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FIXED POINT THEOREM IN ULTRAMETRIC SPACE

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We give a class of generalizations of a number of known fixed point theorems to mappings defined on an ultrametric space and non-Archimedean normed space which are endowed with a graph. We also investigate the relationship between weak connectivity and the existence of fixed points for these mappings.

1. INTRODUCTION AND PRELIMINARIES

The fixed point theorems are used to determine conditions for the existence of solutions of polynomial differential equations of any order, or even of systems of such equations, see Priess-Crampe and Ribenboim [**11, 12**]. Methods of ultrametric dynamics also find applications in the study of differential equations over rings of power series, as in the work of van der Hoeven, for example see his lecture notes [**16**]. A very different and unexpected application of ultrametric dynamics is found in the determination of solutions of the famous Fermat equation in square matrices with entries in a p-adic field, see [**14**]. Programs with positive clauses were shown to have models by means of the fixed point theorem of Knaster and Tarski about monotonic self-maps in a complete lattice. More general programs, involving negation in clauses lead to the ultrametric space of maps from the Herbrand base with values 0, 1; in this space the values of the distance are the subsets of the Herbrand base. The fixed point of the immediate consequence operator gives conditions for the existence of models for the program, see Priess-Crampe and Ribenboim [**9, 13**] and Hitzler and Seda [**4, 5**].

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Fixed point theory has a wide application in almost all fields of quantitative sciences such as economics, biology, physics, chemistry, computer science and many branches of engineering. Fixed point theorems for monotone single-valued mappings in a partially ordered metric spaces have been widely investigated. These theorems are hybrids of the two most fundamental and useful theorems in fixed point theory, namely, the Banach contraction principle [**1**] and Tarski's fixed point theorem [**15**]. In order to generalizing the Banach contraction principle for multivalued mapping mappings in metric spaces:
The founding father of non-Archimedean functional analysis was Monna, who wrote a series of paper in 1943. A milestone was reached in 1978 at the publication of van Rooij's book [**17**], the most extensive treatment on non-Archimedean Banach spaces existing in the literature. For more details the reader is referred to [**17**]. The idea is reasonable to try and generalize ordinary functional analysis by replacing $\mathbb{R}$ and $\mathbb{C}$ by other topological field. This ought to give a new insight in analysis by showing what properties of the scalar field are crucial for certain classical theorems. For this topoligical field Monna choose a field $\mathbb{K}$, provided with real valued absolute value function $|\cdot|$ such that $\mathbb{K}$ is complete relative to the metric induced by $|\cdot|$. Adding the condition that, as a topological field, $\mathbb{K}$ is neither $\mathbb{R}$ nor $\mathbb{C}$, Monna proved the so-called strong triangle inequality

$$|x+y| \leq \max\{|x|, |y|\} \qquad (x, y \in \mathbb{K}).$$

This formula is essential to theorems in non-Archimedean functional analysis. Among other things it implies that $\mathbb{K}$ is totally disconnected and cannot be made into a totally ordered field [**17**]. We first recall some basic notions in ultrametric spaces and non-Archimedean normed spaces. Van Rooij [**17**] introduced the concept of ultrametric space as follows:
Let (X, d) be a metric space. (X, d) is called an ultrametric space if the metric d satisfies the strong triangle inequality, i.e, for all $x, y, z \in X$:

$$d(x, y) \leq \max\{d(x, z), d(y, z)\},$$

in this case d is said to be ultrametric [**17**].

We denote by $B(x, r)$, the closed ball

$$B(x, r) = \{y \in X : d(x, y) \leq r\},$$

where $x \in X$ and we let $r \geq 0$, $B(x, 0) = \{x\}$. A known characteristic property of ultrametric spaces is the following:

If $x, y \in X$, $0 \leq r \leq s$ and $B(x, r) \cap B(y, s) \neq \emptyset$, then $B(x, r) \subset B(y, s)$.

An ultrametric space (X, d) is said to be spherically complete if every shrinking collection of balls in X has a nonempty intersection. A non-Archimedean valued field is a field $\mathbb{K}$ equipped with a function (valuation) $|\cdot|$ from $\mathbb{K}$ into $[0, \infty)$ such that $|x| = 0$ if and only if $x = 0$, $|x + y| \leq \max\{|x|, |y|\}$ and $|xy| = |x||y|$ for all $x, y \in \mathbb{K}$. Clearly, $|1| = |-1| = 1$ and $|n.1_{\mathbb{K}}| \leq 1$ for all $n \in \mathbb{N}$ [**17**].

An example of a non-Archimedean valuation is the mapping $|\cdot|$ taking each point of an arbitrary field but 0 into 1 and $|0| = 0$. This valuation is called trivial. The set $\{|x| : x \in \mathbb{K}, x \neq 0\}$ is a subgroup of the multiplicative group $(0, +\infty)$ and it is called the value group of the valuation. The valuation is called trivial, discrete, or dense accordingly as its value group is $\{1\}$, a discrete subset of $(0, +\infty)$, or a dense subset of $(0, \infty)$, respectively [**17**].

Definition 0.1 ([**17**])**.** Let $\mathbb{K}$ be a non-Archimedean valued field. A norm on a vector space X over $\mathbb{K}$ is a map $\|.\|$ from X into $[0, \infty)$ with the following properties:

1) $\|x\| \neq 0$ if $x \in E \setminus \{0\}$;
2) $\|x + y\| \leq \max\{\|x\|, \|y\|\} \qquad (x, y \in X)$;
3) $\|\alpha x\| = |\alpha|\|x\| \qquad (\alpha \in \mathbb{K}, x \in X)$.

It is clear that the metric induced by a non-Archimedean norm $|\cdot|$ is an ultrametric.

In1993, Petalas and Vidalis in [**8**] presented a generalization of a well-known fixed point theorem for the class of spherically complete non-Archimedean normed spaces, and in 2000 Sibylla Priess-Crampe and Ribenboim in [**10**] obtained similar results in ultrametric space, but the proof of these theorems weren't constructive. In 2012 Kirk and Shahzad in [**7**] gave more constructive proofs of these theorems and strengthened the conclusions as follow:

Theorem 0.2 ([**7**])**.** *Suppose that (X, d) is a spherically complete ultrametric space and suppose $T : X \longrightarrow X$ is strongly contractive. Then every closed ball of the form*

$$B(x, d(x, Tx)) \qquad (x \in X)$$

contains a fixed point of T.

In this paper, motivated by the work of Petalas and Vidalis [**8**], Kirk and Shahzad [**7**] and Jachymski [**6**], we introduce two new strongly contractive conditions for mappings on complete ultrametric spaces and non-Archimedean normed spaces and using these strongly contractive conditions, obtain some fixed point theorems.

2. MAIN RESULTS

Let $G = (V(G), E(G))$ be a directed graph, by $\tilde{G}$ we denote the undirected graph obtained from G by ignoring the direction of edges. If x and y are two vertices in a graph G, then a path in G from x to y of length n is a sequence $(x_i)_{i=0}^n$ of $n+1$ vertices such that $x_0 = x$, $x_n = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, \ldots, n$. A graph G is called connected if there is a path between any two vertices, G is weakly connected if $\tilde{G}$ is connected.

Subsequently, in this paper X is a complete ultrametric spaces or non-Archimedean normed space, and Δ is the diagonal of the Cartesian product $X \times X$. G is directed graph such that the set $V(G)$ of its vertices coincides with X, and the set $E(G)$ of its edges such that $\Delta \subseteq E(G)$ and G has no parallel edges. Moreover, we may treat G as a weighted graph by assigning to each edge the distance between its vertices. We give our first results with constructive proofs. In fact, we extend Kirk and Shahzad's result on strongly contractive mappings on ultrametric spaces and non-Archimedean normed spaces endowed with a graph.

Definition 0.3. Let (X, d) be a metric space endowed with a graph G. We say that a mapping $T : X \longrightarrow X$ is G-strongly contractive if

1) T preserves the edges of G, i.e., $(x, y) \in E(G)$ implies $(Tx, Ty) \in E(G)$ for all $x, y \in X$;

2) $d(Tx, Ty) < d(x, y)$ for all distinct $x, y \in X$ with $(x, y) \in E(G)$.

Definition 0.4. Suppose that (X, d) is an ultrametric space endowed with a graph G and $T : X \longrightarrow X$ a mapping. We would say that a $B(x, r)$ is $G - T$-invariant if for any $u \in B(x, r)$ that $(u, x) \in E(G)$, then

$$Tu \in B(x, r).$$

Theorem 0.5. *Let (X, d) be an ultrametric space endowed with a graph G and a G-strongly contractive mapping $T : X \longrightarrow X$ satisfy the following conditions:*

(a) *There exists an $x_0 \in X$ such that $d(x_0, Tx_0) < 1$;*

(b) *If $x \in X$ is such that $d(x, Tx) < 1$, then there exists a path in $\tilde{G}$ between x and Tx with vertices in $B(x, d(x, Tx))$;*

(c) *If $\{B(x_n, d(x_n, Tx_n))\}$ is a sequence of nonincreasing closed balls in X, and for each $n \geq 1$ there exists a path in $\tilde{G}$ between x_n and x_{n+1} with vertices in $B(x_n, d(x_n, Tx_n)$, then there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ and a $z \in \bigcap_{k=1}^{\infty} B(x_{n_k}, r_{n_k})$ such that for each $k \geq 1$, there exists a path in $\tilde{G}$ between x_{n_k} and z with vertices in $B(x_{n_k}, d(x_{n_k}, Tx_{n_k}))$.*

Then T has a fixed point in each closed ball of the form $B(x, d(x, Tx))$ where $x \in X_0 = \{z \in X : d(z, Tz) < 1\}$.

Proof. Let $x \in X_0$, $r = d(x, Tx)$ and $u \in B(x, r)$ such that $(u, x) \in E(G)$. Then we have

$$\begin{aligned} d(u, Tu) &\leq \max\{d(u, x), d(x, Tx), d(Tx, Tu)\} \\ &\leq \max\{d(u, x), d(x, Tx), d(u, x)\} \\ &= d(x, Tx) \end{aligned}$$

and

$$B(x, d(x, Tx)) \cap B(u, d(u, Tu)) \neq \emptyset.$$

Therefore, $B(u, d(u, Tu)) \subset B(x, d(x, Tx))$, so $Tu \in B(x, d(x, Tx))$. This means that $B(x, d(x, Tx))$ is $G-T$-invariant for all $x \in X_0$. Now fix an $x_0 \in X_0$. Put $x_1 = x_0$, $r_1 = d(x_1, Tx_1)$, If $r_1 = 0$, then x_1 is a fixed point of T, and the proof is complete. Otherwise put

$E_1 = \{x \in B(x_1, r_1) \mid$
there is a path in $\tilde{G}$ between x and x_1 with vertices in $B(x_1, r_1)\}$.

Obviously x_1 and Tx_1 belong to E_1, let

$$\mu_1 = \inf\{d(x, Tx) : x \in E_1\}.$$

If $r_1 = \mu_1$, then x_1 is a fixed point of T, since otherwise, from $d(x_1, Tx_1) < 1$ and (b), there exists a path $(x_1 = y_0, y_1, \ldots, y_n = Tx_1)$ in $B(x_1, r_1)$ from x_1 to Tx_1 and since $B(x_1, r_1)$ is $G-T$-invariant, it follows that $T^2x_1 \in B(x_1, r_1)$ and have

$$\begin{aligned}
\mu_1 &\leq d(T^2x_1, Tx_1) \\
&\leq \max\{d(Tx_1, Ty_1), d(Ty_1, Ty_2), \ldots, d(Ty_{n-1}, T^2x_1)\} \\
&< \max\{d(x_1, y_1), d(y_1, y_2), \ldots, d(y_{n-1}, Tx_1)\} \\
&\leq d(x_1, Tx_1) \\
&= r_1,
\end{aligned}$$

which is a contradiction. So, Finally, let $\mu_1 < r_1$. Suppose $\{\epsilon_n\}$ is a sequence of positive numbers such that $\lim_{n\longrightarrow\infty} \epsilon_n = 0$. Choose $x_2 \in B(x_1, r_1)$ such that there exists a path in $\tilde{G}$ between x_1 and x_2, and

$$r_2 = d(x_2, Tx_2) < \min\{r_1, \mu_1 + \epsilon_1\}.$$

By the same argument, If $r_2 = \mu_2$, then x_2 is a fixed point of T. Otherwise, there exists an $x_3 \in B(x_2, r_2)$ such that there exists a path in $\tilde{G}$ between x_2 and x_3, and

$$r_3 = d(x_3, Tx_3) < \min\{r_2, \mu_2 + \epsilon_2\}.$$

Having defined $x_n \in X$. Let

$E_n = \{x \in B(x_n, r_n) \mid$
there is a path in $\tilde{G}$ between x and x_n with vertices in $B(x_n, r_n)\}$
and

$$\mu_n = \inf\{d(x, Tx) : x \in E_n\}.$$

If $r_n = 0$ or $\mu_n = r_n$, by the same argument for $n = 1$, we are finished. Otherwise, choose $x_{n+1} \in B(x_n, r_n)$ such that exists path between x_n and x_{n+1} and

$$r_{n+1} := d(x_{n+1}, Tx_{n+1}) < \min\{r_n, \mu_n + \epsilon_n\}.$$

If this process ends after a finite number of steps, then we are finished. Otherwise, we obtain a nonincreasing sequence of nontrivial closed $\{B(x_n, d(x_n, Tx_n))\}$. Since $\{r_n\}$ is nonincreasing, $r := \lim_{n\longrightarrow\infty} r_n$ exists. Also, $\{\mu_n\}$ is nondecreasing and bounded above, thus, $\mu := \lim_{n\longrightarrow\infty} \mu_n$ also exists. Hence by (c), there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ and $z \in \cap_{k=1}^{\infty} B\big(x_{n_k}, r_{n_k}\big)$ such that for each $k \in \mathbb{N}$ there exists a path in $\tilde{G}$ between x_{n_k} and z with vertices in $B(x_{n_k}, d(x_{n_k}, Tx_{n_k}))$. Since $B(x_{n_k}, r_{n_k})$ is $G-T$-invariant for all $k \geq 1$, it follows that

$$Tz \in B(x_{n_k}, r_{n_k}),$$

for all $k \geq 1$. Therefore,

$$\begin{aligned} d(z, Tz) &\leq \max\{d(x_{n_k}, z), d(x_{n_k}, Tz)\} \\ &\leq r_{n_k} \end{aligned}$$

for all $k \geq 1$. Thus,

$$\begin{aligned} \mu_{n_k} &\leq d(z, Tz) \\ &\leq r \\ &\leq r_{n_k+1} \\ &\leq \mu_{n_k} + \epsilon_{n_k} \end{aligned}$$

for all $k \geq 1$. Letting $k \longrightarrow \infty$, we obtain $d(z, Tz) = r = \mu$. On the other hand, if $x \in B(z, d(z, Tz))$, then for each $k \in \mathbb{N}$,

$$d(x, z) \leq d(z, Tz) \leq r_{n_k}$$

for all $k \geq 1$. Therefore,

$$d(x, x_{n_k}) \leq \max\{d(x, z), d(x_{n_k}, z)\} \leq r_{n_k}$$

for all $k \geq 1$. Hence, $x \in B(x_{n_k}, r_{n_k})$ all $k \geq 1$. Now let $x \in B(z, d(z, Tz))$ and there exists a path between x and z. Thus there exists a path in $B(x_{n_k}, r_{n_k})$ between x_{n_k} and x for all $k \geq 1$. Hence $\mu_{n_k} \leq d(x, Tx)$ for all $k \geq 1$. Therefore, for each $k \in \mathbb{N}$, $\mu_{n_k} \leq r_{n_k}$. Hence

$$\inf\{d(x, Tx) : x \in B(z, d(z, Tz))\} = d(z, Tz) = r.$$

Since for all $k \geq 1$ we have

$$d(z, Tz) \leq r_{n_k} < 1,$$

thus, it follows by (b) that there exists a path in $B(z, d(z, Tz))$ from z to Tz. We claim that $r = 0$. Suppose on the contrary that $r > 0$ and suppose $(z = y_0, y_1, y_2, \ldots, y_N = Tz)$ be a path in $B(z, d(z, Tz))$ from z to Tz. Since $B(z, d(z, Tz))$ is $G-T$-invariant, it follows that $T^2 z \in B(z, d(z, Tz))$, we have

$$\begin{aligned} d(Tz, T^2z) &\leq \max\{d(Tz, Ty_1), d(Ty_1, Ty_2), \ldots, d(Ty_{n-1}, T^2z)\} \\ &< \max\{d(z, y_1), d(y_1, y_2), \ldots, d(y_{n-1}, T(z)) \\ &\leq d(z, Tz), \end{aligned}$$

which is a contradiction. Hence, $r = 0$ and $z = Tz$. □

Corollary 0.6. *Suppose that $(X, d, \preceq)$ is a partially ordered ultrametric space, $G = (V(G), E(G))$ is a directed graph with $V(G) = X$ and $E(G) = \{(x, y) \in X \times X : x \preceq y\}$ and $T : X \longrightarrow X$ is a G-strongly contractive mapping such that (a), (b) and (c) in Theorem 0.5 hold. Then for every $x \in X$ with $d(x, Tx) < 1$, the closed ball $B\big(x, d(x, Tx)\big)$ contains a fixed point of T.*

Remark 0.7. Recall that the metric induced by a non-Archimedean norm is an ultrametric, so if we replace the ultrametric space (X, d) in Theorem 0.5 and Corollary 0.6 with a non-Archimedean normed space $(X, \|.\|)$, then their results are valid.

In the previous theorem and it's corollary, we obtained some results on the closed balls $B(x, d(x, Tx))$ with $d(x, Tx) < 1$. In the following theorem we obtain these results on every weakly connected ball of the form $B(x, d(x, Tx))$ by adding weak connectivity.

Theorem 0.8. *Let (X, d) be a spherically complete ultrametric space endowed with a graph G and $T : X \longrightarrow X$ be a G-strongly contractive mapping. If every closed ball in X is weakly connected, then every closed ball $B(x, d(x, Tx))$ contains a fixed point of T.*

Proof. Let $x \in X$ and $u \in B(x, d(x, Tx))$. Since $B(x, d(x, Tx))$ is weakly connected, there exists a path $(x = x_0, x_1, \ldots, x_N = u)$ in $\tilde{G}$ from x to u with vertices in $B(x, d(x, Tx))$. Thus, we have

$$\begin{aligned} d(x, Tu) &\leq \max\{d(x, Tx), d(Tx, Tx_1), \ldots, d(Tx_{n-1}, Tu) \\ &\leq \max\{d(x, Tx), d(x, x_1), \ldots, d(x_{n-1}, u) \\ &= d(x, Tx). \end{aligned}$$

So $Tu \in B(x, d(x, Tx))$. Therefore, the closed ball $B(x, d(x, Tx))$ is T-invariant. Now let

$$\Gamma = \{B(y, d(y, Ty)) : y \in B(x, d(x, Tx))\},$$

and consider Γ with the inverse inclusion. Then Zorn's lemma shows that Γ possesses a maximal element, say $B(z, d(z, Tz))$ where $z \in B(x, d(x, Tx))$. We show that $B(z, d(z, Tz))$ is a singleton. To this end, suppose on the contrary that $B(z, d(z, Tz))$ is not a singleton. Since $B(z, d(z, Tz))$ is weakly connected, , there exists a path $(x_i)_{i=0}^{i=n}$ in $\tilde{G}$ from z to Tz with vertices in $B(z, d(z, Tz))$. Therefore,

$$\begin{aligned} d(Tz, T^2z) &\leq \max\{d(Tz, Tx_1), \ldots, d(Tx_{n-1}, Tx_n)\} \\ &< \max\{d(z, x_1), \ldots, d(x_{n-1}, x_n)\} \\ &\leq d(z, Tz). \end{aligned}$$

Hence $B(Tz, d(Tz, T^2z)) \subseteq B(z, d(z, Tz))$ and $z \notin B(Tz, d(Tz, T^2z))$ which contradicts to the maximality of $B(z, d(z, Tz))$. Therefore, $B(z, d(z, Tz))$ is singleton and hence $z = Tz$. □

Remark 0.9. Notice that if we replace (X, d) in Theorem 0.8 with a spherically complete non-Archimedean normed space $(X, \|.\|)$, then again the theorem holds.

3. EXAMPLES

If (X, d) is a spherically complete ultrametric space, then the weakly connectivity of (X, d) implies (c) in Theorem 0.5, so if there exists an $x_0 \in X$ such that $d(x_0, Tx_0) < 1$, then the hypotheses of Theorems 0.5 is fulfilled. In this section, we will give some examples to support our Theorems.

Example. Let $X = \{a, b, c, e\}$ and d be an ultrametric on X defined by

$$d(a, c) = d(a, e) = d(b, c) = d(b, e) = 1$$

$$d(a, b) = d(c, e) = \frac{3}{4}.$$

Consider a graph $G = (V(G), E(G))$ with $V(G) = X$ and $E(G) = \Delta \cup \{(a, b)\}$ and define $T : X \to X$ by

$$Ta = Tb = Tc = a, \qquad Te = b.$$

Obviously, the conditions of Theorem 0.5 hold and T has a fixed point.

In the following example, we present a spherically complete ultrametric space endowed with a weakly connected graph to support Theorems 0.8.

Example. Let X be the space c_0 over a non-Archimedean valued field $\mathbb{K}$ with the discrete valuation and pick a $\pi \in \mathbb{K}$ with $0 < |\pi| < 1$. Consider a graph $G = (V(G), E(G))$ with $V(G) = X$ and

$$E(G) = \{(x, y) \in X \times X \mid \text{ either } x = y \text{ or there exists just one } i \in \mathbb{N} \text{ such that } x_i = y_i\}.$$

Let $x \in X$. Consider the closed ball $B(x, r)$ and let $y, z \in B(x, r)$. If $y = z$, then (z, y) is a path in $\tilde{G}$ from z to y. Otherwise, then we have two cases: Either there exists an $i \in \mathbb{N}$ such that $y_i = z_i$ or not.

1. Let there exists $i \in \mathbb{N}$ such that $y_i = z_i$. Let j be such that $j \neq i$. Put $w_j = z_j + y_j$ if $z_j, y_j \neq 0$ and put $w_j = \pi(z_j + y_j)$ if either $z_j = 0$ or $y_j = 0$, otherwise, choose $n_j \in \mathbb{N}$ such that $|\pi^{n_j}| < r$ and put $w_j = \pi^{n_j}$. So, for each $j \neq i$ $w_j \neq z_j, y_j$ and $|w_j| < r$. Now, put

$$w = (w_1, w_2, ..., w_{i-1}, z_i, w_{i+1}, ...).$$

The process of creating of $\{w_k\}$ shows that for each $k \neq i$, $w_k \neq z_k, y_k$ and $|w_k - x_k| < r$. Since if $\{n_k\}$ is an increasing sequence, then $\lim_{k\to+\infty} |\pi^{n_k}| = 0$, $\lim_{k\to+\infty} |z_k + y_k| = 0$ and $\lim_{k\to+\infty} |\pi(z_k + y_k)| = 0$, therefore $w \in c_0$, and since for each k, $|w_k| < r$, so $w \in B(x, r)$.

2. Let there existn't $i \in \mathbb{N}$ such that $z_i = y_i$. Put

$$w = (z_1, y_2, w_3, w_4, ...)$$

where if $z_j, y_j \neq 0$ then $w_j = z_j + y_j$, if either $z_j = 0$ or $y_j = 0$ then $w_j = \pi(z_j + y_j)$. So for each $k \in \mathbb{N}$, $w_k \neq z_k, y_k$ and $|w_k| < r$.

In both cases, (z, w, y) is a path from z to y with vertices in $B(x, r)$. Therefore, $B(x, r)$ is weakly connected. It is well-known that when $\mathbb{K}$ is discrete, then (X, d) is spherically complete. So all hypotheses of Theorems 0.5 are fulfilled. On the other hand, if $T : c_0 \longrightarrow c_0$ is a G-strongly contractive mapping and there exists an $x_0 \in X$ such that $d(x_0, Tx_0) < 1$, since (X, d, G) is weakly connected, there exists a path in $\tilde{G}$ between x and Tx. Thus the hypothesis (b) in Theorem 0.5 holds.

In the next example, we show that the conditions of Theorems 0.5 are independent of the conditions of Theorems 0.8.

Example. Let X be the space c_0 over a non-Archimedean valued field $\mathbb{K}$ with the discrete valuation and pick a $\pi \in \mathbb{K}$ with $|\pi| > 1$. Suppose a $w \in B(0, 1)$ has just one zero coordinate. Define a graph $G' = (V(G'), E(G'))$, with $V(G') = X$ and

$$E(G') = \{(x, y) \in X \times X : x = y \text{ or } (x, y), (x, w), (y, w) \in E(G)\}.$$

Where G is the graph introduced in Example . Then G' is not weakly connected because if $x \in X$ is such that (x, w) is not an edge of G, then there is no path in G' between x and w. Indeed, if $(x = x_0, x_1, x_2, \ldots, x_N = w)$ is a path in G' between x and w, then $(x, x_1) \in E(G')$, and therefore, $(x, w) \in E(G)$, which is a contradiction. So G' is not weakly connected and the conditions of Theorems 0.8 don't hold. Now, define $T : X \longrightarrow X$ by

$$T(x) = \begin{cases} (\frac{x_1}{\pi}, \frac{x_2}{\pi^2}, \frac{x_3}{\pi^3}, \ldots), & (x, w) \in E(G), \\ (1 + x_1, 2x_2, 2x_3, \ldots), & \text{otherwise.} \end{cases}$$

Then T is G'-strongly contractive mapping and (a) in Theorem 0.5 holds. If $x \in X$ and $d(x, Tx) < 1$, then $(x, w) \in E(G)$. Therefore, (x, Tx) is a path in $\tilde{G}'$ from x to Tx. This means that (b) in of Theorem 0.5 holds. If $\{B(x_n, d(x_n, Tx_n))\}$ is a nonincreasing sequence of closed balls such that for each $n \geq 1$, there exists a path in $\tilde{G}'$ between x_n and x_{n+1}, then $(x_n, w) \in E(G)$ for all $n \geq 1$ and so, $d(x_n, Tx_n) = 0$. Hence there exists a $z \in X$ such that $B(x_n, d(x_n, Tx_n)) = \{z\}$ for each $n \geq 1$. Therefore, the hypotheses of Theorem 0.5 hold.

The following example shows that the corresponding results does not hold in the framework of metric spaces which is not ultrametric space.

Example. Let $\mathbb{R}$ be endowed with the Euclidean metric d and G be a graph with $V(G) = \mathbb{R}$ and $E(G) = \mathbb{R} \times \mathbb{R}$. the mapping

$$T : (\mathbb{R}, d) \to (\mathbb{R}, d)$$
$$Tx = x + \frac{1}{1+e^x},$$

is G-strongly contractive mapping, but has no fixed point.

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