Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **12** (2018), 273–296.
https://doi.org/10.2298/AADM180130011R

TWO KINDS OF THE REVERSE HARDY-TYPE INTEGRAL INEQUALITIES WITH THE EQUIVALENT FORMS RELATED TO THE EXTENDED RIEMANN ZETA FUNCTION

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Applying techniques of real analysis and weight functions, we study some equivalent conditions of two kinds of the reverse Hardy-type integral inequalities with a particular nonhomogeneous kernel. The constant factors are related to the Riemann zeta function and are proved to be best possible. In the form of applications, we deduce a few equivalent conditions of two kinds of the reverse Hardy-type integral inequalities with a particular homogeneous kernel. We also consider some corollaries as particular cases.

1. INTRODUCTION

In 1925, by introducing one pair of conjugate exponents (p, q), Hardy [**1**] proved the following extension of Hilbert's integral inequality:
For $p > 1, \frac{1}{p} + \frac{1}{q} = 1, f(x), g(y) \geq 0,$

$$0 < \int_0^\infty f^p(x)dx < \infty \text{ and } 0 < \int_0^\infty g^q(y)dy < \infty,$$

it holds

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y)dy\right)^{\frac{1}{q}}, \tag{1.0.1}$$

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2010 Mathematics Subject Classification. 26D15, 47A07.
Hardy-type integral inequality; best possible constants; equivalent form; Riemann zeta function

where, the constant factor

$$\frac{\pi}{\sin(\pi/p)}$$

is the best possible.

Inequalities such as (1.0.1), as well as Hilbert's integral inequality (for $p = q = 2$ in (1.0.1), cf. [**2**]) are known for their importance in mathematical analysis and its applications (cf. [**3**], [**4**]). In 1934, Hardy et al. proved an extension of (1.0.1) with the kernel $k_1(x, y)$, where $k_1(x, y)$ is a nonnegative homogeneous function of degree -1 (cf. [**3**], Theorem 319). Additionally, the following Hilbert-type integral inequality with a nonhomogeneous kernel is proved:

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, h(u) > 0,$

$$\phi(\sigma) = \int_0^\infty h(u)u^{\sigma-1}du \in \mathbf{R}_+,$$

then

$$\int_0^\infty \int_0^\infty h(xy)f(x)g(y)dxdy < \phi(\frac{1}{p}) \left(\int_0^\infty x^{p-2}f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y)dy\right)^{\frac{1}{q}}, \tag{1.0.2}$$

where the constant factor $\phi(\frac{1}{p})$ is the best possible (cf. [**3**], Theorem 350).

In 1998, by introducing an independent parameter $\lambda > 0$, Yang proved an extension of (1.0.1) for $p = q = 2$ with the kernel $\frac{1}{(x+y)^\lambda}$ (cf. [**5**], [**6**]). In 2004, by introducing another pair of conjugate exponents (r, s) $(r > 1, \frac{1}{r} + \frac{1}{s} = 1)$, Yang [**7**] proved an extension of (1.0.1) with the kernel $\frac{1}{x^\lambda + y^\lambda}$ $(\lambda > 0)$. In 2005, yet another extension of (1.0.1) as well as of the result of [**5**] was given in [**8**], with the kernel $\frac{1}{(x+y)^\lambda}$ $(\lambda > 0)$. Several authors (cf. [**9**]–[**17**]) have proved some further extensions.

In 2009, Yang gave the following extension of (1.0.1) (cf. [**18**], [**20**]):

If $\lambda_1 + \lambda_2 = \lambda \in \mathbf{R} := (-\infty, \infty)$, $k_\lambda(x, y)$ is a nonnegative homogeneous function of degree $-\lambda$, satisfying

$$k_\lambda(ux, uy) = u^{-\lambda}k_\lambda(x, y) \ (u, x, y > 0),$$

$$k(\lambda_1) = \int_0^\infty k_\lambda(u, 1)u^{\lambda_1-1}du \in \mathbf{R}_+ := (0, \infty),$$

then for $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, we have

$$\int_0^\infty \int_0^\infty k_\lambda(x, y)f(x)g(y)dxdy < k(\lambda_1) \left(\int_0^\infty x^{p(1-\lambda_1)-1}f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty y^{q(1-\lambda_2)-1}g^q(y)dy\right)^{\frac{1}{q}}, \tag{1.0.3}$$

where the constant factor $k(\lambda_1)$ is the best possible; for $0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1$, we obtain the reverse of (1.0.3). The following extension of (1.0.2) was also proved: For $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, we have

$$\begin{aligned}(1.0.4) \qquad & \int_0^\infty \int_0^\infty h(xy)f(x)g(y)dxdy \\ < \; & \phi(\sigma)\left(\int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{q(1-\sigma)-1}g^q(y)dy\right)^{\frac{1}{q}},\end{aligned}$$

where the constant factor $\phi(\sigma)$ is the best possible; for $0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1$, we obtain the reverse of (1.0.4) (cf. [**19**]).
In [**20**], some inequalities equivalent to (1.0.3) and (1.0.4) are considered. In 2013, Yang [**19**] studied also the equivalency of (1.0.3) and (1.0.4). In 2017, Hong [**21**] proved an equivalent condition between (1.0.3) and some parameters.

Remark 1.1. *(cf. [**19**]) If $h(xy) = 0$, for $xy > 1$, then*

$$\phi(\sigma) = \int_0^1 h(u)u^{\sigma-1}du = \phi_1(\sigma) \in \mathbf{R}_+,$$

and the reverse of (1.0.4) reduces to the following reverse Hardy-type integral inequality with the nonhomogeneous kernel:

$$\begin{aligned}(1.0.5) \qquad & \int_0^\infty g(y)\left(\int_0^{\frac{1}{y}} h(xy)f(x)dx\right)dy \\ > \; & \phi_1(\sigma)\left(\int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{q(1-\sigma)-1}g^q(y)dy\right)^{\frac{1}{q}},\end{aligned}$$

where the constant factor $\phi_1(\sigma)$ is the best possible.
If $h(xy) = 0$, for $xy < 1$, then

$$\phi(\sigma) = \int_1^\infty h(u)u^{\sigma-1}du = \phi_2(\sigma) \in \mathbf{R}_+,$$

and the reverse of (1.0.4) reduces to the following other kind of the reverse Hardy-type integral inequality with nonhomogeneous kernel:

$$\begin{aligned}(1.0.6) \qquad & \int_0^\infty g(y)\left(\int_{\frac{1}{y}}^\infty h(xy)f(x)dx\right)dy \\ > \; & \phi_2(\sigma)\left(\int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{q(1-\sigma)-1}g^q(y)dy\right)^{\frac{1}{q}},\end{aligned}$$

where the constant factor $\phi_2(\sigma)$ is the best possible.

In this paper, using techniques of real analysis and weight functions, we obtain a few equivalent conditions of (1.0.5)((1.0.6)) with the particular kernel

$$\frac{|\ln xy|^{\beta}}{|(xy)^{\lambda}-1|}\ (\beta,\lambda>0)\,.$$

The constant factors are related to the extended Riemann zeta function and are the best possible. In the form of applications we obtain some equivalent conditions of two kinds of the reverse Hardy-type integral inequalities with the particular homogeneous kernel

$$\frac{|\ln x/y|^{\beta}}{|x^{\lambda}-y^{\lambda}|}\ (\beta,\lambda>0)\,.$$

We also consider some corollaries as particular cases.

2. AN EXAMPLE AND TWO LEMMAS

Example 1. *Setting* $h(u)=\frac{|\ln u|^{\beta}}{|u^{\lambda}-1|}\ (u>0)$, *then we obtain*

$$h(xy)=\frac{|\ln xy|^{\beta}}{|(xy)^{\lambda}-1|}\ (x,y>0),$$

and for $\beta,\sigma,\lambda>0$,

$$\begin{aligned} k_{\lambda}^{(1)}(\sigma) &:= \int_0^1 \frac{(-\ln u)^{\beta}}{1-u^{\lambda}}u^{\sigma-1}du = \int_0^1 (-\ln u)^{\beta}\sum_{k=0}^{\infty} u^{k\lambda+\sigma-1}du \\ &= \textstyle\sum_{k=0}^{\infty} \displaystyle\int_0^1 (-\ln u)^{\beta} u^{k\lambda+\sigma-1}du. \end{aligned}$$

Setting $v=(k\lambda+\sigma)(-\ln u)$ *in the above integral, we get*

$$\begin{aligned} k_{\lambda}^{(1)}(\sigma) &= \sum_{k=0}^{\infty}\frac{1}{(k\lambda+\sigma)^{\beta+1}}\int_0^{\infty} v^{\beta}e^{-v}dv \\ &= \frac{\Gamma(\beta+1)}{\lambda^{\beta+1}}\zeta(\beta+1,\frac{\sigma}{\lambda})\in \mathbf{R}_{+}, \end{aligned} \tag{2.0.7}$$

where

$$\Gamma(\eta):=\int_0^{\infty} v^{\eta-1}e^{-v}dv\ (\eta>0)$$

is the gamma function and

$$\zeta(s,a):=\sum_{k=0}^{\infty}\frac{1}{(k+a)^{s}}\ (Res>1,a>0)$$

is the extended Riemann zeta function ($\zeta(s,1)=\zeta(s)$ *is the Riemann zeta function*) *(cf. [**24**]).*

For $\lambda > \max\{0, \sigma\}$, setting $v = \frac{1}{u}$, we obtain

$$\begin{aligned} k_{\lambda}^{(2)}(\sigma) &:= \int_1^{\infty} \frac{(\ln u)^{\beta}}{u^{\lambda} - 1} u^{\sigma - 1} du = \int_0^1 \frac{(-\ln v)^{\beta}}{1 - v^{\lambda}} u^{\lambda - \sigma - 1} du \\ &= \tfrac{\Gamma(\beta+1)}{\lambda^{\beta+1}} \zeta(\beta + 1, \tfrac{\lambda - \sigma}{\lambda}) \in \mathbf{R}_+. \end{aligned}$$

In the sequel, we shall always assume that

$$0 < p < 1 \ (q < 0), \frac{1}{p} + \frac{1}{q} = 1, \sigma_1 \in \mathbf{R}, \text{ and } \beta, \lambda, M_1, M_2 > 0.$$

Lemma 2.2. *If $\sigma > 0$ and for any nonnegative measurable functions $f(x)$ and $g(y)$ in $(0, \infty)$ the following inequality*

$$\begin{aligned} (2.0.8) \qquad & \int_0^{\infty} g(y) \left[\int_0^{\frac{1}{y}} \frac{|\ln xy|^{\beta}}{|(xy)^{\lambda} - 1|} f(x) dx \right] dy \\ \geq \quad & M_1 \left[\int_0^{\infty} x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^{\infty} y^{q(1-\sigma_1)-1} g^q(y) dy \right]^{\frac{1}{q}} \end{aligned}$$

holds true, then we have

$$\sigma_1 = \sigma \ \text{and} \ M_1 \leq k_{\lambda}^{(1)}(\sigma).$$

Proof. If $\sigma_1 < \sigma$, $n \in \mathbf{N}$, we define the functions

$$f_n(x) := \begin{cases} x^{\sigma + \frac{1}{pn} - 1}, 0 < x \leq 1 \\ 0, x > 1 \end{cases}, \quad g_n(y) := \begin{cases} 0, 0 < y < 1 \\ y^{\sigma_1 - \frac{1}{qn} - 1}, y \geq 1 \end{cases},$$

and derive that

$$\begin{aligned} J_1 &:= \left[\int_0^{\infty} x^{p(1-\sigma)-1} f_n^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^{\infty} y^{q(1-\sigma_1)-1} g_n^q(y) dy \right]^{\frac{1}{q}} \\ &= \left(\int_0^1 x^{\frac{1}{n} - 1} dx \right)^{\frac{1}{p}} \left(\int_1^{\infty} y^{-\frac{1}{n} - 1} dy \right)^{\frac{1}{q}} = n. \end{aligned}$$

Setting $u = xy$, for $0 < p < 1$, we obtain

$$\begin{aligned} I_1 &:= \int_0^{\infty} g_n(y) \left[\int_0^{\frac{1}{y}} \frac{|\ln xy|^{\beta}}{|(xy)^{\lambda} - 1|} f_n(x) dx \right] dy \\ &= \int_1^{\infty} \left[\int_0^{\frac{1}{y}} \frac{(-\ln xy)^{\beta}}{1 - (xy)^{\lambda}} x^{\sigma + \frac{1}{pn} - 1} dx \right] y^{\sigma_1 - \frac{1}{qn} - 1} dy \\ &= \int_1^{\infty} y^{(\sigma_1 - \sigma) - \frac{1}{n} - 1} dy \int_0^1 \frac{(-\ln u)^{\beta}}{1 - u^{\lambda}} u^{\sigma + \frac{1}{pn} - 1} du \\ &\leq \tfrac{1}{\sigma - \sigma_1 + \frac{1}{n}} \int_0^1 \frac{(-\ln u)^{\beta}}{1 - u^{\lambda}} u^{\sigma - 1} du \leq \frac{k_{\lambda}^{(1)}(\sigma)}{\sigma - \sigma_1}, \end{aligned}$$

and then by (2.0.8), it follows that

$$\frac{k_\lambda^{(1)}(\sigma)}{\sigma-\sigma_1} \geq I_1 \geq M_1 J_1 = M_1 n. \tag{2.0.9}$$

By (2.0.9), for $n \to \infty$, in view of $k_\lambda^{(1)}(\sigma) < \infty$, $\sigma - \sigma_1 > 0$ and $M_1 > 0$, we find that

$$\infty > \frac{k_\lambda^{(1)}(\sigma)}{\sigma-\sigma_1} \geq \infty,$$

which is a contradiction.

If $\sigma_1 > \sigma$, then for

$$n \geq \frac{1}{|q|(\sigma_1-\sigma)} \ (n \in \mathbf{N}),$$

we define the following two functions

$$\widetilde{f}_n(x) := \begin{cases} 0, 0 < x < 1 \\ x^{\sigma-\frac{1}{pn}-1}, x \geq 1 \end{cases}, \quad \widetilde{g}_n(y) := \begin{cases} y^{\sigma_1+\frac{1}{qn}-1}, 0 < y \leq 1 \\ 0, y > 1 \end{cases},$$

and derive that

$$\begin{aligned} \widetilde{J}_1 &:= \left[\int_0^\infty x^{p(1-\sigma)-1}\widetilde{f}_n^p(x)dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma_1)-1}\widetilde{g}_n^q(y)dy\right]^{\frac{1}{q}} \\ &= \left(\int_1^\infty x^{-\frac{1}{n}-1}dx\right)^{\frac{1}{p}} \left(\int_0^1 y^{\frac{1}{n}-1}dy\right)^{\frac{1}{q}} = n. \end{aligned}$$

Setting $u = xy$, in view of $\sigma_1 + \frac{1}{qn} \geq \sigma$ $(q < 0)$, we obtain

$$\begin{aligned} \widetilde{I}_1 &:= \int_0^\infty \widetilde{f}_n(x)\left[\int_0^{\frac{1}{x}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|}\widetilde{g}_n(y)dy\right]dx \\ &= \int_1^\infty \left[\int_0^{\frac{1}{x}} \frac{(-\ln xy)^\beta}{1-(xy)^\lambda} y^{\sigma_1+\frac{1}{qn}-1}dy\right] x^{\sigma-\frac{1}{pn}-1}dx \\ &= \int_1^\infty x^{(\sigma-\sigma_1)-\frac{1}{n}-1}dx \int_0^1 \frac{(-\ln u)^\beta}{1-u^\lambda}u^{\sigma_1+\frac{1}{qn}-1}du \\ &\leq \frac{1}{\sigma_1-\sigma+\frac{1}{n}} \int_0^1 \frac{(-\ln u)^\beta}{1-u^\lambda}u^{\sigma-1}du \leq \frac{k_\lambda^{(1)}(\sigma)}{\sigma_1-\sigma}, \end{aligned}$$

and therefore by Fubini's theorem (cf. [**22**]) and (2.0.8), we find

$$\begin{aligned} \frac{k_1(\sigma)}{\sigma_1-\sigma} &\geq \widetilde{I}_1 = \int_0^\infty \widetilde{g}_n(y)\left[\int_0^{\frac{1}{y}} \frac{|\ln xy|^\beta}{|(xy)^\lambda-1|}\widetilde{f}_n(x)dx\right]dy \\ &\geq M_1\widetilde{J}_1 = M_1 n. \end{aligned} \tag{2.0.10}$$

By (2.0.10), for $n \to \infty$, we obtain that

$$\infty > \frac{k_\lambda^{(1)}(\sigma)}{\sigma_1 - \sigma} \geq \infty,$$

which is a contradiction.
Hence, we conclude that $\sigma_1 = \sigma$.
For $\sigma_1 = \sigma$, we get that $I_1 \geq M_1 J_1$ and thus

$$k_\lambda^{(1)}(\sigma) = \int_0^1 \frac{(-\ln u)^\beta}{1 - u^\lambda} u^{\sigma - 1} du \geq \int_0^1 \frac{(-\ln u)^\beta}{1 - u^\lambda} u^{\sigma + \frac{1}{pn} - 1} du \geq M_1.$$

This completes the proof of the lemma. □

Lemma 2.3. *If $\sigma < \lambda$ and for any nonnegative measurable functions $f(x)$ and $g(y)$ in $(0, \infty)$ the following inequality*

$$\begin{aligned} &\int_0^\infty g(y) \left[\int_{\frac{1}{y}}^\infty \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} f(x) dx \right] dy \\ \geq\ & M_2 \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma_1)-1} g^q(y) dy \right]^{\frac{1}{q}} \end{aligned} \tag{2.0.11}$$

holds true, then we have

$$\sigma_1 = \sigma \text{ and } M_2 \leq k_\lambda^{(2)}(\sigma).$$

Proof. If $\sigma_1 > \sigma$, $n \in \mathbf{N}$, we define two functions $\widetilde{f}_n(x)$ and $\widetilde{g}_n(y)$ as in Lemma 2.2 and derive that

$$\widetilde{J}_1 = \left[\int_0^\infty x^{p(1-\sigma)-1} \widetilde{f}_n^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma_1)-1} \widetilde{g}_n^q(y) dy \right]^{\frac{1}{q}} = n.$$

Setting $u = xy$, we obtain

$$\begin{aligned} \widetilde{I}_2 &:= \int_0^\infty \widetilde{g}_n(y) \left(\int_{\frac{1}{y}}^\infty \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} \widetilde{f}_n(x) dx \right) dy \\ &= \int_0^1 \left[\int_{\frac{1}{y}}^\infty \frac{(\ln xy)^\beta}{(xy)^\lambda - 1} x^{\sigma - \frac{1}{pn} - 1} dx \right] y^{\sigma_1 + \frac{1}{qn} - 1} dy \\ &= \int_0^1 y^{(\sigma_1 - \sigma) + \frac{1}{n} - 1} dy \int_1^\infty \frac{(\ln u)^\beta}{u^\lambda - 1} u^{\sigma - \frac{1}{pn} - 1} du \leq \frac{k_\lambda^{(2)}(\sigma)}{\sigma_1 - \sigma}, \end{aligned}$$

and thus by (2.0.11), it follows that

$$\frac{k_\lambda^{(2)}(\sigma)}{\sigma_1 - \sigma} \geq \widetilde{I}_2 \geq M_2 \widetilde{J}_1 = M_2 n. \tag{2.0.12}$$

By (2.0.12), for $n \to \infty$, we find

$$\infty > \frac{k_\lambda^{(2)}(\sigma)}{\sigma_1 - \sigma} \geq \infty,$$

which is a contradiction.

If $\sigma_1 < \sigma$, then for

$$n \geq \frac{1}{|q|(\sigma - \sigma_1)} \ (n \in \mathbf{N}),$$

we set two functions $f_n(x)$ and $g_n(y)$ as in Lemma 2.2 and find

$$J_1 = \left[\int_0^\infty x^{p(1-\sigma)-1} f_n^p(x)dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma_1)-1} g_n^q(y)dy\right]^{\frac{1}{q}} = n.$$

Setting $u = xy$, in view of $\sigma_1 - \frac{1}{qn} \leq \sigma$, we obtain

$$\begin{aligned} I_2 &:= \int_0^\infty f_n(x) \left[\int_{\frac{1}{x}}^\infty \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} g_n(y)dy\right] dx \\ &= \int_0^1 \left[\int_{\frac{1}{x}}^\infty \frac{(\ln xy)^\beta}{(xy)^\lambda - 1} y^{\sigma_1 - \frac{1}{qn} - 1} dy\right] x^{\sigma + \frac{1}{pn} - 1} dx \\ &= \int_0^1 x^{(\sigma - \sigma_1) + \frac{1}{n} - 1} dx \int_1^\infty \frac{(\ln u)^\beta}{u^\lambda - 1} u^{\sigma_1 - \frac{1}{qn} - 1} du \leq \frac{k_\lambda^{(2)}(\sigma)}{\sigma - \sigma_1}, \end{aligned}$$

and then by Fubini's theorem (cf. [**22**]) and (2.0.11), it follows that

$$\text{(2.0.13)} \quad \begin{aligned} \frac{k_\lambda^{(2)}(\sigma)}{\sigma - \sigma_1} &\geq I_2 = \int_0^\infty g_n(y) \left[\int_{\frac{1}{y}}^\infty \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} f_n(x)dx\right] dy \\ &\geq M_2 J_1 = M_2 n. \end{aligned}$$

By (2.0.13), for $n \to \infty$, we obtain that

$$\infty > \frac{k_\lambda^{(2)}(\sigma)}{\sigma - \sigma_1} \geq \infty,$$

which is a contradiction.

Hence, we conclude the fact that $\sigma_1 = \sigma$.

For $\sigma_1 = \sigma$, we get $\widetilde{I}_2 \geq M_2 \widetilde{J}_2$ and therefore it follows that

$$\text{(2.0.14)} \quad k_\lambda^{(2)}(\sigma) = \int_1^\infty \frac{(\ln u)^\beta}{u^\lambda - 1} u^{\sigma - 1} du \geq \int_1^\infty \frac{(\ln u)^\beta}{u^\lambda - 1} u^{\sigma - \frac{1}{pn} - 1} \geq M_2.$$

This completes the proof of the lemma. □

3. FIRST KIND OF THE REVERSE HARDY-TYPE INEQUALITIES

Theorem 3.4. *If $\sigma > 0$, then the following conditions are equivalent:*

(i) For any $f(x) \geq 0$, satisfying

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty,$$

we have the following reverse Hardy-type integral inequality of the first kind with the nonhomogeneous kernel:

$$\begin{aligned} J &:= \left\{ \int_0^\infty y^{p\sigma_1 - 1} \left[\int_0^{\frac{1}{y}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} f(x) dx \right]^p dy \right\}^{\frac{1}{p}} \\ &> M_1 \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}. \end{aligned}$$

(ii) For any $g(y) \geq 0$, satisfying

$$0 < \int_0^\infty y^{q(1-\sigma_1)-1} g^q(y) dy < \infty,$$

we have the following reverse Hardy-type integral inequality of the first kind with the nonhomogeneous kernel:

$$\begin{aligned} & \left\{ \int_0^\infty x^{q\sigma - 1} \left[\int_0^{\frac{1}{x}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} g(y) dy \right]^q dx \right\}^{\frac{1}{q}} \\ > \quad & M_1 \left[\int_0^\infty y^{q(1-\sigma_1)-1} g^q(y) dy \right]^{\frac{1}{q}}. \end{aligned} \tag{3.0.15}$$

(iii) For any $f(x), g(y) \geq 0$, satisfying

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty,$$

and

$$0 < \int_0^\infty y^{q(1-\sigma_1)-1} g^q(y) dy < \infty,$$

we have the following inequality:

$$\begin{aligned} I &:= \int_0^\infty g(y) \left[\int_0^{\frac{1}{y}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} f(x) dx \right] dy \\ &> M_1 \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma_1)-1} g^q(y) dy \right]^{\frac{1}{q}}. \end{aligned}$$

(iv) It holds

$$\sigma_1 = \sigma,\ M_1 \leq k_\lambda^{(1)}(\sigma).$$

If $\sigma_1 = \sigma$, then the constant

$$M_1 = k_\lambda^{(1)}(\sigma)$$

in (3.0.15), (3.0.15) and (3.0.16) is the best possible.

Proof. For $(i) \Rightarrow (iii)$:
By the reverse Hölder inequality (cf. [**23**]), we have

$$\begin{aligned} (3.0.16) \qquad I &= \int_0^\infty \left[y^{\sigma_1 - \frac{1}{p}} \int_0^{\frac{1}{y}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} f(x)dx \right] \left(y^{\frac{1}{p} - \sigma_1} g(y) \right) dy \\ &\geq J \left[\int_0^\infty y^{q(1-\sigma_1)-1} g^q(y)dy \right]^{\frac{1}{q}}. \end{aligned}$$

Then by (3.0.15), we deduce (3.0.16).

For $(iii) \Rightarrow (iv)$: By Lemma 2.2, we have $\sigma_1 = \sigma, M_1 \leq k_\lambda^{(1)}(\sigma)$.

For $(iv) \Rightarrow (i)$: Setting $u = xy$, we obtain the following weight function:

$$\begin{aligned} \omega_1(\sigma, y) &:= y^\sigma \int_0^{\frac{1}{y}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} x^{\sigma-1} dx \\ &= \int_0^1 \frac{(-\ln u)^\beta}{1 - u^\lambda} u^{\sigma-1} du = k_\lambda^{(1)}(\sigma)(y > 0). \end{aligned}$$

By the reverse weighted Hölder inequality and (3.0.17), for $y \in (0, \infty)$, we have

$$\begin{aligned} & \left(\int_0^{\frac{1}{y}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} f(x)dx \right)^p \\ &= \left\{ \int_0^{\frac{1}{y}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} \left[\frac{y^{(\sigma-1)/p}}{x^{(\sigma-1)/q}} f(x) \right] \left[\frac{x^{(\sigma-1)/q}}{y^{(\sigma-1)/p}} \right] dx \right\}^p \\ &\geq \int_0^{\frac{1}{y}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x)dx \left[\int_0^{\frac{1}{y}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} \frac{x^{\sigma-1}}{y^{(\sigma-1)q/p}} dx \right]^{p-1} \\ &= \left[\omega_1(\sigma, y) y^{q(1-\sigma)-1} \right]^{p-1} \int_0^{\frac{1}{y}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x)dx \\ &= (k_\lambda^{(1)}(\sigma))^{p-1} y^{-p\sigma+1} \int_0^{\frac{1}{y}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x)dx. \end{aligned}$$

If (3.0.17) takes the form of equality for some $y \in (0, \infty)$, then (cf. [**23**]) it follows that there exist constants A and B, such that they are not both zero, and

$$A \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x) = B \frac{x^{\sigma-1}}{y^{(\sigma-1)q/p}}\ a.e. \text{ in } \mathbf{R}_+.$$

Let us assume that $A \neq 0$ (otherwise $B = A = 0$). Then, it follows that

$$x^{p(1-\sigma)-1} f^p(x) = y^{q(1-\sigma)} \frac{B}{Ax} \; a.e. \text{ in } \mathbf{R}_+,$$

which contradicts the fact that

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty \, .$$

Hence, (3.0.17) takes the form of strict inequality.

For $\sigma_1 = \sigma$, by Fubini's theorem (cf. [**22**]) and the above result, we obtain that

$$\begin{aligned} J &> (k_\lambda^{(1)}(\sigma))^{\frac{1}{q}} \left\{ \int_0^\infty \left[\int_0^{\frac{1}{y}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x) dx \right] dy \right\}^{\frac{1}{p}} \\ &= (k_\lambda^{(1)}(\sigma))^{\frac{1}{q}} \left\{ \int_0^\infty \left[\int_0^{\frac{1}{x}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} \frac{y^{\sigma-1}}{x^{(\sigma-1)(p-1)}} dy \right] f^p(x) dx \right\}^{\frac{1}{p}} \\ &= (k_\lambda^{(1)}(\sigma))^{\frac{1}{q}} \left[\int_0^\infty \omega_1(\sigma, x) x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \\ &= k_\lambda^{(1)}(\sigma) \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} . \end{aligned}$$

Since

$$0 < M_1 \leq k_\lambda^{(1)}(\sigma) (< \infty),$$

(3.0.15) follows.

Therefore, the conditions (i), (iii) and (iv) are equivalent.

Since the conditions (i) and (iii) are equivalent, by Fubini's theorem we similarly obtain that

$$I = \int_0^\infty f(x) \left(\int_0^{\frac{1}{x}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} g(y) dy \right) dx,$$

and therefore it follows that the conditions (ii) and (iii) are also equivalent. Hence, the conditions (i), (ii), (iii) and (iv) are equivalent.

For the case when $\sigma_1 = \sigma$, if there exists a constant $M_1 \geq k_\lambda^{(1)}(\sigma)$, such that (3.0.15) is satisfied, then by Lemma 2.2 we obtain that $M_1 \leq k_\lambda^{(1)}(\sigma)$. Hence, the constant factor $M_1 = k_\lambda^{(1)}(\sigma)$ in (3.0.15) is the best possible.

The constant factor $M_1 = k_\lambda^{(1)}(\sigma)$ in (3.0.15) is still the best possible. Otherwise, by (3.0.16) (for $\sigma_1 = \sigma$), it would follow that the constant factor $M_1 = k_\lambda^{(1)}(\sigma)$ in (3.0.16) is not the best possible. Similarly, we can prove that the constant factor $M_1 = k_\lambda^{(1)}(\sigma)$ in (3.0.15) is the best possible.

This completes the proof of the theorem □

Remark 3.5. *For $\sigma_1 = \sigma = \lambda > 0$, the constant*

$$M_1 = k_\lambda^{(1)}(\lambda) = \frac{\Gamma(\beta+1)}{\lambda^{\beta+1}}\zeta(\beta+1)$$

in (4.0.30), (4.0.31) and (4.0.32) is the best possible.

In particular, for $\sigma = \sigma_1 = \frac{1}{p}\,(>0)$ in Theorem 3.4, we have:

Corollary 3.6. *The following conditions are equivalent:*

(i) For any $f(x) \geq 0$, satisfying

$$0 < \int_0^\infty x^{p-2} f^p(x)dx < \infty,$$

the following inequality holds true:

$$\left[\int_0^\infty \left(\int_0^{\frac{1}{y}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} f(x)dx\right)^p dy\right]^{\frac{1}{p}} > M_1 \left(\int_0^\infty x^{p-2} f^p(x)dx\right)^{\frac{1}{p}}. \tag{3.0.17}$$

(ii) For any $g(y) \geq 0$, satisfying

$$0 < \int_0^\infty g^q(y)dy < \infty,$$

the following inequality holds true:

$$\left[\int_0^\infty x^{q-2} \left(\int_0^{\frac{1}{x}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} g(y)dy\right)^q dx\right]^{\frac{1}{q}} > M_1 \left(\int_0^\infty g^q(y)dy\right)^{\frac{1}{q}}. \tag{3.0.18}$$

(iii) For any $f(x), g(y) \geq 0$, satisfying

$$0 < \int_0^\infty x^{p-2} f^p(x)dx < \infty,$$

and

$$0 < \int_0^\infty g^q(y)dy < \infty,$$

the following inequality holds true:

$$\int_0^\infty g(y) \left(\int_0^{\frac{1}{y}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} f(x)dx\right) dy > M_1 \left(\int_0^\infty x^{p-2} f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y)dy\right)^{\frac{1}{q}}. \tag{3.0.19}$$

(iv) We have

$$M_1 \leq k_\lambda^{(1)}(\frac{1}{p}).$$

The constant

$$M_1 = k_\lambda^{(1)}(\frac{1}{p}) = \frac{\Gamma(\beta+1)}{\lambda^{\beta+1}}\zeta(\beta+1, \frac{1}{p\lambda})$$

in (3.0.17), (3.0.18) and (3.0.19) is the best possible.

Setting

$$y = \frac{1}{Y},\ G(Y) = g(\frac{1}{Y})\frac{1}{Y^2}$$

in Theorem 3.4, and then replacing Y by y, we obtain the corollary below:

Corollary 3.7. *If $\sigma > 0$, then the following conditions are equivalent:*

(i) For any $f(x) \geq 0$, satisfying

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx < \infty,$$

the following inequality holds true:

$$\left[\int_0^\infty y^{-p\sigma_1-1}\left(\int_0^y \frac{|\ln x/y|^\beta}{|(x/y)^\lambda - 1|} f(x)dx\right)^p dy\right]^{\frac{1}{p}} > M_1\left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}. \tag{3.0.20}$$

(ii) For any $G(y) \geq 0$, satisfying

$$0 < \int_0^\infty y^{q(1+\sigma_1)-1} G^q(y)dy < \infty,$$

the following reverse Hardy-type integral inequality holds true:

$$\left[\int_0^\infty x^{q\sigma-1}\left(\int_0^x \frac{|\ln x/y|^\beta}{|(x/y)^\lambda - 1|} G(y)dy\right)^q dx\right]^{\frac{1}{q}} > M_1\left[\int_0^\infty y^{q(1+\sigma_1)-1} G^q(y)dy\right]^{\frac{1}{q}}. \tag{3.0.21}$$

(iii) For any $f(x), G(y) \geq 0$, satisfying

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx < \infty,$$

and

$$0 < \int_0^\infty y^{q(1+\sigma_1)-1} G^q(y)dy < \infty,$$

the following inequality holds true:

$$\int_0^\infty G(y)\left(\int_0^y \frac{|\ln x/y|^\beta}{|(x/y)^\lambda - 1|} f(x)dx\right) dy$$
$$> M_1 \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1+\sigma_1)-1} G^q(y)dy\right]^{\frac{1}{q}}. \tag{3.0.22}$$

(iv) We have

$$\sigma_1 = \sigma, M_1 \leq k_\lambda^{(1)}(\sigma).$$

If $\sigma_1 = \sigma$, then the constant $M_1 = k_\lambda^{(1)}(\sigma)$ in (3.0.20), (3.0.21) and (3.0.22) is the best possible.

For

$$g(y) = y^\lambda G(y) \text{ and } \mu = \lambda - \sigma_1$$

in Corollary 3.7, we have

Theorem 3.8. *If $\sigma > 0$, then the following conditions are equivalent:*

(i) For any $f(x) \geq 0$, satisfying

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx < \infty,$$

the following reverse Hardy-type inequality of the first kind with a homogeneous kernel holds true:

$$\left[\int_0^\infty y^{p\mu-1}\left(\int_0^y \frac{|\ln x/y|^\beta}{|x^\lambda - y^\lambda|} f(x)dx\right)^p dy\right]^{\frac{1}{p}}$$
$$> M_1 \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}. \tag{3.0.23}$$

(ii) For any $g(y) \geq 0$, satisfying

$$0 < \int_0^\infty y^{q(1-\mu)-1} g^q(y)dy < \infty,$$

the following reverse Hardy-type inequality of the first kind with a homogeneous kernel holds true:

$$\left[\int_0^\infty x^{q\sigma-1}\left(\int_0^x \frac{|\ln x/y|^\beta}{|x^\lambda - y^\lambda|} g(y)dy\right)^q dx\right]^{\frac{1}{q}}$$
$$> M_1 \left[\int_0^\infty y^{q(1-\mu)-1} g^q(y)dy\right]^{\frac{1}{p}}. \tag{3.0.24}$$

(iii) For any $f(x), g(y) \geq 0$, satisfying

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx < \infty,$$

and

$$0 < \int_0^\infty y^{q(1-\mu)-1} g^q(y)dy < \infty,$$

the following inequality holds true:

$$\begin{aligned} &\int_0^\infty g(y) \left(\int_0^y \frac{|\ln x/y|^\beta}{|x^\lambda - y^\lambda|} f(x)dx \right) dy \\ &> M_1 \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\mu)-1} g^q(y)dy \right]^{\frac{1}{q}}. \end{aligned} \tag{3.0.25}$$

(iv) We have

$$\mu + \sigma = \lambda, \ M_1 \leq k_\lambda^{(1)}(\sigma).$$

If $\mu+\sigma = \lambda$, then the constant $M_1 = k_\lambda^{(1)}(\sigma)$ in (3.0.23), (3.0.24) and (3.0.25) is the best possible.

In particular, for

$$\lambda = 1, \ \sigma = \frac{1}{p}, \ \mu = \frac{1}{q}$$

in Theorem 3.8, we obtain the corollary below:

Corollary 3.9. *The following conditions are equivalent:*

(i) For any $f(x) \geq 0$, satisfying

$$0 < \int_0^\infty x^{p-2} f^p(x)dx < \infty,$$

we have the following inequality:

$$\left[\int_0^\infty y^{p-2} \left(\int_0^y \frac{|\ln \frac{x}{y}|^\beta}{|x-y|} f(x)dx \right)^p dy \right]^{\frac{1}{p}} > M_1 \left(\int_0^\infty x^{p-2} f^p(x)dx \right)^{\frac{1}{p}}. \tag{3.0.26}$$

(ii) For any $g(y) \geq 0$, satisfying

$$0 < \int_0^\infty y^{q-2} g^q(y)dy < \infty,$$

we have the following inequality:

$$\left[\int_0^\infty x^{q-2} \left(\int_0^x \frac{|\ln \frac{x}{y}|^\beta}{|x-y|} g(y)dy \right)^q dx \right]^{\frac{1}{q}} > M_1 \left(\int_0^\infty y^{q-2} g^q(y)dy \right)^{\frac{1}{p}}. \tag{3.0.27}$$

(iii) For any $f(x), g(y) \geq 0$, satisfying

$$0 < \int_0^\infty x^{p-2} f^p(x)dx < \infty,$$

and

$$0 < \int_0^\infty y^{q-2} g^q(y)dy < \infty,$$

we have the following inequality:

$$\begin{aligned} I &= \int_0^\infty g(y)\left(\int_0^y \frac{|\ln x/y|^\beta}{|x-y|} f(x)dx\right) dy \\ &> M_1 \left(\int_0^\infty x^{p-2} f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty y^{q-2} g^q(y)dy\right)^{\frac{1}{q}}. \end{aligned} \tag{3.0.28}$$

(iv) We have $M_1 \leq k_1^{(1)}(\frac{1}{p})$.

The constant

$$M_1 = k_1^{(1)}(\frac{1}{p}) = \Gamma(\beta+1)\zeta(\beta+1, \frac{1}{p})$$

in (3.0.26), (3.0.27) and (3.0.28) is the best possible.

4. SECOND KIND OF THE REVERSE HARDY-TYPE INEQUALITIES

Similarly to as we worked previously, we obtain the following weight function:

$$\begin{aligned} \omega_2(\sigma, y) &:= y^\sigma \int_{\frac{1}{y}}^\infty \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} x^{\sigma-1} dx \\ &= \int_1^\infty \frac{(\ln u)^\beta}{u^\lambda - 1} u^{\sigma-1} du = k_\lambda^{(2)}(\sigma) \ (y > 0). \end{aligned} \tag{4.0.29}$$

In view of Lemma 2.3 and following the same method, we have:

Theorem 4.10. *If $\sigma < \lambda$, then the following conditions are equivalent:*

(i) For any $f(x) \geq 0$, satisfying

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx < \infty,$$

the following reverse Hardy-type inequality of the second kind with a nonhomogeneous kernel holds true:

$$\begin{aligned} &\left[\int_0^\infty y^{p\sigma_1 - 1}\left(\int_{\frac{1}{y}}^\infty \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} f(x)dx\right)^p dy\right]^{\frac{1}{p}} \\ &> M_2 \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}. \end{aligned} \tag{4.0.30}$$

(ii) For any $g(y) \geq 0$, satisfying

$$0 < \int_0^\infty y^{q(1-\sigma_1)-1} g^q(y) dy < \infty,$$

the following reverse Hardy-type integral inequality of the second kind with a non-homogeneous kernel holds true:

$$\left[\int_0^\infty x^{q\sigma-1}\left(\int_{\frac{1}{x}}^\infty \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} g(y)dy\right)^q dx\right]^{\frac{1}{q}} > M_2 \left[\int_0^\infty y^{q(1-\sigma_1)-1} g^q(y)dy\right]^{\frac{1}{q}}. \tag{4.0.31}$$

(iii) For any $f(x), g(y) \geq 0$, satisfying

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty,$$

and

$$0 < \int_0^\infty y^{q(1-\sigma_1)-1} g^q(y) dy < \infty,$$

the following inequality holds true:

$$\int_0^\infty g(y)\left(\int_{\frac{1}{y}}^\infty \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} f(x)dx\right) dy > M_2 \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma_1)-1} g^q(y)dy\right]^{\frac{1}{q}}. \tag{4.0.32}$$

(iv) We have

$$\sigma_1 = \sigma, M_2 \leq k_\lambda^{(2)}(\sigma).$$

If $\sigma_1 = \sigma$, then the constant $M_2 = k_\lambda^{(2)}(\sigma)$ in (4.0.30), (4.0.31) and (4.0.32) is the best possible.

Remark 4.11. *For $\sigma_1 = \sigma = 0 < \lambda$, the constant*

$$M_2 = k_\lambda^{(2)}(0) = \frac{\Gamma(\beta+1)}{\lambda^{\beta+1}}\zeta(\beta+1)$$

in (4.0.30), (4.0.31) and (4.0.32) is the best possible.

In particular, for $\sigma = \sigma_1 = \frac{1}{p}$ in Theorem 4.10, we have

Corollary 4.12. *If $\lambda > \frac{1}{p}$, then the following conditions are equivalent:*

(i) For any $f(x) \geq 0$, satisfying

$$0 < \int_0^\infty x^{p-2} f^p(x) dx < \infty,$$

we have the following inequality:

$$\left[\int_0^\infty \left(\int_{\frac{1}{y}}^\infty \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} f(x) dx\right)^p dy\right]^{\frac{1}{p}} > M_2 \left(\int_0^\infty x^{p-2} f^p(x) dx\right)^{\frac{1}{p}}. \tag{4.0.33}$$

(ii) For any $g(y) \geq 0$, satisfying

$$0 < \int_0^\infty g^q(y) dy < \infty,$$

we have the following inequality:

$$\left[\int_0^\infty x^{q-2} \left(\int_{\frac{1}{x}}^\infty \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} g(y) dy\right)^q dx\right]^{\frac{1}{q}} > M_2 \left[\int_0^\infty g^q(y) dy\right]^{\frac{1}{q}}. \tag{4.0.34}$$

(iii) For any $f(x), g(y) \geq 0$, satisfying

$$0 < \int_0^\infty x^{p-2} f^p(x) dx < \infty,$$

and

$$0 < \int_0^\infty g^q(y) dy < \infty,$$

we have the following inequality:

$$\begin{aligned} &\int_0^\infty g(y) \left(\int_{\frac{1}{y}}^\infty \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} f(x) dx\right) dy \\ > \quad & M_2 \left(\int_0^\infty x^{p-2} f^p(x) dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y) dy\right)^{\frac{1}{q}}. \end{aligned} \tag{4.0.35}$$

(iv) We have

$$M_2 \leq k_\lambda^{(2)}(\frac{1}{p}).$$

The constant

$$M_2 = k_\lambda^{(2)}(\frac{1}{p}) = \frac{\Gamma(\beta+1)}{\lambda^{\beta+1}} \zeta(\beta+1, \frac{p\lambda - 1}{p\lambda})$$

in (4.0.33), (4.0.34) and (4.0.35) is the best possible.

Setting

$$y = \frac{1}{Y}, \quad G(Y) = g(\frac{1}{Y}) \frac{1}{Y^2}$$

in Theorem 4.10, and then replacing Y by y, we obtain the corollary below:

Corollary 4.13. *If $\sigma < \lambda$, then the following conditions are equivalent:*

(i) For any $f(x) \geq 0$, satisfying

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx < \infty,$$

we have the following inequality:

$$\left[\int_0^\infty y^{-p\sigma_1-1}\left(\int_y^\infty \frac{|\ln x/y|^\beta}{|(x/y)^\lambda - 1|} f(x)dx\right)^p dy\right]^{\frac{1}{p}} > M_2\left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}. \tag{4.0.36}$$

(ii) For any $G(y) \geq 0$, satisfying

$$0 < \int_0^\infty y^{q(1+\sigma_1)-1} G^q(y)dy < \infty,$$

we have the following reverse Hardy-type integral inequality:

$$\left[\int_0^\infty x^{q\sigma-1}\left(\int_x^\infty \frac{|\ln x/y|^\beta}{|(x/y)^\lambda - 1|} G(y)dy\right)^q dx\right]^{\frac{1}{q}} > M_2\left[\int_0^\infty y^{q(1+\sigma_1)-1} G^q(y)dy\right]^{\frac{1}{q}}. \tag{4.0.37}$$

(iii) For any $f(x), G(y) \geq 0$, satisfying

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx < \infty,$$

and

$$0 < \int_0^\infty y^{q(1+\sigma_1)-1} G^q(y)dy < \infty,$$

we have the following inequality:

$$\int_0^\infty G(y)\left(\int_y^\infty \frac{|\ln x/y|^\beta}{|(x/y)^\lambda - 1|} f(x)dx\right) dy > M_2\left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{q(1+\sigma_1)-1} G^q(y)dy\right]^{\frac{1}{q}}. \tag{4.0.38}$$

(iv) We have

$$\sigma_1 = \sigma, M_2 \leq k_\lambda^{(2)}(\sigma).$$

If $\sigma_1 = \sigma$, then the constant $M_2 = k_\lambda^{(2)}(\sigma)$ in (4.0.36), (4.0.37) and (4.0.38) is the best possible.

For $g(y) = y^\lambda G(y)$ and $\mu = \lambda - \sigma_1$ in Corollary 4.13, we obtain the theorem below:

Theorem 4.14. *If $\sigma < \lambda$, then the following conditions are equivalent:*

(i) For any $f(x) \geq 0$, satisfying

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx < \infty,$$

the following reverse Hardy-type integral inequality of the second kind with a homogeneous kernel holds true:

$$\left[\int_0^\infty y^{p\mu-1}\left(\int_y^\infty \frac{|\ln x/y|^\beta}{|x^\lambda - y^\lambda|} f(x)dx\right)^p dy\right]^{\frac{1}{p}} > M_2\left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}. \tag{4.0.39}$$

(ii) For any $g(y) \geq 0$, satisfying

$$0 < \int_0^\infty y^{q(1-\mu)-1} g^q(y)dy < \infty,$$

the following reverse Hardy-type inequality of the second kind with a homogeneous kernel holds true:

$$\left[\int_0^\infty x^{q\sigma-1}\left(\int_x^\infty \frac{|\ln x/y|^\beta}{|x^\lambda - y^\lambda|} g(y)dy\right)^q dx\right]^{\frac{1}{q}} > M_2\left[\int_0^\infty y^{q(1-\mu)-1} g^q(y)dy\right]^{\frac{1}{p}}. \tag{4.0.40}$$

(iii) For any $f(x), g(y) \geq 0$, satisfying

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx < \infty,$$

and

$$0 < \int_0^\infty y^{q(1-\mu)-1} g^q(y)dy < \infty,$$

we have the following inequality:

$$\int_0^\infty g(y)\left(\int_y^\infty \frac{|\ln x/y|^\beta}{|x^\lambda - y^\lambda|} f(x)dx\right) dy > M_2\left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{q(1-\mu)-1} g^q(y)dy\right]^{\frac{1}{q}}. \tag{4.0.41}$$

(iv) We have

$$\mu + \sigma = \lambda,\ M_2 \leq k_\lambda^{(2)}(\sigma).$$

If $\mu+\sigma=\lambda$, then the constant $M_2=k_\lambda^{(2)}(\sigma)=k_\lambda^{(1)}(\mu)$ in (4.0.39), (4.0.40) and (4.0.41) is the best possible.

In particular, for $\lambda=1, \sigma=\frac{1}{q}, \mu=\frac{1}{p}$ in Theorem 4.14, we deduce the corollary below:

Corollary 4.15. *The following conditions are equivalent:*

(i) For any $f(x)\geq 0$, satisfying

$$0<\int_0^\infty f^p(x)dx<\infty,$$

we have the following inequality:

$$\left[\int_0^\infty \left(\int_y^\infty \frac{|\ln x/y|^\beta}{|x-y|} f(x)dx\right)^p dy\right]^{\frac{1}{p}} > M_2\left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}}. \tag{4.0.42}$$

(ii) For any $g(y)\geq 0$, satisfying

$$0<\int_0^\infty g^q(y)dy<\infty,$$

we have the following inequality:

$$\left[\int_0^\infty \left(\int_x^\infty \frac{|\ln x/y|^\beta}{|x-y|} g(y)dy\right)^q dx\right]^{\frac{1}{q}} > M_2\left(\int_0^\infty g^q(y)dy\right)^{\frac{1}{p}}. \tag{4.0.43}$$

(iii) For any $f(x), g(y)\geq 0$, satisfying

$$0<\int_0^\infty f^p(x)dx<\infty,$$

and

$$0<\int_0^\infty g^q(y)dy<\infty,$$

we have the following inequality:,

$$\begin{aligned} &\int_0^\infty g(y)\left(\int_y^\infty \frac{|\ln x/y|^\beta}{|x-y|} f(x)dx\right)dy \\ > \ & M_2\left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty g^q(y)dy\right)^{\frac{1}{q}}. \end{aligned} \tag{4.0.44}$$

(iv) We have

$$M_2\leq k_1^{(2)}(\frac{1}{q}).$$

The constant

$$M_2 = k_1^{(2)}(\frac{1}{q}) = \Gamma(\beta+1)\zeta(\beta+1, \frac{1}{p})$$

in (4.0.42), (4.0.43) and (4.0.44) is the best possible.

Acknowledgments.
M. Th. Rassias: I would like to express my gratitude to the J. S. Latsis Foundation for their financial support provided under the auspices of my current "Latsis Foundation Senior Fellowship" position.
B. C. Yang: This work is supported by the National Natural Science Foundation (No. 61370186, No. 61640222), and Appropriative Researching Fund for Professors and Doctors, Guangdong University of Education (No. 2015ARF25). We are grateful for their help.
A. Raigorodskii: I would like to acknowledge financial support from the grant NSh-6760.2018.1.

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(Received 30.01.2018)
(Revised 22.05.2018)

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