem for a pair of single-valued operators satisfying to some contraction and expansion type conditions. Existence, uniqueness and qualitative properties of the solution will be presented. The results are based on some fixed point theorems for nonlinear contractions in complete b-metric spaces. An application illustrates the theoretical results.

1. INTRODUCTION

An extension of the Banach's contraction principle was given, in the framework of b-metric spaces (also called, in some papers, quasi-metric spaces or metric type spaces), by S. Czerwik in [**4**]. For several fixed point results in this framework see [**1**], [**2**], [**7**].

Let (X, d) and (Y, ρ) be two metric spaces and $g, t : X \to Y$ be two operators. The coincidence point problem for t and g means to find $x^* \in X$ such that $t(x^*) = g(x^*)$. We will denote by $CP(g, t)$ the coincidence point set for g and t.

The aim of this paper is to present, in the context of b-metric spaces, two types of coincidence point theorems under some contraction and expansion type conditions. The method is based on the application of some fixed point point theorems of Ran-Reurings type in ordered b-metric spaces. Our coincidence results are in connection with some nice previous theorems given in A. Buică [**3**], J. Garcia Falset, O. Mleşniţe [**5**], O. Mleşniţe [**8**] and I. A. Rus [**16**].

*Corresponding author. Gabriela Petruşel
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2. PRELIMINARIES

Throughout this paper $\mathbb{N}$ stands for the set of natural numbers, while $\mathbb{N}^*$ for the set of natural numbers except 0. By $\mathbb{R}_+$ we will denote the set of real non-negative numbers. We will recall now the definition of a b-metric space.

Definition 2.1. (Bakhtin [**1**], Czerwik [**4**]) Let X be a nonempty set and let $s \geq 1$ be a given real number. A functional $d : X \times X \to \mathbb{R}_+$ is said to be a b-metric if the usual axioms of the metric take place with the following modification of the triangle inequality axiom $d(x, z) \leq s[d(x, y) + d(y, z)]$, for all $x, y, z \in X$. A pair (X, d) with the above properties is called a b-metric space.

Some examples of b-metric spaces are given in [**2**], [**4**], [**7**] and in many other papers.

It is worth to mention that the b-metric structure produces some differences to the classical case of metric spaces: the b-metric on a nonempty set X need not be continuous, open balls in such spaces need not be open sets and so on.
In this context, we notice that a set $Y \subset X$ is said to be closed if for any sequence (x_n) in Y which is convergent to some x, we have that $x \in Y$.

We also mention some continuity concepts. Let (X, d) and (Y, ρ) be two b-metric spaces. Then $f : X \to Y$ is called:

a) continuous on X if for every $x \in X$ and any sequence $(x_n)_{n\in\mathbb{N}}$ in X which converges to x in (X, d), the sequence $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(x)$ in (Y, ρ);

b) with closed graph if for every sequence $(x_n)_{n\in\mathbb{N}}$ in X which converges, with respect to d, to an element x such that the sequence $(f(x_n))_{n\in\mathbb{N}}$ converges to y in (Y, ρ) as $n \to \infty$, we have that $x \in X$ and $y = f(x)$;

c) uniformly continuous on X if for any $\epsilon > 0$ there is $\delta = \delta(\epsilon) > 0$ such that

$$x_1, x_2 \in X \text{ and } d(x_1, x_2) < \delta \text{ implies } \rho(f(x_1), f(x_2)) < \epsilon.$$

d) orbitally continuous on X if $(\forall\, x \in X)(\forall\, \{x_n\} \subseteq O(x, f))$ with $x_n \to y \in X$, $n \to \infty$, implies $f(x_n) \to f(y)$, $n \to \infty$, where $O(x, f) = \{f^n(x) \mid n \in \mathbb{N}\}$ is the orbit of point $x \in X$ with respect to a mapping f.

Notice that any uniformly continuous mapping is continuous and any continuous mapping is with closed graph.

If X is a nonempty set and $f : X \to X$ is a single-valued operator, then we denote $Fix(f) := \{x \in X \mid x = f(x)\}$ the fixed point set for f, by $Graph(f) := \{(x, f(x)) \mid x \in X\}$ the graph of the operator f.

If X, Y are two nonempty sets and $f, g : X \to Y$ are two mappings, then we denote by

$$C(f, g) := \{x \in X \mid f(x) = g(x)\}$$

the coincidence point set for f and g.

3. RAN-REURINGS TYPE FIXED POINT THEOREMS FOR NONLINEAR CONTRACTIONS

We will prove first some fixed point results which are important tools in our coincidence point problem approach.

Our first result is an extension to the case of b-metric spaces of the well known fixed point theorem given by Ran and Reurings and, in the same time, an extension of Czerwik's fixed point theorem for nonlinear contractions to the case of b-metric spaces endowed with a partial order relation.

Recall that a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a comparison function (see [**17**]) if it is increasing and $\varphi^n(t) \to 0$ as $n \to \infty$, for all $t \geq 0$. Several examples of comparison mappings can be found, for example, in [**17**] and [**19**].

Theorem 3.1. *Let X be a nonempty set endowed with a partial order "$\preceq$" and $d : X \times X \to X$ be a complete b-metric with constant $s \geq 1$. Let $f : X \to X$ be an operator which has closed graph with respect to d and is increasing with respect to "$\preceq$". Suppose that there exist a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ and an element $x_0 \in X$ such that:*

(i) $d(f(x), f(y)) \leq \varphi(d(x, y))$, for all $x, y \in X$ with $x \preceq y$;

(ii) $x_0 \preceq f(x_0)$;

(iii) for every $x, y \in X$ there exists $z \in X$ which is comparable to x and y.

Then, f is a Picard operator, i.e., $Fix(f) = \{x^*\}$ and the sequence of successive approximations $(f^n(x))_{n\in\mathbb{N}}$ starting from any point $x \in X$ converges to x^*.

Proof. Let $x_0 \in X$ such that $x_0 \preceq f(x_0)$. Then $x_0 \preceq f^n(x_0)$, for every $n \in \mathbb{N}^*$. Denote $x_n := f^n(x_0)$, $n \in \mathbb{N}^*$. Then we have:

(a) $x_{n+1} = f(x_n)$, $n \in \mathbb{N}$;

(b) all the elements of the sequence (x_n) are comparable with respect to $\preceq$;

(c) for each $n \in \mathbb{N}^*$ we have $d(x_n, x_{n+1}) \leq \varphi^n(d(x_0, f(x_0))) \to 0$ as $n \to \infty$.

Let $\epsilon > 0$. Since $\varphi^n(\epsilon) \to 0$ as $n \to \infty$, there exists $n(\epsilon) > 0$ such that $\varphi^n(\epsilon) < \dfrac{\epsilon}{4s^2}$, for each $n \geq n(\epsilon)$. Let $g := f^{n(\epsilon)}$ and $y_m := g^m(x_0)$, $m \in \mathbb{N}$. Then we have

$$d(y_n, y_{m+1}) = d(f^{n(\epsilon)m}(x_0), f^{n(\epsilon)m}(g(x_0))) \leq \varphi^{n(\epsilon)m}(d(x_0, g(x_0))) \to 0,\ n \to \infty.$$

Hence, for $\epsilon > 0$ there exists $m(\epsilon) > 0$ such that $d(y_m, y_{m+1}) < \dfrac{\epsilon}{2s}$, for each $m \geq m(\epsilon)$. Let $\tilde{B}(y_{m(\epsilon)}; \epsilon) := \{y \in X \mid d(y, y_{m(\epsilon)}) \leq \epsilon\}$. We will show that $g : \tilde{B}(y_{m(\epsilon)}; \epsilon) \to \tilde{B}(y_{m(\epsilon)}; \epsilon)$. Indeed, let $u \in \tilde{B}(y_{m(\epsilon)}; \epsilon)$. Then

$$\begin{aligned} d(g(u), y_{m(\epsilon)}) &\leq s(d(g(u), g(y_{m(\epsilon)})) + d(g(y_{m(\epsilon)}), y_{m(\epsilon)})) \\ &= s(d(g(u), g(y_{m(\epsilon)})) + d(y_{m(\epsilon)+1}, y_{m(\epsilon)})). \end{aligned}$$

If $u, y_{m(\epsilon)} \in X$ are comparable, then we can write directly

$$d(g(u), g(y_{m(\epsilon)})) \leq \varphi^{n(\epsilon)}(d(u, y_{m(\epsilon)})),$$

if not then there exists $z \in X$ which is comparable with $u, y_{m(\epsilon)}$. Then

$$\begin{aligned} d(g(u), g(y_{m(\epsilon)})) &\leq s(d(g(u), g(z)) + d(g(z), g(y_{m(\epsilon)}))) \\ &\leq s(\varphi^{n(\epsilon)}(d(u,z)) + \varphi^{n(\epsilon)}(d(z, y_{m(\epsilon)}))). \end{aligned}$$

Hence

$$d(g(u), y_{m(\epsilon)}) \leq s[s\left(\varphi^{n(\epsilon)}(d(u,z)) + \varphi^{n(\epsilon)}(d(z, y_{m(\epsilon)}))\right) + d(y_{m(\epsilon)+1}, y_{m(\epsilon)})] \leq \epsilon.$$

As a consequence, for every $i, j \in \mathbb{N}$ with $i, j \geq m(\epsilon)$, we get

$$d(y_i, y_j) \leq s(d(y_i, y_{m(\epsilon)}) + d(y_j, y_{m(\epsilon)})) \leq 2s\epsilon,$$

which proves that the sequence (y_m) is Cauchy. By the completeness of the space there exists $x^* \in X$ such that $(y_m) \to x^*$ as $m \to \infty$. Since f has closed graph, it follows that g has closed graph too and thus $x^* \in Fix(g)$. Moreover,

$$y_m = g^m(x_0) \to x^* \text{ as } m \to \infty.$$

We will show now that for each $x \in X$ we have that $g^m(x) \to x^*$ as $m \to \infty$. Let $x \in X$. We have two cases:

1. If x and x_0 are comparable, then

$$d(g^m(x), g^m(x_0)) = d(f^{n(\epsilon)m}(x), f^{n(\epsilon)m}(x_0)) \leq \varphi^{n(\epsilon)m}(d(x, x_0)) \to 0, m \to \infty.$$

2. If x and x_0 are not comparable, then there exists $w \in X$ which is comparable to x and x_0. Then, we have

$$\begin{aligned} d(g^m(x), g^m(x_0)) &\leq s(d(g^m(x), g^m(w)) + d(g^m(w), g^m(x_0))) \\ &\leq s\left(\varphi^{n(\epsilon)m}(d(x,w)) + \varphi^{n(\epsilon)m}(d(w, x_0))\right) \to 0, m \to \infty. \end{aligned}$$

In both cases, we get that $g^m(x) \to x^*$ as $m \to \infty$, for each $x \in X$.

We will show now that x^* is a fixed point for f too. For each $x \in X$, we have

$$\lim_{m\to\infty} f(g^m(x)) = \lim_{m\to\infty} g^m(f(x)) = x^* \text{ and } g^m(x) \to x^* \text{ as } n \to \infty.$$

Since f has closed graph, we get that $x^* \in Fix(f)$. The uniqueness of the fixed point follows in a similar way to Nieto et al.' fixed point theorem, see [**9**]-[**11**]. □

Remark 3.2. 1) In particular, if (X, d) is a complete metric space and $\varphi(t) = kt$, $t \in \mathbb{R}_+$ (where $k \in [0, 1)$), then we obtain Ran-Reurings' fixed point theorem, see Theorem 2.1 in [**15**]. See also [**9**].

2) If (X, d) is a complete metric space and the contraction condition holds for all $x, y \in X$, then (without the assumptions (ii) and (iii) in the above theorem) we obtain Czerwik's fixed point theorem in [**4**]. See also [**7**].

Another result of this type can be established for a nonlinear contraction with respect to a b-comparison function. In this case, an approximation result and an apriori estimation for a solution can be additionally obtained.
Recall that a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a (b)-comparison function if:

(a) φ is increasing;

(ii) there exist $k_0 \in \mathbb{N}$, $\alpha \in (0,1)$ and a convergent series of non-negative terms $\sum_{k\geq 1} v_k$ such that $\varphi^{k+1}(t) \leq \alpha\varphi^k(t) + v_k$, for $k \geq k_0$ and for any $t \in \mathbb{R}_+$.

As a consequence of this definition, we have the following properties.

Lemma 3.3. *If $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a (b)-comparison function, then:*

(a) the series $\sum_{k\geq 0} s^k\varphi^k(t)$ converges for any $t \in \mathbb{R}_+$;

(b) the mapping $s : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $s(t) := \sum_{k\geq 0} s^k\varphi^k(t)$ is increasing and continuous at 0.

The following concept is also well-known in fixed point theory in ordered structures.

Definition 3.4. Let X be a nonempty set, let " $\preceq$ " be a partial order on X and d be a b-metric on X with constant $s \geq 1$. Then the triple $(X, \preceq, d)$ is called an ordered b-metric space if:

(i) " $\preceq$ " be a partial order on X;

(ii) d is a b-metric on X with constant $s \geq 1$;

(iii) for any sequence $(x_n)_{n\in\mathbb{N}}$ monotone increasing and convergent in (X, d) to $x^* \in X$, we have that $x_n \preceq x^*$, for all $n \in \mathbb{N}$.

Our second fixed point result, under a stronger condition on the function φ, is the following.

Theorem 3.5. *Let $(X, \preceq, d)$ be an ordered b-metric space such that $d : X \times X \to X$ is a complete b-metric with constant $s \geq 1$. Let $f : X \to X$ be an operator which has closed graph with respect to d and is increasing with respect to "$\preceq$". Suppose that there exist a (b)-comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ and an element $x_0 \in X$ such that:*

(i) $d(f(x), f(y)) \leq \varphi(d(x, y))$, for all $x, y \in X$ with $x \preceq y$;

(ii) $x_0 \preceq f(x_0)$.

Then:

(1) $Fix(f) \neq \emptyset$ and the sequence of successive approximations $(f^n(x))_{n\in\mathbb{N}}$, starting from any point $x \in X$ which is comparable to x_0, converges to a fixed point of f.

(2) If additionally, $t - s\varphi(t) \to +\infty$ as $t \to \infty$, then

$$d(f^n(x_0), x^*(x_0)) \leq \varphi^n(t_{x_0}),\ \textit{for each } n \in \mathbb{N}^*,$$

where $\lim_{n\to\infty} f^n(x_0) := x^*(x_0) \in Fix(f)$ and $t_{x_0} := \sup\{t \in \mathbb{R}_+ \ |t - s\varphi(t) \leq sd(x_0, f(x_0))\}$

Proof. (1) Let $x_0 \in X$ such that $x_0 \preceq f(x_0)$. Then $x_0 \preceq f^n(x_0)$ for every $n \in \mathbb{N}^*$. Denote $x_n := f^n(x_0)$, $n \in \mathbb{N}^*$. Then we have:

(a) $x_{n+1} = f(x_n)$, $n \in \mathbb{N}$ and (x_n) is increasing;

(b) all the elements of the sequence (x_n) are comparable with respect to $\preceq$;

(c) for each $n \in \mathbb{N}^*$ we have $d(x_n, x_{n+1}) \leq \varphi^n(d(x_0, f(x_0))) \to 0$ as $n \to \infty$.

Now we have

$$d(x_n, x_{n+p}) \leq \frac{1}{s^{n-1}} \sum_{k=n}^{n+p-1} s^k \varphi^k(d(x_0, f(x_0))).$$

If we denote $S_n := \sum_{k=0}^{n} s^k \varphi^k(d(x_0, f(x_0)))$, then we have that

$$d(x_n, x_{n+p}) \leq \frac{1}{s^{n-1}}(S_{n+p-1} - S_n), \ \forall \ n \in \mathbb{N}, \ \forall \ p \in \mathbb{N}^*.$$

By Lemma 3.3, we get that the sequence (x_n) is Cauchy and hence it converges to an element $x^*(x_0) \in X$. Since f has closed graph, by (a), we immediately get that $x^*(x_0) \in Fix(f)$.

Moreover, if $x \leq x_0$ (or $x \geq x_0$) the monotonicity condition on f implies that $f^n(x) \leq f^n(x_0)$ (or reversely), for each $n \in \mathbb{N}$. By the contraction condition (i) we get that

$$d(f^n(x), f^n(x_0)) \leq \varphi^n(d(x, x_0)) \to 0 \text{ as } n \to \infty.$$

Thus

$$\begin{aligned} d(f^n(x), x^*) &\leq s(d(f^n(x), f^n(x_0)) + d(f^n(x_0), x^*)) \\ &\leq s(\varphi^n(d(x, x_0)) + d(f^n(x_0), x^*)) \to 0 \text{ as } n \to \infty, \end{aligned}$$

which immediately yields that $(f^n(x))_{n \in \mathbb{N}} \to x^*$ as $n \to \infty$, for every $x \in X$ which is comparable to x_0.

(2) For our last part of the proof, notice first that $x_n \preceq x^*(x_0)$, for all $n \in \mathbb{N}$. Hence we obtain

$$d(x_n, x^*(x_0)) = d(f^n(x_0), f^n(x^*(x_0))) \leq \varphi^n(d(x_0, x^*(x_0))), \text{ for all } n \in \mathbb{N}^*.$$

On the other hand, since $d(x_0, x^*(x_0)) \leq s\,(d(x_0, f(x_0)) + d(f(x_0), f(x^*(x_0)))) \leq sd(x_0, f(x_0)) + s\varphi(d(x_0, x^*(x_0)))$ we immediately get that $d(x_0, x^*(x_0)) \leq t_{x_0}$. Thus

$$d(x_n, x^*(x_0)) \leq \varphi^n(t_{x_0}), \text{ for each } n \in \mathbb{N}^*. \ \square$$

Remark 3.6. The above results take also please if, instead of te closed graph condition, we suppose the orbital continuity of the mapping. Moreover, a dual result (for decreasing operators) takes place under dual conditions on the space (axiom (iii) in Definition 3.4) and on the operator (hypothesis (ii) of te above theorem).

4. COINCIDENCE POINT RESULTS IN b-METRIC SPACES

We present first an auxiliary result in the context of b-metric spaces.

Lemma 4.1. *Let (X, d) and (Y, ρ) be two complete b-metric spaces. Let $f : X \to Y$ be an injective and continuous mapping such that $f^{-1} : f(X) \to X$ is uniformly continuous. Then $f(X)$ is a closed subset of Y.*

Proof. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence of elements of $f(X)$ such that (y_n) converges to y^*. We will prove that $y^* \in f(X)$. Since (y_n) is Cauchy and f^{-1} is uniformly continuous it follows that $(f^{-1}(y_n))_{n\in\mathbb{N}}$ is a Cauchy sequence too. Thus $(f^{-1}(y_n))$ converges to $x^* \in X$. Finally, since f is continuous we can conclude that (y_n) converges to $f(x^*)$, which means that $y^* = f(x^*) \in f(X)$. □

The following result is a coincidence point theorem in a complete b-metric spaces.

Theorem 4.2. *Let (X, d) be a b-metric space with constant $s_1 \geq 1$ and Y be a nonempty set. Let ρ be a complete b-metric on Y with constant $s_2 \geq 1$ and $g, t : X \to Y$ be two operators. Suppose that the following assumptions take place:*

(i) $g(X) \subset t(X)$;

(ii) $g : X \to Y$ is a φ-contraction, i.e., $\varphi : \mathbb{R} \to \mathbb{R}_+$ is a comparison function and

$$d(g(x_1), g(x_2)) \leq \varphi(d(x_1, x_2)), \text{ for all } x_1, x_2 \in X;$$

(iii) $t : X \to Y$ is expansive, i.e.,

$$\rho(t(x_1), t(x_2)) \geq d(x_1, x_2), \text{ for all } x_1, x_2 \in X;$$

(iv) one of the following conditions hold:

(iv)-a) $t : X \to Y$ is continuous;

(iv)-b) $t(X)$ is closed with respect to the b-metric ρ;

(iv)-c) the b-metrics d and ρ are continuous.

Then $C(g, t) = \{x^*\}$.

Proof. By (iii) the operator t is an injection. Thus $t : X \to t(X)$ is a bijection. Let $t^{-1} : t(X) \to X$. By (iii), using the notation $x_1 := t^{-1}(y_1)$ and $x_2 := t^{-1}(y_2)$, we have

$$d(t^{-1}(y_1), t^{-1}(y_2)) \leq \rho(t(t^{-1}(y_1)), t(t^{-1}(y_2))) = \rho(y_1, y_2), \text{ for all } y_1, y_2 \in t(X).$$

Thus t^{-1} is a nonexpansive mapping and hence t^{-1} is also uniformly continuous.

a) We suppose first that $t : X \to Y$ is continuous. Then, by Lemma 4.1 we obtain that $t(X)$ is closed in (Y, ρ) and hence $(t(X), \rho)$ is complete too. Consider now the function $h : t(X) \to t(X)$ defined by $h := g \circ t^{-1}$. Notice that h is a single-valued operator by the above remarks and it is a self operator by condition (i). Moreover, h is a φ-contraction since, for $y_1, y_2 \in t(X)$, we have

$$\rho(h(y_1), h(y_2)) = \rho(g(t^{-1}(y_1)), g(t^{-1}(y_2))) \leq \varphi(t^{-1}(y_1), t^{-1}(y_2)) \leq \varphi(\rho(y_1, y_2)).$$

Thus, by Czerwik's fixed point theorem (see [**4**] or [**7**]) there exists a unique $y^* \in t(X)$ with $y^* = h(y^*)$. If we denote $x^* = t^{-1}(y^*)$, then we get $y^* = g(x^*) = t(x^*)$. Hence $x^* \in C(g,t)$. Uniqueness of the coincidence point follows by the uniqueness of the fixed point of h.

b) The case when $t(X)$ is closed with respect to the b-metric ρ follows in a similar way.

c) If $t : X \to Y$ is not necessarily continuous, suppose that the b-metrics d and ρ are continuous. Notice that, in this case, the pair $(\overline{t(X)}, \rho)$ is complete in (Y, ρ). Since t^{-1} is uniformly continuous we may define an operator $\tilde{t}^{-1} : \overline{t(X)} \to X$ by

$$\tilde{t}^{-1}(y) = \begin{cases} t^{-1}(y) & \text{if} \quad y \in t(X) \\ \lim\limits_{n\to\infty} t^{-1}(y_n) & \text{if} \quad y \in \overline{t(X)} \setminus t(X), \end{cases}$$

where $(y_n) \subset t(X)$ is such that $y_n \to y$ as $n \to \infty$. It is easy to see (by the continuity of the b-metrics d and ρ) that $\tilde{t}^{-1}$ is nonexpansive. Consider now the operator $\tilde{h}$ defined by $\tilde{h} := g \circ \tilde{t}^{-1}$. Then as before we can prove that $\tilde{h} : \overline{t(X)} \to \overline{t(X)}$ and it is a φ-contraction. Hence, by Czerwik's fixed point theorem we get that there exists a unique $y^* \in \overline{t(X)}$ such that $\tilde{h}(y^*) = y^*$. Let us show that $y^* \in t(X)$. Since $y^* = \tilde{h}(y^*)$ we get that $y^* = (g \circ \tilde{t}^{-1})(y^*) \in g(X) \subset t(X)$. Next, if we denote $x^* = t^{-1}(y^*)$, then we obtain that $y^* = g(x^*) = t(x^*)$. Uniqueness of the coincidence point follows as before by the uniqueness of the fixed point of h. □

The following theorem is a coincidence point theorem in an ordered complete b-metric spaces.

Theorem 4.3. *Let (X,d) be a b-metric space with constant $s_1 \geq 1$, Y be a nonempty set and "$\preceq$" be a partial order relation on X and on Y. Let ρ be a complete b-metric on Y with constant $s_2 \geq 1$ and $g, t : X \to Y$ be two operators. Suppose that the following assumptions take place:*

(i) $g(X) \subset t(X)$;

(ii) there exists a comparison function $\varphi : \mathbb{R} \to \mathbb{R}_+$ such that

$$\rho(g(x_1), g(x_2)) \leq \varphi(d(x_1,x_2)), \text{ for all } x_1, x_2 \in X \text{ with } x_1 \preceq x_2;$$

(iii) $t : X \to Y$ is increasing with respect to $\preceq$ and expansive, i.e.,

$$\rho(t(x_1), t(x_2)) \geq d(x_1, x_2), \text{ for all } x_1, x_2 \in X;$$

(iv) g has closed graph with respect to d and ρ and it is increasing with respect to $\preceq$;

(v) one of the following conditions hold:

(v)-a) $t : X \to Y$ is continuous;

(v)-b) $t(X)$ is closed with respect to the b-metric ρ;

(v)-c) the b-metrics d and ρ are continuous;

(vi) there exists $x_0 \in X$ such that $t(x_0) \preceq g(x_0)$;

(vii) for every $y, w \in Y$ there exists $z \in Y$ which is comparable to y and w. Then $C(g,t) = \{x^*\}$.

Proof. By (iii) the operator t is an injection. Thus $t : X \to t(X)$ is a bijection. Hence, using again (iii) for $t^{-1} : t(X) \to X$, we have

$$d(t^{-1}(y_1), t^{-1}(y_2)) \leq \rho(t(t^{-1}(y_1)), t(t^{-1}(y_2))) = \rho(y_1, y_2), \text{ for all } y_1, y_2 \in t(X).$$

Thus t^{-1} is a nonexpansive mapping and hence t^{-1} is uniformly continuous. Moreover, t^{-1} is also increasing.

a) We suppose first that $t : X \to Y$ is continuous. Then, by Lemma 4.1 we obtain again that $t(X)$ is closed in (Y, ρ) and hence $(t(X), \rho)$ is complete too. Consider now the function $h : t(X) \to t(X)$ defined by $h := g \circ t^{-1}$. Notice that h is single-valued and increasing by the above remarks and it is a self operator by condition (i). Additionally, if we denote $y_0 := t(x_0)$, then we have $y_0 \preceq h(y_0)$. Moreover, for $y_1, y_2 \in t(X)$ with $y_1 \preceq y_2$, we can prove that

$$\rho(h(y_1), h(y_2)) \leq \varphi(\rho(y_1, y_2)).$$

Indeed, let $y_1, y_2 \in t(X)$ such that $y_1 \preceq y_2$. Then, there exist $x_1, x_2 \in X$ such that $y_1 = t(x_1)$ and $y_2 = t(x_2)$. Since t^{-1} is increasing we get that $t^{-1}(y_1) \preceq t^{-1}(y_2)$. Then, by (ii) and (iii), we get

$$\rho(h(y_1), h(y_2)) \leq \varphi(d(t^{-1}(y_1), t^{-1}(y_2))) \leq \varphi(\rho(y_1, y_2)).$$

Then, by Theorem 3.1, there exists a unique $y^* \in t(X)$ such that $y^* = h(y^*)$. As a consequence, if we denote $x^* := t^{-1}(y^*)$, then we obtain $y^* = g(x^*) = t(x^*)$.

b) The case when $t(X)$ is closed with respect to the b-metric ρ follows in a similar way.

c) If $t : X \to Y$ is not necessarily continuous, suppose that the b-metrics d and ρ are continuous. Notice that the pair $(\overline{t(X)}, \rho)$ is complete in (Y, ρ). Since t^{-1} is uniformly continuous we may define an operator $\tilde{t}^{-1} : \overline{t(X)} \to X$ by

$$\tilde{t}^{-1}(y) = \begin{cases} t^{-1}(y) & \text{if } \; y \in t(X) \\ \lim\limits_{n\to\infty} t^{-1}(y_n) & \text{if } \; y \in \overline{t(X)} \setminus t(X), \end{cases}$$

where $(y_n) \subset t(X)$ is such that $y_n \to y$ as $n \to \infty$. It is easy to see (by the continuity of the b-metrics d and ρ) that $\tilde{t}^{-1}$ is nonexpansive. Consider now the operator $\tilde{h}$ defined by $\tilde{h} := g \circ t^{-1}$. Then, as before, we can prove that $\tilde{h} : \overline{t(X)} \to \overline{t(X)}$ and it satisfies the following relation

$$\rho(\tilde{h}(y_1), \tilde{h}(y_2)) \leq \varphi(\rho(y_1, y_2)), \text{ for all } y_1, y_2 \in t(X) \text{ with } y_1 \preceq y_2.$$

Hence, again by Theorem 3.1 there exists a unique $y^* \in \overline{t(X)}$ such that $\tilde{h}(y^*) = y^*$. Let us show that $y^* \in t(X)$. Since $y^* = \tilde{h}(y^*)$ we get that

$$y^* = (g \circ \tilde{t}^{-1})(y^*) \in g(X) \subset t(X).$$

Now, if we denote $x^* = t^{-1}(y^*)$, then we obtain that

$$y^* = g(x^*) = t(x^*).$$

Finally notice that uniqueness follows by the assumption (vii). □

A data dependence theorem for the coincidence point problem is the following result.

Theorem 4.4. *Let (X, d) be a b-metric space with constant $s_1 \geq 1$, Y be a nonempty set and "$\preceq$" be a partial order relation on Y. Let ρ be a complete b-metric on Y with constant $s_2 \geq 1$ and $g, t : X \to Y$ be two operators satisfying all the assumptions of Theorem 4.2. Denote by x^* the unique coincidence point of g and t. Let $g_1, t_1 : X \to Y$ be two operators having at least one coincidence point $x_1^* \in X$. We also suppose that:*

(i) $t_1 : X \to Y$ is injective, $t_1(X) \subset t(X)$ and $t(X)$ is a closed subset of (Y, ρ);

(ii) there exist $\eta_1, \eta_2, \eta_3 > 0$ such that

$$\rho(g(x), g_1(x)) \leq \eta_1, \text{ for all } x \in X;$$

$$\rho(t(x), t_1(x)) \leq \eta_2, \text{ for all } x \in X;$$

$$d(t^{-1}(y), t_1^{-1}(y)) \leq \eta_3, \text{ for all } y \in t_1(X);$$

(iii) the function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$, $\gamma(t) := t - s_2\varphi(t)$ satisfies the condition $\lim_{t\to\infty} \gamma(t) = \infty$.

Then, the following estimation holds

$$d(x^*, x_1^*) \leq s_2(\psi(\eta_1, \eta_3) + \eta_2),$$

where $\psi : \mathbb{R}_+^2 \to \mathbb{R}_+$ is given by

$$\psi(\eta_1, \eta_3) := \sup\{t \geq 0 \mid t - s_2\varphi(t) \leq s_2^2(\varphi(\eta_3) + \eta_1)\}.$$

Proof. Let us consider $h : t(X) \to t(X)$ defined by $h := g \circ t^{-1}$ and $h_1 : t_1(X) \to t_1(X)$ defined by $h_1 := g_1 \circ t_1^{-1}$. Denote $y^* = t(x^*) = g(x^*)$ and $y_1^* = t_1(x_1^*) = g_1(x_1^*)$. Then y^* and y_1^* are fixed points for h and respectively h_1 and, by the proof of Theorem 4.2, the operator h is a φ-contraction. Then, we have the following estimation

$$\begin{aligned}\rho(y^*, y_1^*) = \rho(h(y^*), h_1(y_1^*)) &\leq s_2(\rho(h(y^*), h(y_1^*)) + \rho(h(y_1^*), h_1(y_1^*))) \\ &\leq s_2(\varphi(\rho(y^*, y_1^*)) + \eta),\end{aligned}$$

where $\eta > 0$ is given by the following relation

$$\rho(h(y_1^*), h_1(y_1^*)) = \rho((g \circ t^{-1})(y_1^*), (g_1 \circ t_1^{-1})(y_1^*))$$

$$\le s_2(\rho((g\circ t^{-1})(y_1^*),(g\circ t_1^{-1})(y_1^*))+\rho((g\circ t_1^{-1})(y_1^*),(g_1\circ t_1^{-1})(y_1^*)))$$
$$\le s_2(\varphi(d(t^{-1}(y_1^*),t_1^{-1}(y_1^*)))+\eta_1)\le s_2(\varphi(\eta_3)+\eta_1):=\eta.$$

Hence
$$\rho(y^*,y_1^*)-s_2\varphi(\rho(y^*,y_1^*))\le s_2^2(\varphi(\eta_3)+\eta_1).$$

We conclude that
$$\rho(y^*,y_1^*)\le\psi(\eta_1,\eta_3):=\sup\{t\ge 0\mid t-s_2\varphi(t)\le s_2^2(\varphi(\eta_3)+\eta_1)\}.$$

Since t is expansive, we get that
$$\begin{aligned} d(x^*,x_1^*) &\le \rho(t(x^*),t(x_1^*))\\ &\le s_2(\rho(t(x^*),t_1(x_1^*))+\rho(t_1(x_1^*),t(x_1^*)))\\ &\le s_2(\rho(y^*,y_1^*)+\eta_2).\end{aligned}$$

As a conclusion $d(x^*,x_1^*)\le s_2(\psi(\eta_1,\eta_3)+\eta_2)$. □

A well-posedness result for the coincidence problem is given in the next theorem.

Definition 4.5. Let (X,d) and (Y,ρ) be two b-metric spaces with constants $s_1\ge 1$ and respectively $s_2\ge 1$. Let $g,t:X\to Y$ be two operators. By definition, the coincidence problem for g and t is well-posed if:

(i) $C(g,t)=\{x^*\}$;

(ii) for any sequence $(x_n)_{n\in\mathbb{N}}$ in X for which $\rho(g(x_n),t(x_n))\to 0$ as $n\to\infty$, we have that $x_n\to x^*$ as $n\to\infty$.

Theorem 4.6. *Let (X,d) be a b-metric space with constant $s_1\ge 1$, Y be a nonempty set and "$\preceq$" be a partial order relation on Y. Let ρ be a complete b-metric on Y with constant $s_2\ge 1$ and $g,t:X\to Y$ be two operators satisfying all the assumptions of Theorem 4.2. Additionally suppose that the mapping $\psi:\mathbb{R}_+\to\mathbb{R}_+$, $\psi(t)=t-s_2^2\varphi(t)$ is a bijection such that $\psi^{-1}(u_n)\to 0$ as $u_n\to 0$, for $n\to\infty$. Then the coincidence problem for g and t is well-posed.*

Proof. By Theorem 4.2 we have that $C(g,t)=\{x^*\}$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that $\rho(g(x_n),t(x_n))\to 0$ as $n\to\infty$. Then, we have
$$\begin{aligned} d(x_n,x^*) &\le \rho(t(x_n),t(x^*))\\ &\le s_2(\rho(t(x_n),g(x_n))+\rho(g(x_n),t(x^*)))\\ &\le s_2\rho(t(x_n),g(x_n))+s_2^2(\rho(g(x_n),g(x^*))+\rho(g(x^*),t(x^*)))\\ &\le s_2\rho(t(x_n),g(x_n))+s_2^2\varphi(d(x_n,x^*)).\end{aligned}$$

Thus
$$d(x_n,x^*)-s_2^2\varphi(d(x_n,x^*))\le s_2\rho(t(x_n),g(x_n))$$

and so
$$d(x_n,x^*)\le\psi^{-1}(s_2\rho(t(x_n),g(x_n)))\to 0 \text{ as } n\to\infty.$$ □

We will study now the Ulam-Hyers stability of the coincidence point problem. For a general study of this problem in generalized metric spaces see I.A. Rus [**18**].

Definition 4.7. Let (X, d) and (Y, ρ) be two b-metric spaces with constants $s_1 \geq 1$ and respectively $s_2 \geq 1$. Let $g, t : X \to Y$ be two operators. By definition, the coincidence problem for g and t is Ulam-Hyers stable if there exists an increasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ continuous in 0 with $\psi(0) = 0$ such that for each $\epsilon > 0$ and each ϵ-solution $\tilde{x} \in X$ of the coincidence problem for g and t (i.e., $\rho(t(\tilde{x}), g(\tilde{x})) \leq \epsilon$), there exists a solution $x^* \in X$ of the coincidence problem for g and t such that $d(x^*, \tilde{x}) \leq \psi(\epsilon)$.

Theorem 4.8. *Let (X, d) be a b-metric space with constant $s_1 \geq 1$, Y be a nonempty set and "$\preceq$" be a partial order relation on Y. Let ρ be a complete b-metric on Y with constant $s_2 \geq 1$ and $g, t : X \to Y$ be two operators satisfying all the assumptions of Theorem 4.2. Additionally, suppose that the mapping $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$, $\gamma(t) = t - s_2^2\varphi(t)$ is strictly increasing and onto. Then the coincidence problem for g and t is Ulam-Hyers stable.*

Proof. By Theorem 4.2 we have that $C(g, t) = \{x^*\}$. Let $\epsilon > 0$ and $\tilde{x} \in X$ such that $\rho(t(\tilde{x}), g(\tilde{x})) \leq \epsilon$. Then we have

$$\begin{aligned} d(x^*, \tilde{x}) &\leq \rho(t(x^*), t(\tilde{x})) \\ &\leq s_2(\rho(t(x^*), g(\tilde{x})) + \rho(g(\tilde{x}), t(\tilde{x}))) \\ &\leq s_2^2(\rho(t(x^*), g(x^*)) + \rho(g(x^*), g(\tilde{x}))) + s_2\epsilon \\ &\leq s_2^2\varphi(d(x^*, \tilde{x})) + s_2\epsilon. \end{aligned}$$

Hence

$$d(x^*, \tilde{x}) - s_2^2\varphi(d(x^*, \tilde{x})) \leq s_2\epsilon$$

and so

$$d(x^*, \tilde{x}) \leq \gamma^{-1}(s_2\epsilon). \qquad \square$$

The last result of this section is another coincidence point theorem of Ran-Reurings type. The result is a slight extension of Theorem 1 in [**13**] and a generalization of Theorem 3 in [**8**].

Theorem 4.9. *Let (X, d) be a b-metric space with constant $\lambda \geq 1$, Y be a nonempty set and "$\preceq$" be a partial order relation on Y. Let ρ be a b-metric on Y with constant $s \geq 1$ and $g, t : X \to Y$ be two operators with closed graph. Suppose that the following assumptions take place:*

(i) $t(X) \subset g(X)$;

(ii) $(t(X), \rho)$ is a complete subset of Y;

(iii) there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\rho(t(x), t(y)) \leq \varphi(\rho(g(x), g(y))), \text{ for all } x, y \in X \text{ with } g(x) \preceq g(y);$$

(iv) there exists $x_0 \in X$ such that $g(x_0) \in t(X)$ and $g(x_0) \preceq t(x_0)$;

(v) t is increasing with respect to g, i.e.,

$$x_1, x_2 \in X \text{ and } g(x_1) \preceq g(x_2) \Rightarrow t(x_1) \preceq t(x_2).$$

Then, there exists $x^* \in X$ such that $g(x^*) = t(x^*)$ and the sequence z_n defined by $g(z_{n+1}) = t(z_n)$ (where $n \in \mathbb{N}$ and $z_0 := x_0 \in X$) converges to x^* as $n \to \infty$.

If, in addition:

(vi) for every $y, w \in Y$ there exists $z \in Y$ which is comparable to y and w;

(vii) g is an injection,

then $C(t, g) = \{x^*\}$ and the sequence $(z_n)_{n\in\mathbb{N}}$ defined by $g(z_{n+1}) = t(z_n)$, starting from any point $z_0 \in X$ converges to x^*.

Proof. Let $x_0 \in X$ such that $g(x_0) \preceq t(x_0)$. Let us define $f := t \circ g^{-1}$. Then, we have for f the following properties:

1) f is a single-valued operator on $t(X)$;
2) $f : t(X) \to t(X)$;
3) f has closed graph;
4) $\rho(f(y_1), f(y_2)) \leq \varphi(\rho(y_1, y_2))$, for all $y_1, y_2 \in t(X)$ with $y_1 \preceq y_2$;
5) f is increasing on $t(X)$;
6) If $y_0 := g(x_0)$, then $y_0 \preceq (t \circ g^{-1})(y_0) = f(y_0)$.

By Theorem 3.1, we obtain that f is a Picard operator. Thus $Fix(f) = \{y^*\}$. Then $(t \circ g^{-1})(y^*) = y^*$. Thus, if we denote $x^* := g^{-1}(y^*)$, then we have $t(x^*) = g(x^*) = y^*$, showing that x^* is a coincidence point for t and g. Moreover, the sequence $y_{n+1} = f(y_n)$ (where $n \in \mathbb{N}$), starting from $y_0 := g(x_0) \in t(X)$ converges to y^* as $n \to \infty$, while the sequence z_n defined by $g(z_{n+1}) = t(z_n)$ (where $n \in \mathbb{N}$ and $z_0 := x_0 \in X$) converges to x^* as $n \to \infty$.

The uniqueness of the coincidence point follows by (vi) and (vii). Indeed, by the first part of this theorem there exist $x^* \in X$ and $y^* \in t(X)$ such that $t(x^*) = g(x^*) = y^*$. Suppose that there exist $u^* \in X$ and $v^* \in t(X)$ such that $t(u^*) = g(u^*) = v^*$. We have two cases:

Case 1. If $g(x^*)$ and $g(u^*)$ are comparable, i.e., $g(x^*) \preceq g(u^*)$ or reversely.

Let $f : t(X) \to t(X)$, $f(y) := t \circ g^{-1}(y)$ with the above six properties. Suppose, for example, that $g(x^*) \preceq g(u^*)$. Then $t(x^*) \preceq t(u^*)$ and so

$$\rho(y^*, v^*) = \rho(f(y^*), f(v^*)) \leq \varphi(\rho(y^*, v^*)).$$

By the properties of the comparison function φ we get that $\rho(y^*, v^*) = 0$. Thus $y^* = v^*$ which implies $g(x^*) = g(u^*)$. By the injectivity of g we get that $x^* = u^*$.

Case 2. Suppose that $g(x^*)$ and $g(u^*)$ are not comparable. Then, there exists $z \in Y$ such that z is comparable to $g(x^*)$ and $g(u^*)$. Suppose $g(x^*) \preceq z \preceq g(u^*)$. Thus $y^* \preceq z \preceq v^*$. Consider $f : t(X) \to t(X)$, $f(y) := t \circ g^{-1}(y)$. Then, since f is increasing, we get that $f^n(y^*) \preceq f^n(z) \preceq f^n(v^*)$, for all $n \in \mathbb{N}^*$.

Let $y_1, y_2 \in t(X)$ with $y_1 \preceq y_2$. By the monotonicity of f we obtain that $f^n(y_1) \preceq f^n(y_2)$, for all $n \in \mathbb{N}^*$. Now, by induction, we get

$$\rho(f^n(y_1), f^n(y_2)) \leq \varphi^n(\rho(y_1, y_2)), \text{ for all } n \in \mathbb{N}^*.$$

Applying the above relation we get

$$\begin{aligned}\rho(y^*, v^*) &= \rho(f^n(y^*), f^n(v^*)) \\ &\leq s(\rho(f^n(y^*), f^n(z)) + \rho(f^n(z), f^n(v^*))) \\ &\leq s(\varphi^n(\rho(y^*, z)) + \varphi^n(\rho(z, v^*))) \to 0 \text{ as } n \to \infty.\end{aligned}$$

Thus $y^* = v^*$. As before, by the injectivity of g we obtain $x^* = u^*$. The rest of the cases can be treated similarly. □

5. AN APPLICATION

Let us consider, as an illustration of the previous results, an integral equation of the following form:

$$(1) \qquad \begin{cases} T(x(s)) = \displaystyle\int_0^s g(p, x(p))dp, \text{ for } s \in [0, \alpha] \text{ (with } \alpha > 0), \\ x(0) = 0, \end{cases}$$

where:

(i) $T : \mathbb{R}_+ \to \mathbb{R}_+$ and $T(0) = 0$;

(ii) $g : [0, \alpha] \times \mathbb{R}_+ \to \mathbb{R}_+$

are two continuous mappings.

If we consider

$$X := C_+([0, \alpha]) = \{x : [0, \alpha] \to \mathbb{R} \mid x(0) = 0,\ x(s) \geq 0,\ \forall\ s \in [0, \alpha]\}$$

and the operators $t, G : X \to X$ given by

$$tx(s) := T(x(s)) \text{ and } Gx(s) := \int_0^s g(p, x(p))dp,$$

then our problem can be re-written as a coincidence point problem of the following form

$$tx = Gx,\ x \in X.$$

Notice that on $C_+([0, \alpha])$ we can define a partial order relation by

$$x \leq_C y \text{ if and only if } x(s) \leq y(s),\ \forall\ s \in [0, \alpha]$$

and a Bielecki type norm given by

$$\|x\|_B := \max_{s \in [0, \alpha]} (|x(s)|e^{-\tau s}),\ \text{where } \tau > 0,$$

with respect to which the space X is complete.

We have the following existence and uniqueness result for (1).

Theorem 5.1 *Consider the functional-integral equation (1). We suppose that:*

(i) $T : \mathbb{R}_+ \to \mathbb{R}_+$ *and* $g : [0, \alpha] \times \mathbb{R}_+ \to \mathbb{R}_+$ *are two continuous mappings;*

(ii) T *is onto, increasing and expansive;*

(iii) there exist $\tau > 0$ *and a function* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that for arbitrary* $q > 1$ *we have* $\psi(qt) \leq q\psi(t)$, *for all* $t \in \mathbb{R}_+$, *the function* $\varphi(t) = \frac{1}{\tau}\psi(t)$ *is a comparison function and*

$$|f(s,u) - f(s,v)| \leq \psi(|u - v|), \ \forall \ s \in [0, \alpha] \text{ and } \forall \ u, v \in \mathbb{R}_+ \text{ with } u \leq v;$$

(iv) $f(s, \cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ *is increasing, for all* $s \in [0, \alpha]$*;*

(v) there exists $x_0 \in C_+([0, \alpha])$ *a lower solution of (1), i.e.,*

$$T(x_0(s)) \leq \int_0^s g(p, x_0(p))dp, \text{ for } s \in [0, \alpha].$$

Then, the functional-integral equation (1) has a unique solution in $C_+([0, \alpha])$.

Proof. Consider the space X endowed with the partial order $\leq_C$ and the Bielecki type norm $\|x\|_B$ and the operators t, G defined as above. Then, we have:

(a) $G(X) \subset t(X) = X$;

(b) for $x, y \in X$ with $x \leq_C y$, for each $s \in [0, \alpha]$ and any $\tau > 0$ we successively obtain

$$\begin{aligned} |Gx(s) - Gy(s)| &\leq \int_0^s |f(p, x(p)) - f(p, y(p))|dp \leq \int_0^s \psi(|x(p) - y(p)|)dp \\ &\leq \int_0^s e^{\tau p}\psi(\|x - y\|_B)dp \leq \frac{1}{\tau}\psi(\|x - y\|_B)e^{\tau s} \\ &= \varphi(\|x - y\|_B)e^{\tau s}. \end{aligned}$$

Thus $\|Gx - Gy\|_B \leq \varphi(\|x - y\|_B), \ \forall \ x, y \in X$, with $x \leq_C y$.

(c) t and G are continuous and increasing with respect to $\leq_C$;

(d) there exists $x_0 \in X$ such that $tx_0 \leq_C Gx_0$.

Thus, all the conditions of Theorem 4.2 are satisfied and the conclusion follows by Theorem 4.2. □

Remark 5.2. In particular, if $T(u) := e^u - 1$ then the assumptions on T of the above theorem are satisfied and under the conditions on g given in Theorem 4.1 there exists a unique solution $x^* \in C_+([0, \alpha])$ of the equation

$$e^{x(s)} = \int_0^s g(p, x(p))dp + 1, \text{ for } s \in [0, \alpha] \text{ (with } \alpha > 0).$$

See [**5**] for more details concerning this example.

6. FURTHER RESEARCH DIRECTIONS

Fixed point theorems of Ran-Reurings type are a very useful tool for proving coupled fixed point theorems of Gnana Bhaskar-Lakshmikantham type, see [**6**]. For this approach see, for example, [**13, 14**].

Following this idea, an useful approach in the coupled coincidence theory is based on the above coincidence theorems. More precisely, if we consider the following coupled coincidence problem: find $(x, y) \in X \times X$ satisfying

$$\begin{cases} g(x) = T(x, y) \\ g(y) = T(y, x), \end{cases}$$

(where X is a nonempty set and $g : X \to X$ and $T : X \times X \to X$ are two given operators), then the above problem could be transposed in a coincidence problem of the following form

$$G(x, y) = S(x, y),$$

where $G, S : X \times X \to X \times X$ are given by the following expressions

$$G(x, y) = (g(x), g(y)) \text{ and } S(x, y) := (T(x, y), T(y, x)).$$

This will be the subject of our forthcoming work.

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*Faculty of Mathematics and Computer Science,
Babeş-Bolyai University,
400084 Cluj-Napoca, Romania
Academy of Romanian Scientists,
Romania
E-mail: petrusel@math.ubbcluj.ro

Faculty of Business
Babeş-Bolyai University,
400084 Cluj-Napoca, Romania
E-mail: gabi.petrusel@tbs.ubbcluj.ro

Research Center for Interneural Computing,
China Medical University Hospital,
Taichung 40447, Taiwan
E-mail: yaojc@mail.cmu.edu.tw