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OSCILLATION CRITERIA FOR THREE DIMENSIONAL LINEAR DIFFERENCE SYSTEMS

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In this work, sufficient conditions for oscillation of solutions of three dimensional difference systems of the form

$$X(n+1) = A(n)X(n)$$

are established. Here $X(n) = [x_1(n), x_2(n), x_3(n)]^T$, $A(n) = [a_{ij}(n)]$ is a given 3×3 matrix, $x_i\colon \mathbb{N} \to \mathbb{R}$, $a_{ij}\colon \mathbb{N} \to \mathbb{R}$ for $i, j \in \{1, 2, 3\}$.

1. Introduction

Let's consider the 3-dimensional difference system

(1)
$$X(n+1) = A(n)X(n),$$

where $X(n) = [x_1(n), x_2(n), x_3(n)]^T$, $A(n) = [a_{ij}(n)]$ is a given 3×3 matrix, $x_i\colon \mathbb{N} \to \mathbb{R}$, $a_{ij}\colon \mathbb{N} \to \mathbb{R}$ for $i, j \in \{1, 2, 3\}$. Here $\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{N}_{n_0} = \{n_0, n_0 + 1, n_0 + 2, \dots\}, n_0 \in \mathbb{N}$ and $\mathbb{R}$ denotes the set of real numbers.

If $a_{ij}(n) \equiv a_{ij} \in \mathbb{R}$ for any $i, j \in \{1, 2, 3\}$, then equation (1) is equivalent to

(2)
$$X(n+1) = AX(n),$$

where $X(n) = [x(n), y(n), z(n)]^T$ and $A(n) = [a_{ij}]_{3\times3}$. The characteristic equation of (2) is given by $\det(\lambda I - A) = 0$, that is,

(3)
$$\lambda^3 - (\operatorname{tr} A)\lambda^2 + m\lambda - \det A = 0,$$

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where $m = a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{13}a_{31} - a_{23}a_{32} - a_{12}a_{21}$. If we denote

$$G = -\det A + \frac{m(\operatorname{tr} A)}{3} - \frac{2(\operatorname{tr} A)^3}{27},\ H = \frac{1}{3}\big(m - \frac{(\operatorname{tr} V)^2}{3}\big),\ \mu = \lambda - \frac{(\operatorname{tr} A)}{3},$$

then (3) reduces to

$$\mu^3 + 3H\mu + G = 0.$$

From the theory of algebraic equations (see e.g. [**3**]), it follows that (3) admits at least one real root λ_1 such that the sign of λ_1 is opposite to that of the last term, namely $-\det A$. Hence, we have the following result:

Proposition 1. *Let* $\det A < 0$. *If* $G^2 + 4H^3 > 0$, *then equation* (3) *admits a negative real root and two imaginary roots. If* $G^2 + 4H^3 < 0$, *then* (3) *admits at least one negative real root. If* $m = (\operatorname{tr} A)^2$ *and* $7(\operatorname{tr} A)^3 - 27\det A = 0$, *then* (3) *admits a negative real root with multiplicity 3. If* $\det A > 0$, *then* (3) *admits a positive real root.*

In [**11**], the author has studied the oscillatory behaviour of solutions of the systems

$$\begin{bmatrix} x(n+1) \\ y(n+1) \end{bmatrix} = \begin{bmatrix} a(n) & b(n) \\ c(n) & d(n) \end{bmatrix} \begin{bmatrix} x(n) \\ y(n) \end{bmatrix}$$

and

$$\begin{bmatrix} x(n+1) \\ y(n+1) \end{bmatrix} = \begin{bmatrix} a(n) & b(n) \\ c(n) & d(n) \end{bmatrix} \begin{bmatrix} x(n) \\ y(n) \end{bmatrix} + \begin{bmatrix} f_1(n) \\ f_2(n) \end{bmatrix}.$$

In this paper, the oscillation criteria of the above systems are established unlike the oscillation criteria for the differential systems (see e.g. [**1**], [**5**]). Keeping in view of the above purpose as in [**11**], an attempt is made here to study the oscillatory behaviour of solutions of equation (1) and also (2) .

Thandapani et al. [**10**] have studied the oscillation properties of solutions of three dimensional difference systems of the form:

$$\begin{aligned} \Delta x(n) &= a(n)y^{\alpha}(n) \\ \Delta y(n) &= b(n)z^{\beta}(n) \\ \Delta z(n) &= -c(n)x^{\gamma}(n) \end{aligned}$$

which is the discrete analogue of its continuous counterpart. Also in an another work [**9**], Schmeidel has investigated the oscillation properties of solutions of the systems

$$\begin{aligned} \Delta x(n) &= a(n)a_{23}(y(n-l)) \\ \Delta y(n) &= b(n)g(z(n-m)) \\ \Delta z(n) &= \delta c(n)h(x(n-k)). \end{aligned}$$

A close observation reveals that the methods incorporated in both works [**9**] and [**10**] are similar. Unlike the methods of [**9**] and [**10**], our objective in this work is to establish the oscillation criteria for the systems (1). Meanwhile, we use some of the results of [**7**] and [**8**]. Concerning difference equations and system of difference equations, we refer the monographs by Agarwal et al. [**2**] and Elyadi [**4**].

Definition 1. *By a solution of* (1)/(2) *we mean a vector* $X(n)$ *which satisfies* (1)/(2) *for* $n \in \mathbb{N}$. *We say that the solution* $X(n) = [x_1(n), x_2(n), x_3(n)]^T$ *oscillates componentwise or simply oscillates if each component oscillates. Otherwise, the solution* $X(n)$ *is called nonoscillatory. Therefore, a solution of* (1)/(2) *is nonoscillatory if it has a component which is eventually positive or eventually negative.*

2. Preliminaries

In [**7**] and [**8**], Parhi and Tripathy have discussed the oscillation and nonoscillation of third order difference equations of the form:

$$y(n+3) + \alpha(n)y(n+2) + \beta(n)y(n+1) + \gamma(n)y(n) = 0 \tag{4}$$

and

$$y(n+3) + \alpha y(n+2) + \beta y(n+1) + \gamma y(n) = 0, \tag{5}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\gamma \neq 0$ and $\{\alpha(n)\}$, $\{\beta(n)\}$, $\{\gamma(n)\}$ are real valued sequences defined on $\mathbb{N}_{n_0}$.

A nontrivial solution $\{y(n)\}$ of (4) is said to be oscillatory, if for every positive integer N there exists $n \geq N$ such that $y(n)y(n+1) \leq 0$. Otherwise, the solution is nonoscillatory. In other words, a solution $\{y(n)\}$ is oscillatory if it is neither eventually positive nor eventually negative. Equation (4) is said to be oscillatory if all its solutions are oscillatory and strongly nonoscillatory if all its solutions are nonoscillatory.

In the following, we state some of the main results of [**7**] and [**8**] which will be useful for our next discussion.

Proposition 2. *Let* $\gamma > 0$. *If* $G^2 + 4H^3 > 0$ *or* $G = 0$ *and* $H = 0$, *then* (5) *is oscillatory. If* $G^2 + 4H^3 \leq 0$, *then* (5) *admits an oscillatory solution, where*

$$G = \gamma - \frac{\alpha\beta}{3} + \frac{2\alpha^3}{27}, H = \frac{1}{3}\left(\beta - \frac{\alpha^2}{3}\right).$$

Corollary 1. *Let* $\gamma > 0$. *If one of the cases*

(i) $3\beta > \alpha^2$;

(ii) $\beta \leq 0,\ \alpha \geq 0,\ \gamma - \frac{\alpha\beta}{3} + \frac{2\alpha^3}{27} - \frac{2}{3\sqrt{3}}(\frac{\alpha^2}{3} - \beta)^{\frac{3}{2}} > 0$;

(iii) $\beta \geq 0,\ \alpha \leq 0,\ 3\beta \leq \alpha^2,\ \gamma - \frac{\alpha\beta}{3} + \frac{2\alpha^3}{27} - \frac{2}{3\sqrt{3}}(\frac{\alpha^2}{3} - \beta)^{\frac{3}{2}} > 0$;

(iv) $3\beta = \alpha^2, \gamma = \frac{\alpha\beta}{3} - \frac{2\alpha^3}{27}$

holds, then (5) *is oscillatory.*

Remark 1. *We may notice that* $\gamma > 0$, $3\beta = \alpha^2$ *and* $\gamma = \frac{\alpha\beta}{3} - \frac{2\alpha^3}{27}$ *imply that* $\gamma > 0$ *and* $\beta > 0$. *If* $\alpha \geq 0$, $\beta \geq 0$ *and* $\gamma \geq 0$ *such that* $\alpha + \beta + \gamma > 0$, *then* (5) *is oscillatory.*

Theorem 1. *Let* $\gamma < 0$ *and* $\alpha > 0$. *If one of the conditions*

(i) $\gamma - \frac{\alpha\beta}{3} + \frac{2\alpha^3}{27} < \frac{2}{3\sqrt{3}}(\frac{\alpha^2}{3} - \beta)^{\frac{3}{2}}$, $\beta < \frac{\alpha^2}{3}$;

(ii) $0 < \gamma - \frac{\alpha\beta}{3} + \frac{2\alpha^3}{27} < \frac{2}{3\sqrt{3}}(\frac{\alpha^2}{3} - \beta)^{\frac{3}{2}}$, $0 \leq \beta < \frac{\alpha^2}{3}$;

(iii) $\gamma - \frac{\alpha\beta}{3} + \frac{2\alpha^3}{27} = 0$, $0 \leq \beta < \frac{2\alpha^2}{9}$

holds, then (5) *admits one nonoscillatory solution.*

Theorem 2. *Let* $\gamma < 0$, $\alpha > 0$ *and* $\beta < \frac{\alpha^2}{3}$. *If*

$$\frac{2}{3\sqrt{3}}\Big(\frac{\alpha^2}{3} - \beta\Big)^{\frac{3}{2}} = \gamma - \frac{\alpha\beta}{3} + \frac{2\alpha^3}{27} > 0,$$

then (5) *admits two nonoscillatory solutions.*

Theorem 3. *Let* $\gamma < 0$ *and* $\alpha < 0$. *If one of the conditions*

(i) $0 < \frac{\alpha\beta}{3} - \gamma - \frac{2\alpha^3}{27} < \frac{2}{3\sqrt{3}}(\frac{\alpha^2}{3} - \beta)^{\frac{3}{2}}$, $\frac{\gamma}{\alpha} \leq \beta < \frac{\alpha^2}{3}$;

(ii) $0 < \frac{\alpha\beta}{3} - \gamma - \frac{2\alpha^3}{27} = \frac{2}{3\sqrt{3}}(\frac{\alpha^2}{3} - \beta)^{\frac{3}{2}}$, $\frac{\gamma}{\alpha} \leq \beta < \frac{\alpha^2}{3}$

holds, then (5) *is strongly nonoscillatory.*

Theorem 4. *Let* $\gamma < 0$, $\alpha < 0$ *and* $0 < \frac{\alpha\beta}{3} - \gamma - \frac{2\alpha^3}{27} < \frac{2}{3\sqrt{3}}(\frac{\alpha^2}{3} - \beta)^{\frac{3}{2}}$. *If* $\beta < \frac{\gamma}{\alpha} < \frac{\alpha^2}{3}$ *or* $\beta < \frac{\alpha^2}{3} \leq \frac{\gamma}{\alpha}$ *holds, then* (5) *admits two oscillatory solutions.*

Theorem 5. *Suppose that* $\gamma < 0$, $\alpha < 0$ *and*

$$0 < \frac{\alpha\beta}{3} - \gamma - \frac{2\alpha^3}{27} = \frac{2}{3\sqrt{3}}\left(\frac{\alpha^2}{3} - \beta\right)^{\frac{3}{2}}.$$

is satisfied. If $\beta < \frac{\gamma}{\alpha} < \frac{\alpha^2}{3}$ *or* $\beta < \frac{\alpha^2}{3} \leq \frac{\gamma}{\alpha}$ *is satisfied, then* (5) *admits two oscillatory solutions.*

Theorem 6. *Let* $\gamma(n) > 0$, $\beta(n) < 0$, *and* $\alpha(n) < 0$ *for* $n \in \mathbb{N}$. *If*

$$\alpha(n+1)(\alpha(n-1)\gamma(n) - \gamma(n) - \beta(n)\beta(n-1))$$
$$\geq \beta(n-1)(\beta(n+1) - \gamma(n+1) - \alpha(n)\alpha(n+1))$$

and

$$\gamma(n+1)\beta(n-1) \leq \alpha(n+1)(\beta(n)\beta(n-1) - \gamma(n)\alpha(n-1))$$

hold for large n, *then* (4) *is oscillatory.*

Theorem 7. *Suppose that* $\gamma(n) > 0$, $\beta(n) > 0$, *and* $\alpha(n) < 0$ *for* $n \in \mathbb{N}$. *If* $\inf_{n\geq 0} \alpha(n) = l < 0$, $\liminf_{n\longrightarrow\infty} \beta(n) = m > 0$ *and* $\liminf_{n\longrightarrow\infty} \gamma(n) = s > 0$ *such that*

$$\frac{2m^3}{27s^2} - \frac{ml}{3s^2} + \frac{1}{s} - \frac{2}{3\sqrt{3}}\left(\frac{m^2}{3s^2} - \frac{l}{s}\right)^{\frac{3}{2}} > 0,$$

then (4) *is oscillatory.*

Theorem 8. *Let* $\gamma(n) \geq 0$, $\beta(n) \geq 0$ *and* $\alpha(n) < 0$ *for* $n \in \mathbb{N}$. *If* $\liminf_{n\longrightarrow\infty} \beta(n) = m \geq 0$ *and*

$$\limsup_{n\longrightarrow\infty} \beta(n) > \limsup_{n\longrightarrow\infty} \alpha(n-1)\left(\alpha(n) - \frac{m}{\alpha(n+1)}\right)$$

holds, then (4) *is oscillatory.*

Theorem 9. *Let* $\gamma(n) \geq 0$, $\beta(n) < 0$ *and* $\alpha(n) > 0$ *for* $n \in \mathbb{N}$. *If* $\liminf_{n\longrightarrow\infty} \gamma(n) = s \geq 0$ *and*

$$\limsup_{n\longrightarrow\infty} \gamma(n) > \limsup_{n\longrightarrow\infty} \frac{\beta(n-1)}{\alpha(n-1)}\left(\beta(n) - \frac{s\alpha(n)}{\beta(n+1)}\right)$$

hold, then (4) *is oscillatory.*

Theorem 10. *Assume that* $\gamma(n) \geq 0$, $\beta(n) > 0$ *and* $\alpha(n) \leq 0$ *for* $n \in \mathbb{N}$. *If* $4m > l^2$, *then* (4) *is oscillatory, where* $m = \liminf_{n\longrightarrow\infty} \beta(n)$ *and* $l = \liminf_{n\longrightarrow\infty} \alpha(n)$.

Let us denote
$s = \liminf_{n\longrightarrow\infty} \gamma(n), m = \liminf_{n\longrightarrow\infty} \beta(n)$ and $l = \liminf_{n\longrightarrow\infty} \alpha(n)$.

Theorem 11. *Suppose that* $\gamma(n) \geq 0$, $\beta(n) > 0$ *and* $\alpha(n) \leq 0$ *for* $n \in \mathbb{N}$. *If* $l^2 > 3m$ *and*

$$s - \frac{lm}{3} + \frac{2l^3}{27} - \frac{2}{3\sqrt{3}}\left(\frac{l^2}{3} - m\right)^{\frac{3}{2}} > 0,$$

then (4) *is oscillatory.*

Theorem 12. *Let* $\gamma(n) > 0$, $\beta(n) < 0$ *and* $\alpha(n) > 0$ *for* $n \in \mathbb{N}$. *If*

$$s - \frac{lm}{3} + \frac{2l^3}{27} - \frac{2}{3\sqrt{3}}\left(\frac{l^2}{3} - m\right)^{\frac{3}{2}} > 0,$$

then (4) *is oscillatory.*

Theorem 13. *If* $\alpha(n) \geq 0$, $\beta(n) \geq 0$, *and* $\gamma(n) \geq 0$ *for* $n \in \mathbb{N}$ *such that* $\alpha(n) + \beta(n) + \gamma(n) > 0$, *than* (4) *is oscillatory.*

Theorem 14. *If* $\gamma(n) \geq 0$, $\beta(n) \geq 0$, $\alpha(n) < 0$ *and*

$$\frac{\gamma(n+1)}{\alpha(n+1)\alpha(n-1)} > \frac{\beta(n+1)}{\alpha(n+1)} + \frac{\beta(n)}{\alpha(n-1)} - \alpha(n)$$

hold for large n, *then* (4) *is oscillatory.*

Theorem 15. *If* $\gamma(n) \geq 0$, $\beta(n) < 0$, $\alpha(n) \geq 0$ *and*

$$\beta(n) > \frac{\alpha(n)\gamma(n+1)}{\beta(n+1)} + \frac{\gamma(n)\alpha(n-1)}{\beta(n-1)}$$

holds for large n*, then* (4) *is oscillatory.*

Theorem 16. *If*

$$\sum_{n=1}^{\infty} n\big(|\alpha(n)+2| + |\beta(n)-1| + |\gamma(n)|\big) < \infty,$$

then (4) *admits a bounded nonoscillatory solution.*

3. Oscillation Criteria for the System (2)

In this section, sufficient conditions are established for oscillation and nonoscillation of the system (2) .

Theorem 17. *Let* $\det A < 0$*. If* $G^2 + 4H^3 > 0$ *or* $G = 0$ *and* $H = 0$*, then the system* (2) *is oscillatory. If* $G^2 + 4H^3 \leq 0$*, then* (2) *admits an oscillatory component.*

Proof. The proof of the theorem follows from Proposition 2 and Proposition 1. Hence, the details are omitted. □

Corollary 2. *Let* $\det A < 0$*. If one of the cases*

(i) $3m > (\operatorname{tr} A)^2$;

(ii) $m \leq 0,\ \operatorname{tr} A \leq 0,\ -\det A + \frac{m(\operatorname{tr} A)}{3} - \frac{2(\operatorname{tr} A)^3}{27} - \frac{2}{3\sqrt{3}}\left[\frac{(\operatorname{tr} A)^2}{3} - m\right]^{\frac{3}{2}} > 0$;

(iii) $m \geq 0,\ \operatorname{tr} A \geq 0,\ 3m \leq (\operatorname{tr} A)^2$,

$$-\det A + \frac{m(\operatorname{tr} A)}{3} - \frac{2(\operatorname{tr} A)^3}{27} - \frac{2}{3\sqrt{3}}\left[\frac{(\operatorname{tr} A)^2}{3} - m\right]^{\frac{3}{2}} > 0;$$

(iv) $3m = (\operatorname{tr} A)^2,\ \frac{2(\operatorname{tr} A)^3}{27} - \frac{m(\operatorname{tr} A)}{3} + \det A = 0$

holds, then the system (2) *is oscillatory.*

Proof. For Cases **(i)**-**(iii)**, there is $G^2 + 4H^3 > 0$. For Case (**iv**), we have $G = 0$, $H = 0$. Hence, by Corollary 2 and Proposition 1, equation (3) admits a negative real root and two complex roots for Cases **(i)**-**(iii)**, whereas Case **(iv)** comes out with one negative real root repeated thrice. This implies that $X(n)$ is oscillatory in each of cases. This completes the proof of the corollary. □

Example 1. *Let us consider the following system of difference equations*

$$\begin{cases} x_1(n+1) = -x_1(n) \\ x_2(n+1) = -x_3(n), \\ x_3(n+1) = \quad x_2(n) \end{cases} \tag{6}$$

with initial condition $[x_1(0), x_2(0), x_3(0)]^T = [c_1, c_2, c_3]^T$. *Here*

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix},$$

$\det A = -1$, $\operatorname{tr} A = -1$ *and* $m = 1$. *Hence, the assumptions of Corollary 2 Case **(i)** are satisfied. On the virtue of this corollary the system is oscillatory. In fact, it is easy to see that a solution of* (6) *is the following*

$$\begin{cases} x_1(n) = c_1(-1)^n \\ x_2(n) = c_2 \cos \frac{n\pi}{2} - c_3 \sin \frac{n\pi}{2}, \\ x_3(n) = c_3 \cos \frac{n\pi}{2} + c_2 \sin \frac{n\pi}{2} \end{cases}$$

and it is oscillatory.

Remark 2. *From Corollary 2 and Remark 1 it follows that the system* (2) *is oscillatory when*

$$m - \operatorname{tr} A - \det A > 0.$$

Theorem 18. *Let* $\det A > 0$ *and* $\operatorname{tr} A < 0$. *If one of the conditions*

(i) $-\det A + \frac{m(\operatorname{tr} A)}{3} - \frac{2(\operatorname{tr} A)^3}{27} < \frac{2}{3\sqrt{3}} \left[\frac{(\operatorname{tr} A)^2}{3} - m\right]^{\frac{3}{2}}$, $m < \frac{(\operatorname{tr} A)^2}{3}$;

(ii) $0 < -\det A + \frac{m(\operatorname{tr} A)}{3} - \frac{2(\operatorname{tr} A)^3}{27} < \frac{2}{3\sqrt{3}} \left[\frac{(\operatorname{tr} A)^2}{3} - m\right]^{\frac{3}{2}}$, $0 \le m < \frac{(\operatorname{tr} A)^2}{3}$;

(iii) $-\det A + \frac{m(\operatorname{tr} A)}{3} - \frac{2(\operatorname{tr} A)^3}{27} = 0$, $0 \le m < \frac{2(\operatorname{tr} A)^2}{9}$

is satisfied, then the system (2) *is nonoscillatory.*

Proof. Since $\det A > 0$, then (3) admits a positive real root which then implies that one component of $X(n)$ is nonoscillatory. The rest of the proof follows from Theorem 1. This completes the proof of the theorem. □

Theorem 19. *Let* $\det A > 0$, $\operatorname{tr} A < 0$ *and* $m < \frac{(\operatorname{tr} A)^2}{3}$. *If*

$$0 < -\det A + \frac{m(\operatorname{tr} A)}{3} - \frac{2(\operatorname{tr} A)^3}{27} = \frac{2}{3\sqrt{3}} \left[\frac{(\operatorname{tr} A)^2}{3} - m\right]^{\frac{3}{2}},$$

then the system (2) *is nonoscillatory.*

Proof. By Theorem 2, it follows that (3) admits two positive real roots and hence two components of $X(n)$ are nonoscillatory. Therefore, (2) is nonoscillatory. Thus the proof of the theorem is complete. □

Theorem 20. *Let* $\det A > 0$ *and* $\operatorname{tr} A < 0$. *If one of the conditions*

(i) $0 < \det A - \frac{m(\operatorname{tr} A)}{3} + \frac{2(\operatorname{tr} A)^3}{27} < \frac{2}{3\sqrt{3}}\left[\frac{(\operatorname{tr} A)^2}{3} - m\right]^{\frac{3}{2}}, \; \frac{\det A}{\operatorname{tr} A} \leq m < \frac{(\operatorname{tr} A)^2}{3}$;

(ii) $0 < \det A - \frac{m(\operatorname{tr} A)}{3} + \frac{2(\operatorname{tr} A)^3}{27} = \frac{2}{3\sqrt{3}}\left[\frac{(\operatorname{tr} A)^2}{3} - m\right]^{\frac{3}{2}}, \; \frac{\det A}{\operatorname{tr} A} \leq m < \frac{(\operatorname{tr} A)^2}{3}$

is satisfied, then the system (2) *is nonoscillatory.*

Proof. Due to $\det A > 0$ and Theorem 3, equation (3) admits three positive real roots. Therefore, $X(n)$ has three nonoscillatory components. Hence, the theorem is proved. □

Theorem 21. *Let* $\det A > 0, \operatorname{tr} A < 0$ *and* $0 < \det A - \frac{m(\operatorname{tr} A)}{3} + \frac{2(\operatorname{tr} A)^3}{27} < \frac{2}{3\sqrt{3}}\left[\frac{(\operatorname{tr} A)^2}{3} - m\right]^{\frac{3}{2}}$. *If* $m < \frac{\det A}{\operatorname{tr} A} < \frac{(\operatorname{tr} A)^2}{3}$ *or* $m < \frac{(\operatorname{tr} A)^2}{3} \leq \frac{\det A}{\operatorname{tr} A}$ *is satisfied, then the system* (2) *is nonoscillatory.*

Proof. Since $\det A > 0$, then (3) admits one positive real root. By Theorem 4, system (3) admits two complex roots. Hence, $X(n)$ is nonoscillatory. This completes the proof of the theorem. □

Theorem 22. *Let* $\det A > 0, \operatorname{tr} A < 0$ *and* $0 < \det A - \frac{m(\operatorname{tr} A)}{3} + \frac{2(\operatorname{tr} A)^3}{27} = \frac{2}{3\sqrt{3}}\left[\frac{(\operatorname{tr} A)^2}{3} - m\right]^{\frac{3}{2}}$. *If* $m < \frac{\det A}{\operatorname{tr} A} < \frac{(\operatorname{tr} A)^2}{3}$ *or* $m < \frac{(\operatorname{tr} A)^2}{3} \leq \frac{\det A}{\operatorname{tr} A}$ *is satisfied, then* (2) *is nonoscillatory.*

Proof. Due to Theorem 5 and $\det A > 0$, it is easy to verify that two components of $X(n)$ are oscillatory and one component is nonoscillatory. Hence, (2) is nonoscillatory. □

Example 2. *Consider the system of equations*

$$X(n+1) = AX(n), A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}. \tag{7}$$

Clearly, $\det A = -1, \operatorname{tr} A = 3$ *and* $G^2 + 4H^3 = -\frac{512}{729}$. *Hence, by Theorem 3.1,* (7) *has an oscillatory component. Indeed,* $\lambda = 1, 1+\sqrt{2}, 1-\sqrt{2}$ *are the roots of the characteristic equation of the system* (7) *and the corresponding components of* $X(n)$ *are given by*

$$x(n) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; y(n) = (1+\sqrt{2})^n \begin{bmatrix} 1 \\ \sqrt{2} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \; z(n) = (1-\sqrt{2})^n \begin{bmatrix} 1 \\ -\sqrt{2} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

4. Oscillation Criteria for the System (1)

In this section, sufficient conditions are established for oscillation and nonoscillation of all solutions of the system (1). From the system (1), it is easy to see that

$$\begin{aligned}
&x_1(n+3) - a_{11}(n+2)x_1(n+2) - u_1(n+1)x_1(n+1)\\
&- [a_{21}(n)v_1(n+1) + a_{31}(n)s_1(n+1)]x_1(n)\\
&= [a_{22}(n)v_1(n+1) + a_{32}(n)s_1(n+1)]x_2(n)\\
&+ [a_{23}(n)v_1(n+1) + a_{33}(n)s_1(n+1)]x_3(n),
\end{aligned} \tag{8}$$

where

$$\begin{aligned}
u_1(n) &= a_{21}(n)a_{12}(n+1) + a_{31}(n)a_{13}(n+1),\\
v_1(n) &= a_{22}(n)a_{12}(n+1) + a_{32}(n)a_{13}(n+1),\\
s_1(n) &= a_{23}(n)a_{12}(n+1) + a_{33}(n)a_{13}(n+1).
\end{aligned}$$

Let $x_2(n)$ and $x_3(n)$ be nontrivial sequences. If we assume that

$$\begin{aligned}
&[a_{22}(n)v_1(n+1) + a_{32}(n)s_1(n+1)] = 0,\\
&[a_{23}(n)v_1(n+1) + a_{33}(n)s_1(n+1)] = 0,
\end{aligned} \tag{9}$$

then (8) becomes a third order difference equation of the form

$$\begin{aligned}
&x_1(n+3) - a_{11}(n+2)x_1(n+2) - u_1(n+1)x_1(n+1)\\
&- [a_{21}(n)v_1(n+1) + a_{31}(n)s_1(n+1)]x_1(n) = 0.
\end{aligned} \tag{10}$$

If we assume that

$$\begin{aligned}
&[a_{11}(n)u_2(n+1) + a_{31}(n)s_2(n+1)] = 0,\\
&[a_{13}(n)u_2(n+1) + a_{33}(n)s_2(n+1)] = 0,
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
&[a_{12}(n)u_3(n+1) + a_{22}(n)v_3(n+1)] = 0,\\
&[a_{11}(n)u_3(n+1) + a_{21}(n)v_3(n+1)] = 0,
\end{aligned} \tag{12}$$

where

$$\begin{aligned}
u_2(n) &= a_{11}(n)a_{21}(n+1) + a_{31}(n)a_{23}(n+1),\\
v_2(n) &= a_{12}(n)a_{21}(n+1) + a_{32}(n)a_{23}(n+1),\\
s_2(n) &= a_{13}(n)a_{21}(n+1) + a_{33}(n)a_{23}(n+1),\\
u_3(n) &= a_{11}(n)a_{31}(n+1) + a_{21}(n)a_{32}(n+1),\\
v_3(n) &= a_{12}(n)a_{31}(n+1) + a_{31}(n)a_{32}(n+1),\\
s_3(n) &= a_{13}(n)a_{31}(n+1) + a_{23}(n)a_{32}(n+1),
\end{aligned}$$

hold, then we find the third order difference equations of the form

$$x_2(n+3) - a_{22}(n+2)x_2(n+2) - v_2(n+1)x_2(n+1) \\ - [a_{12}(n)u_2(n+1) + a_{32}(n)s_2(n+1)]x_2(n) = 0 \tag{13}$$

and

$$x_3(n+3) - a_{33}(n+2)x_3(n+2) - s_3(n+1)x_3(n+1) \\ - [a_{13}(n)u_3(n+1) + a_{23}(n)v_3(n+1)]x_3(n) = 0. \tag{14}$$

For all n, if we denote

$$\begin{aligned} u_{11}(n) &= [a_{21}(n)v_1(n+1) + a_{31}(n)s_1(n+1)], \\ v_{12}(n) &= [a_{12}(n)u_2(n+1) + a_{32}(n)s_2(n+1)], \\ s_{13}(n) &= [a_{13}(n)u_3(n+1) + a_{23}(n)v_3(n+1)], \end{aligned}$$

then (10), (13) and (14) can be rewritten as

$$x_1(n+3) - a_{11}(n+2)x_1(n+2) - u_1(n+1)x_1(n+1) - u_{11}(n)x_1(n) = 0, \tag{15}$$

$$x_2(n+3) - a_{22}(n+2)x_2(n+2) - v_2(n+1)x_2(n+1) - v_{12}(n)x_2(n) = 0, \tag{16}$$

and

$$x_3(n+3) - a_{33}(n+2)x_3(n+2) - s_3(n+1)x_3(n+1) - s_{13}(n)x_3(n) = 0, \tag{17}$$

respectively.

Theorem 23. *Let* $a_{11}(n) > 0, a_{22}(n) > 0, a_{33}(n) > 0, u_1(n) > 0, v_2(n) > 0, s_3(n) > 0, u_{11}(n) < 0, v_{12}(n) < 0$ *and* $s_{13}(n) < 0$ *for all* $n \in \mathbb{N}$. *Assume that conditions* (9), (11) *and* (12) *hold for all* $n \in \mathbb{N}$. *If*

$$(I_1) \begin{cases} a_{11}(n+3)[u_1(n)u_1(n+1) - a_{11}(n+1)u_{11}(n) - u_{11}(n)] \\ \qquad \geq u_1(n)[a_{11}(n+2)a_{11}(n+3) + u_1(n+2) - u_{11}(n+1)], \\ u_1(n)u_{11}(n+1) \leq a_{11}(n+3)[u_{11}(n)a_{11}(n+1) - u_1(n)u_1(n+1)] \end{cases}$$

$$(I_2) \begin{cases} a_{22}(n+3)[v_2(n)v_2(n+1) - a_{22}(n+1)v_{12}(n) - v_{12}(n)] \\ \qquad \geq v_2(n)[a_{22}(n+2)a_{22}(n+3) + v_2(n+2) - v_{12}(n+1)], \\ v_2(n)v_{12}(n+1) \leq a_{22}(n+3)[v_{12}(n)a_{22}(n+1) - v_2(n)v_2(n+1)] \end{cases}$$

and

$$(I_3) \begin{cases} a_{33}(n+3)[s_3(n)s_3(n+1) - a_{33}(n+1)s_{13}(n) - s_{13}(n)] \\ \qquad \geq s_3(n)[a_{33}(n+2)a_{33}(n+3) + s_3(n+2) - s_{13}(n+1)], \\ s_3(n)s_{13}(n+1) \leq a_{33}(n+3)[s_{13}(n)a_{33}(n+1) - s_3(n)s_3(n+1)] \end{cases}$$

hold for all $n \in \mathbb{N}$, *then the system* (1) *is oscillatory.*

Proof. Since (I_1) holds, in the view of (9), we obtain (15). Due to Theorem 6, equation (15) is oscillatory. It means that $x_1(n)$ is oscillatory. Analogously, we get the thesis for $x_2(n)$ and $x_3(n)$. This completes the proof of the theorem. □

Theorem 24. *Let* $a_{11}(n) > 0, a_{22}(n) > 0, a_{33}(n) > 0, u_1(n) < 0, v_2(n) < 0, s_3(n) < 0, u_{11}(n) < 0, v_{12}(n) < 0$ *and* $s_{13}(n) < 0$ *for all* $n \in \mathbb{N}$. *Assume that conditions* (9), (11) *and* (12) *hold for all* $n \in \mathbb{N}$. *If*

$$\inf_{n\geq 0}(-a_{11}(n+2)) = \eta_{11} < 0, \quad \inf_{n\geq 0}(-a_{22}(n+2)) = \eta_{21} < 0,$$
$$\inf_{n\geq 0}(-a_{33}(n+2)) = \eta_{31} < 0, \quad \liminf_{n\to\infty}(-u_1(n+1)) = \eta_{12} > 0,$$
$$\liminf_{n\to\infty}(-v_2(n+1)) = \eta_{22} > 0, \quad \liminf_{n\to\infty}(-s_3(n+1)) = \eta_{32} > 0,$$
$$\liminf_{n\to\infty}(-u_{11}(n)) = \eta_{13} > 0, \quad \liminf_{n\to\infty}(-v_{12}(n)) = \eta_{23} > 0,$$
$$\liminf_{n\to\infty}(-s_{13}(n)) = \eta_{33} > 0 \text{ such that}$$

(18) $$\frac{2\eta_{12}^3}{27\eta_{13}^2} - \frac{\eta_{11}\eta_{12}}{3\eta_{13}^2} + \frac{1}{\eta_{13}} - \frac{2}{3\sqrt{3}}\left(\frac{\eta_{12}^2}{3\eta_{13}^2} - \frac{\eta_{11}}{\eta_{13}}\right)^{\frac{3}{2}} > 0,$$

(19) $$\frac{2\eta_{22}^3}{27\eta_{23}^2} - \frac{\eta_{21}\eta_{22}}{3\eta_{23}^2} + \frac{1}{\eta_{23}} - \frac{2}{3\sqrt{3}}\left(\frac{\eta_{22}^2}{3\eta_{23}^2} - \frac{\eta_{21}}{\eta_{23}}\right)^{\frac{3}{2}} > 0$$

and

(20) $$\frac{2\eta_{32}^3}{27\eta_{33}^2} - \frac{\eta_{31}\eta_{32}}{3\eta_{33}^2} + \frac{1}{\eta_{33}} - \frac{2}{3\sqrt{3}}\left(\frac{\eta_{32}^2}{3\eta_{33}^2} - \frac{\eta_{31}}{\eta_{33}}\right)^{\frac{3}{2}} > 0$$

are satisfied, then every solution of the system (1) *oscillates.*

Proof. The proof is analogous to the proof of Theorem 23 and hence is omitted. □

Theorem 25. *Let* $a_{11}(n) > 0, a_{22}(n) > 0, a_{33}(n) > 0, u_1(n) \leq 0, v_2(n) \leq 0, s_3(n) \leq 0, u_{11}(n) \leq 0, v_{12}(n) \leq 0$ *and* $s_{13}(n) \leq 0$ *for all* $n \in \mathbb{N}$. *Suppose also that conditions* (9), (11) *and* (12) *hold for all* $n \in \mathbb{N}$. *Furthermore, assume that*

$$\liminf_{n\to\infty}(-u_1(n+1)) = \eta_{12} \geq 0,$$
$$\liminf_{n\to\infty}(-v_2(n+1)) = \eta_{22} \geq 0 \text{ and}$$
$$\liminf_{n\to\infty}(-s_3(n+1)) = \eta_{32} \geq 0. \text{ If}$$

$$\limsup_{n\longrightarrow\infty}(-u_1(n+1)) > \limsup_{n\longrightarrow\infty}(-a_{11}(n+1))\left(\frac{\eta_{12}}{a_{11}(n+3)} - a_{11}(n+2)\right),$$

$$\limsup_{n\longrightarrow\infty}(-v_2(n+1)) > \limsup_{n\longrightarrow\infty}(-a_{22}(n+1))\left(\frac{\eta_{22}}{a_{22}(n+3)} - a_{22}(n+2)\right)$$

and

$$\limsup_{n\longrightarrow\infty}(-s_3(n+1)) > \limsup_{n\longrightarrow\infty}(-a_{33}(n+1))\left(\frac{\eta_{32}}{a_{33}(n+3)} - a_{33}(n+2)\right)$$

are satisfied, then (1) *is oscillatory.*

Proof. Applying Theorem 8 to (15), (16) and (17), the proof is analogous to the proof of Theorem 23. Hence the details are omitted. □

Theorem 26. *Let* $a_{11}(n) < 0, a_{22}(n) < 0, a_{33}(n) < 0, u_1(n) > 0, v_2(n) > 0, s_3(n) > 0, u_{11}(n) \leq 0, v_{12}(n) \leq 0$ *and* $s_{13}(n) \leq 0$ *for all* $n \in \mathbb{N}$. *Suppose that conditions* (9), (11) *and* (12) *hold for all* $n \in \mathbb{N}$. *Furthermore, assume that*
$\liminf_{n\to\infty}(-u_{11}(n)) = \eta_{13} \geq 0$,
$\liminf_{n\to\infty}(-v_{12}(n)) = \eta_{23} \geq 0$ *and*
$\liminf_{n\to\infty}(-s_{13}(n)) = \eta_{33} \geq 0$. *If*

$$\limsup_{n\longrightarrow\infty}(-u_{11}(n)) > \limsup_{n\longrightarrow\infty} \frac{u_1(n)}{a_{11}(n+1)} \left(-u_1(n+1) - \frac{\eta_{13}a_{11}(n+2)}{u_1(n+2)}\right),$$

$$\limsup_{n\longrightarrow\infty}(-v_{12}(n)) > \limsup_{n\longrightarrow\infty} \frac{v_2(n)}{a_{22}(n+1)} \left(-v_2(n+1) - \frac{\eta_{23}a_{22}(n+2)}{v_2(n+2)}\right)$$

and

$$\limsup_{n\longrightarrow\infty}(-s_{13}(n)) > \limsup_{n\longrightarrow\infty} \frac{s_3(n)}{a_{33}(n+1)} \left(-s_3(n+1) - \frac{\eta_{33}a_{33}(n+2)}{s_3(n+2)}\right),$$

then every vector solution of (1) *oscillates.*

Proof. The proof follows directly from Theorems 10–16. □

Theorem 27. *Let* $a_{11}(n) \geq 0, a_{22}(n) \geq 0, a_{33}(n) \geq 0, u_1(n) < 0, v_2(n) < 0, s_3(n) < 0, u_{11}(n) \leq 0, v_{12}(n) \leq 0$ *and* $s_{13}(n) \leq 0$ *for all* $n \in \mathbb{N}$. *Assume that conditions* (9), (11) *and* (12) *hold for all* $n \in \mathbb{N}$. *Furthermore, suppose that*
$\liminf_{n\to\infty}(-a_{11}(n+2)) = \eta_{11}$, $\quad \liminf_{n\to\infty}(-a_{22}(n+2)) = \eta_{21}$,
$\liminf_{n\to\infty}(-a_{33}(n+2)) = \eta_{31}$, $\quad \liminf_{n\to\infty}(-u_1(n+1)) = \eta_{12}$,
$\liminf_{n\to\infty}(-v_2(n+1)) = \eta_{22}$, $\quad \liminf_{n\to\infty}(-s_3(n+1)) = \eta_{32}$.
If $4\eta_{12} > \eta_{11}^2, 4\eta_{22} > \eta_{21}^2$ *and* $4\eta_{32} > \eta_{31}^2$, *then the system* (1) *is oscillatory.*

Theorem 28. *Let* $a_{11}(n) \geq 0, a_{22}(n) \geq 0, a_{33}(n) \geq 0, u_1(n) < 0, v_2(n) < 0, s_3(n) < 0, u_{11}(n) \leq 0, v_{12}(n) \leq 0$ *and* $s_{13}(n) \leq 0$ *for all* $n \in \mathbb{N}$. *Suppose that conditions* (9), (11) *and* (12) *hold for all* $n \in \mathbb{N}$. *Furthermore, assume that*
$\liminf_{n\to\infty}(-a_{11}(n+2)) = \eta_{11}$, $\quad \liminf_{n\to\infty}(-a_{22}(n+2)) = \eta_{21}$,
$\liminf_{n\to\infty}(-a_{33}(n+2)) = \eta_{31}$, $\quad \liminf_{n\to\infty}(-u_1(n+1)) = \eta_{12}$,
$\liminf_{n\to\infty}(-v_2(n+1)) = \eta_{22}$, $\quad \liminf_{n\to\infty}(-s_3(n+1)) = \eta_{32}$,
$\liminf_{n\to\infty}(-u_{11}(n)) = \eta_{13}$, $\quad \liminf_{n\to\infty}(-v_{12}(n)) = \eta_{23}$,
$\liminf_{n\to\infty}(-s_{13}(n)) = \eta_{33}$.

If $3\eta_{12} < \eta_{11}^2, 3\eta_{22} < \eta_{21}^2, 3\eta_{32} < \eta_{31}^2$ *and*

$$\eta_{13} - \frac{\eta_{11}\eta_{12}}{3} + \frac{2\eta_{11}^3}{27} - \frac{2}{3\sqrt{3}}\left(\frac{\eta_{11}^2}{3} - \eta_{12}\right)^{\frac{3}{2}} > 0, \tag{21}$$

$$\eta_{23} - \frac{\eta_{21}\eta_{22}}{3} + \frac{2\eta_{21}^3}{27} - \frac{2}{3\sqrt{3}}\left(\frac{\eta_{21}^2}{3} - \eta_{22}\right)^{\frac{3}{2}} > 0, \tag{22}$$

$$\eta_{33} - \frac{\eta_{31}\eta_{32}}{3} + \frac{2\eta_{31}^3}{27} - \frac{2}{3\sqrt{3}}\left(\frac{\eta_{31}^2}{3} - \eta_{32}\right)^{\frac{3}{2}} > 0 \tag{23}$$

are satisfied, then every vector solution of (1) *is oscillatory.*

Theorem 29. *Let* $a_{11}(n) > 0, a_{22}(n) > 0, a_{33}(n) > 0, u_1(n) < 0, v_2(n) < 0, s_3(n) < 0, u_{11}(n) > 0, v_{12}(n) > 0$ *and* $s_{13}(n) > 0$ *for all* $n \in \mathbb{N}$. *Assume that conditions* (9), (11) *and* (12) *hold for all* $n \in \mathbb{N}$.
Let $\liminf_{n\to\infty}(-a_{11}(n+2)) = \eta_{11}$, $\liminf_{n\to\infty}(-a_{22}(n+2)) = \eta_{21}$,
$\liminf_{n\to\infty}(-a_{33}(n+2)) = \eta_{31}$, $\liminf_{n\to\infty}(-u_1(n+1)) = \eta_{12}$,
$\liminf_{n\to\infty}(-v_2(n+1)) = \eta_{22}$, $\liminf_{n\to\infty}(-s_3(n+1)) = \eta_{32}$,
$\liminf_{n\to\infty}(-u_{11}(n)) = \eta_{13}$, $\liminf_{n\to\infty}(-v_{12}(n)) = \eta_{23}$,
$\liminf_{n\to\infty}(-s_{13}(n)) = \eta_{33}$.
If conditions (21), (22) *and* (23) *hold, then* (1) *is oscillatory.*

Theorem 30. *Assume that conditions* (9), (11) *and* (12) *are satisfied for all* $n \in \mathbb{N}$. *If* $a_{11}(n) \le 0$, $a_{22}(n) \le 0$, $a_{33}(n) \le 0$, $u_1(n) \le 0$, $v_2(n) \le 0$, $s_3(n) \le 0$, $u_{11}(n) \le 0$, $v_{12}(n) \le 0$ *and* $s_{13}(n) \le 0$ *for all* $n \in \mathbb{N}$ *such that* $a_{11}(n+2)+u_1(n+1)+u_{11}(n) < 0$, $a_{22}(n+2) + v_2(n+1) + v_{12}(n) < 0$ *and* $a_{33}(n+2) + s_3(n+1) + s_{13}(n) < 0$ *for all* $n \in \mathbb{N}$, *then* (1) *is oscillatory.*

Example 3. *Let us consider the following system of difference equations*

$$\begin{cases} x_1(n+1) = -x_1(n) + (-1)^n x_2(n) \\ x_2(n+1) = (-1)^n x_1(n) \\ x_3(n+1) = -x_3(n) \end{cases} \tag{24}$$

Here

$$A = \begin{bmatrix} -1 & (-1)^n & 0 \\ (-1)^n & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

where $a_{11}(n) \le 0$, $a_{22}(n) \le 0$, $a_{33}(n) \le 0$. *We see that* $u_1(n) = -1 \le 0$, $v_2(n) = -1 \le 0$, $s_3(n) = 0 \le 0$, $u_{11}(n) = 0 \le 0$, $v_{12}(n) = -1 \le 0$, $s_{13}(n) = 0 \le 0$,

$a_{11}(n+2)+u_1(n+1)+u_{11}(n) = -2 < 0$, $a_{22}(n+2)+v_2(n+1)+v_{12}(n) = -2 < 0$ *and* $a_{33}(n+2)+s_3(n+1)+s_{13}(n) = -1 < 0$ *for all* $n \in \mathbb{N}$. *On the virtue of Theorem 30, the system* (24) *is oscillatory. In fact, it is easy check that a solution of* (24) *is the following*

$$\begin{cases} x_1(n) = c_1 \cos \frac{2n\pi}{3} + c_2 \sin \frac{2n\pi}{3} \\ x_2(n) = (-1)^n \big(c_1 \cos \frac{2(n-1)\pi}{3} + c_2 \sin \frac{2(n-1)\pi}{3}\big), \\ x_3(n) = c_3(-1)^n \end{cases}$$

for $n \geq 1$, *and it is oscillatory.*

Theorem 31. *Let* $a_{11}(n) > 0, a_{22}(n) > 0, a_{33}(n) > 0, u_1(n) \leq 0, v_2(n) \leq 0, s_3(n) \leq 0, u_{11}(n) \leq 0, v_{12}(n) \leq 0$ *and* $s_{13}(n) \leq 0$ *for all* $n \in \mathbb{N}$. *Assume that conditions* (9), (11) *and* (12) *hold for all* $n \in \mathbb{N}$.
If the following inequalities

$$\frac{(-u_{11}(n+1))}{a_{11}(n+3)a_{11}(n+1)} > \frac{u_1(n+2)}{a_{11}(n+3)} + \frac{u_1(n+1)}{a_{11}(n+1)} + a_{11}(n+2),$$

$$\frac{(-v_{12}(n+1))}{a_{22}(n+3)a_{22}(n+1)} > \frac{v_2(n+2)}{a_{22}(n+3)} + \frac{v_2(n+1)}{a_{22}(n+1)} + a_{22}(n+2),$$

and

$$\frac{(-s_{13}(n+1))}{a_{33}(n+3)a_{33}(n+1)} > \frac{s_3(n+2)}{a_{33}(n+3)} + \frac{s_3(n+1)}{a_{33}(n+1)} + a_{33}(n+2)$$

hold for large n, *then* (1) *is oscillatory.*

Theorem 32. *Let* $a_{11}(n) \leq 0, a_{22}(n) \leq 0, a_{33}(n) \leq 0, u_1(n) > 0, v_2(n) > 0, s_3(n) > 0, u_{11}(n) \leq 0, v_{12}(n) \leq 0$ *and* $s_{13}(n) \leq 0$ *for all* $n \in \mathbb{N}$. *Assume that conditions* (9), (11) *and* (12) *hold for all* $n \in \mathbb{N}$.
If the following inequalities

$$u_1(n+1) < \frac{a_{11}(n+2)u_{11}(n+1)}{u_1(n+2)} + \frac{u_{11}(n)a_{11}(n+1)}{u_1(n)},$$

$$v_2(n+1) < \frac{a_{22}(n+2)v_{12}(n+1)}{v_2(n+2)} + \frac{v_{12}(n)a_{22}(n+1)}{v_2(n)}$$

and

$$s_3(n+1) < \frac{a_{33}(n+2)s_{13}(n+1)}{s_3(n+2)} + \frac{s_{13}(n)n+1)}{s_3(n)}$$

hold for any large n, *then the system* (1) *is oscillatory.*

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