

## EXISTENCE RESULTS FOR SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS IN BANACH SPACES

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In this paper we are concerned with the existence of solutions for certain classes of second order differential equations. First we deal with an infinite system of second order linear differential equations, which is reduced to an ordinary differential equation posed in the space of convergent sequences. Next we investigate the problem of existence for a second order differential equation posed on an arbitrary Banach space. The used approach is based on the measures of noncompactness concept, the use of Darbo's fixed point theorem and Kamke comparison functions.

### 1. INTRODUCTION AND PRELIMINARIES

Fixed point theory is an important tool in Nonlinear Analysis, in particular, in obtaining existence result for a variety of mathematical problems. One of the most used results of Fixed Point Theory, is the Banach Contraction Principle, which asserts that any contraction defined on a Banach space (or a complete metric space) has a unique fixed point. However, in many cases, the operator arising in problem under consideration does not belong to the class of contraction mappings (nonexpansive mappings, discontinuous mappings etc.). In such cases Banach Contraction Principle cannot be applied. Similarly another celebrated fixed point theorem, is due to Schauder, (which is an extension of Brouwer's fixed point result from finite

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dimension to infinite dimensional Banach spaces) where the operator is supposed to be compact as well as continuous. Compactness condition in this result appears to be strong enough to be applicable to a large class of problems. For operators on infinite dimensional Banach spaces and locally convex topological spaces, one encounters such a situation very often. A significant generalization of compactness condition, was done by K. Kuratowski in 1930 [17], defining a characteristic value for a set, named as Kuratowski or alpha measure of noncompactness, defined below.

$$(1) \quad \alpha(B) = \inf \{ \delta > 0 : B = \cup_{i=1}^n B_i, \text{ diam } (B_i) \leq \delta, 1 \leq i \leq n, n \in \mathbb{N} \}$$

This concept was first used by G. Darbo in 1955 [10], where a class of operator namely  $\alpha$ -condensing operator was introduced to obtain a fixed point result. This theorem relied on a weaker assumption than compactness, which introduced the so called condensing operator. Later on, Hausdorff MNC was introduced by Goldenstein et. al. [19] and its properties were studied in [20], which is defined below for a bounded subset  $B$ ,

$$(2) \quad \chi(B) = \inf \{ \epsilon > 0 : B \text{ has finite } \epsilon \text{-net in } E \}$$

In 1972, Istrătescu gave another important MNC, known as Istrătescu or  $\beta$ -measure of noncompactness defined as follows [27]

$$(3) \quad \beta(B) = \inf \{ \epsilon > 0 : B \text{ has no infinite } \epsilon \text{- discrete subset} \}.$$

A measure of noncompactness, is a non-negative real valued map, defined on a collection of bounded subsets of a metric space. It assigns the value zero to each member of class of relatively compact sets. There are several ways to define a measure of noncompactness on a given space, the widely used approach is the axiomatic approach, introduced by Banas and Goebel [11], which is discussed below.

Let  $(X, \|\cdot\|)$  be a normed space. The unit sphere and closed ball in  $X$  are denoted by  $S_X := \{x \in X : \|x\| = 1\}$  and  $B_X := \{x \in X : \|x\| \leq 1\}$  respectively. A ball of radius  $r$  and centered at  $x_0$  is denoted by  $B(x_0, r) := \{x \in X : \|x - x_0\| \leq r\}$ .

For a Banach space  $E$ , let  $\mathcal{M}_E$  denote the class of all non-empty, bounded subsets of  $E$  and  $\mathcal{N}_E$  denote the class of all non-empty, relatively compact subsets of  $E$ . Let  $\mathbb{R}_+ = [0, \infty)$ ,  $\bar{X}$  and  $\text{Conv}(X)$  denote the closure and convex closure of a set  $X$ , respectively. Now we define measure of noncompactness axiomatically.

**Definition 1.** A mapping  $\mu : \mathcal{M}_E \rightarrow \mathbb{R}_+$  is said to be measure of noncompactness on  $E$ , if it is monotonic, convex, and invariant under passage to closed convex hull, the relatively compact sets are assigned the value 0, and, satisfies cantor intersection type property w.r.t  $\mu$ .

For details of measure of noncompactness and above definition refer to [11, 14].

**Definition 2.** Let  $\mu$  be a measure of noncompactness in a Banach space  $E$ . We call the measure to be sublinear if

$\mu(\lambda X) = |\lambda|\mu(X)$  for  $\lambda \in \mathbb{R}$ . (homogeneous)

$\mu(X + Y) \leq \mu(X) + \mu(Y)$ . (subadditive)

**Definition 3.** Let  $E$  be a Banach space and  $C = C([0, T], E)$  denote the space of all continuous functions acting from  $[0, T]$  into  $E$  with maximum norm. A set  $X \in \mathcal{M}_C$  will be called regular if  $X$  is equicontinuous on the interval  $[0, T]$ .

**Theorem 1.** [Darbo [10]] Let  $M$  be a non-empty, closed, convex, bounded subset of a Banach space, and  $T : M \rightarrow M$  is a  $\mu$ -condensing operator with contraction constant  $k < 1$ . Then  $T$  has at least one fixed point and the set of all fixed points of  $T$  belongs to  $\ker_\mu$ , where  $\mu$  is an arbitrary measure of noncompactness.

**Theorem 2.** [Schauder [25]] Let  $C$  be a closed, convex subset of a Banach space  $E$ . Then every compact, continuous map  $F : C \rightarrow C$  has at least one fixed point.

Now we briefly describe the concept of Kamke comparison function, fix  $T > 0$  and put  $J = [0, T]$ ,  $J_0 = (0, T]$ . Let  $\Omega$  denote an arbitrary open subset of Banach space  $E$ . For detail on various classes of Kamke type functions refer to [14]. We define Kamke comparison function as follows

**Definition 4.** A function  $\omega : J_0 \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is called a Kamke comparison function provided  $\omega$  should be measurable in  $t$  and continuous and non-decreasing with respect to the second argument, having the property  $\omega(t, 0) = 0$  and such that  $u(t) \equiv 0$  is the only continuous function on  $J$  satisfying the inequality  $u(t) \leq \int_0^t \omega(s, u(s))ds$  and the condition  $u(0) = 0$ .

**Note:** A function  $f : J_0 \times \Omega \rightarrow \mathbb{R}_+$  belongs to Kamke comparison class  $\omega$ , if  $\|f(t, x) - f(t, y)\| \leq \omega(t, \|x - y\|)$ , is valid for  $x, y \in \Omega$ ,  $t \in J$  (or  $t \in J_0$ )

**Lemma 1.** (pp. 222 [14]) Let  $X \in \mathcal{M}_C$  be a regular set and let  $x_0 \in E$  be arbitrarily fixed. Then

$$\mu\left(x_0 + \int_0^t X(s)ds\right) \leq \int_0^t \mu(x_0 + X(s))ds.$$

for any  $t \in [0, T]$  provided  $T \leq 1$ .

## 2. SECOND ORDER IVP IN BANACH SEQUENCE SPACE

In this section we introduce an infinite system of second order linear differential equations, with initial conditions. The theory of infinite system of ordinary differential equation, was initiated by Persidskii [18]. One can come across the infinite systems of differential equations, in variety of situations, such as, numerical methods for PDE [7], certain problems in neural networks and branching process [16] and some problems in mechanics [16], [26]. An interesting fact is, that the

theory of infinite systems of ODE can be considered as a particular case of the theory of ordinary differential equations in Banach spaces. As it is readily observable, an infinite system of differential equation can be rewritten, as an ordinary differential equation in an appropriate Banach space. This lucid fact allows us to make a suitable adaptation of infinite system of differential equation to a Banach space, thus enabling one to exploit the tools and rich theory of differential equations in Banach spaces itself.

The solution of first order Cauchy problem  $x'(t) = f(t, x(t))$ ,  $t \in [0, T]$ ,  $x(t) \in E$  for all  $t \in [0, T]$ ,  $x(0) = x_0 \in E$ , in various Banach space of infinite dimension has been discussed in ([12],[21], [13], [22], [24]). The integral equations in such Banach spaces have been studied in ([9],[2],[14],[3],[1]). Second order boundary value problems for the sequence space  $\ell_1$  has been discussed in [4] and [23].

Now we introduce our problem which is a second order differential equation with two initial conditions. Let  $(E, \|\cdot\|)$  be a Banach sequence space, consider the second order linear initial value problem (IVP) given below

$$(4) \quad y''(t) = F(t, y(t)), \quad y(0) = \alpha, \quad y'(0) = \beta, \quad t \in [0, T].$$

where  $y(0) = (\alpha_i)$  and  $y'(0) = (\beta_i) \in E$ .

The function  $F(t, y) = (f_1(t, y), f_2(t, y), \dots)$  is defined over  $[0, T] \times B \mapsto B$ , where  $B$  is bounded subset of  $E$ . The real valued functions  $f_i$  are defined on  $I \times \mathbb{R}^\infty$ . Integrating (4) the above equation w.r.t  $t$ , we get

$$y'_i(t) - \beta_i = \int_0^t f_i(s, y(s)) ds$$

Again integrating w.r.t  $t$  we have

$$(5) \quad \begin{aligned} y_i(t) - t\beta_i - \alpha_i &= \int_0^t dt \int_0^t f_i(s, y(s)) ds \\ y_i(t) &= \alpha_i + \int_0^t (\beta_i + (t-s)f_i(s, y(s))) ds \end{aligned}$$

Thus the system of differential equations (4) is equivalent to the system of integral equations given by (5). However, (5) is meaningful if  $f_i(s, y(s))$  is Lebesgue integrable.

Next we discuss the existence results for the solution of the system of second order linear IVP (5), in the sequence space  $c_0$  equipped with norm  $\|x\|_{c_0} = \max\{|x_k| : k = 1, 2, \dots\}$ . The Hausdorff measure of noncompactness (2) for a nonempty bounded subset  $B$  of space  $c_0$  is given as

$$\chi(B) = \lim_{i \rightarrow \infty} \left\{ \sup_{x \in B} \{ \max\{|x_k| : k \geq i\} \} \right\}$$

We have the assumptions that  $y(t) = (y_i(t)) \in c_0$  for all  $t \in [0, T]$ . Also  $y'(t)$  and  $y''(t) \in c_0$  for all  $t \in [0, T]$  and  $(\alpha_i), (\beta_i) \in c_0$ . We can rewrite the system (5) as follows

$$(6) \quad y(t) = \left( \alpha_i + t\beta_i + \int_0^t (t-s)f_i(s, y(s))ds \right)$$

We will investigate the system (6) under the following hypotheses:

(i) The functions  $f_i$  are defined on the set  $I \times \mathbb{R}^\infty$  with values in  $\mathbb{R}$ . The operator  $f$  defined on the space  $I \times c_0$  as follows

$$(t, y) \mapsto (fy)(t) = (f_1(t, y(t)), f_2(t, y(t)), \dots)$$

transforms the space  $I \times c_0$  into  $c_0$ , such that the family of functions  $\{(fy)(t)_{t \in I}\}$  is equicontinuous at every point of the space  $c_0$ .

(ii) There exist non-negative functions  $a_i(t)$  and  $b_i(t)$  satisfying

$$|f_i(t, y_1, y_2, y_3, \dots)| \leq a_i(t) + b_i(t) \sup\{|y_k| : k \geq 1\}$$

for each  $t \in I, i = 1, 2, \dots$  and each  $y = (y_i) \in c_0$ .

Moreover the functions  $a_i(t), b_i(t)$  are Lebesgue integrable, bounded on  $I$  such that  $\int_0^T b_i(s)ds < 1$ , and the function sequence  $\left(\int_0^t a_i(s)ds\right)$  converges monotonically to zero at each point  $t \in I$ , the sequence  $\left(\int_0^t b_i(s)ds\right)$  is non-increasing at each point  $t \in I$ .

**Remark 1.** The assumption (i) on the equicontinuity of the family of functions  $\{(fy)(t)_{t \in I}\}$  at each point in the space  $c_0$  means that for any  $y_0 \in c_0$  and for any arbitrarily fixed  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $\|((t-s)fy)(t) - ((t-s)fy_0)(t)\|_{c_0} \leq \|(fy)(t) - (fy_0)(t)\|_{c_0} \leq \epsilon$  for each  $t \in I$  and for each  $y \in c_0$  such that  $\|y - y_0\|_{c_0} \leq \delta$ .

**Theorem 3.** Under the assumptions (i)-(ii), the infinite system (6) has at least one solution  $y(t) = (y_i(t))$  such that  $y(t) \in c_0$  for each  $t \in I = [0, T]$  where  $T < 1$ .

*Proof.* Consider  $C = C(I, c_0)$  the space of all continuous functions from  $I$  to  $c_0$ . Let  $Y_0$  be the subset of  $C$  which contains all the functions  $y(t) = (y_i(t))$  such that

$$\sup\{|y_k(t)| : k \geq i\} \leq h_i(t) + g_i(t) \quad i \in \{1, 2, \dots\}, t \in I$$

where  $h_i(t)$  and  $g_i(t)$  are defined as

$$h_i(t) = \frac{\int_0^t a_i(s)ds}{1 - \int_0^t b_i(s)ds}, \quad g_i(t) = \frac{\sup\{|\alpha_i + \beta_i s| : 0 \leq s \leq t\}}{1 - \int_0^t b_i(s)ds}$$

The functions  $h_i(t)$  and  $g_i(t)$  are non-decreasing on the interval  $I$ . From our assumptions for  $a_i(t)$  and  $b_i(t)$ , it clearly follows that the function sequences  $(h_i(t))$  and  $(g_i(t))$  converge uniformly on  $I$  to the function vanishing identically on  $I$ . Now let us consider the operator  $\mathcal{F}$  defined on the space  $C(I, c_0)$  as follows

$$(\mathcal{F}y)(t) = ((\mathcal{F}y)_i(t)) = \left( \alpha_i + t\beta_i + \int_0^t (t-s)f_i(s, y_1, y_2, \dots)ds \right)$$

The operator  $\mathcal{F}$  maps the set  $Y_0$  into itself. Let us arbitrarily fix  $i$  and  $y_0 \in Y_0$ . Then for  $k \geq i$ , we have

$$\begin{aligned} |(\mathcal{F}y)_k(t)| &\leq |\alpha_k + t\beta_k| + \left| \int_0^t (t-s)f_k(s, y_1, y_2, \dots)ds \right| \\ &\leq |\alpha_i + t\beta_i| + \int_0^t |t-s| |f_k(s, y_1, y_2, \dots)|ds \end{aligned}$$

as  $T < 1$  hence  $|t-s| < 1$ .

$$\begin{aligned} |(\mathcal{F}y)_k(t)| &\leq |\alpha_i + t\beta_i| + \int_0^t |f_k(s, y_1, y_2, \dots)|ds \\ &\leq |\alpha_i + t\beta_i| + \int_0^t [a_k(s) + b_k(s) \sup\{|y_n(s)| : n \geq k\}]ds \\ &\leq |\alpha_i + t\beta_i| + \int_0^t a_k(s)ds + \int_0^t b_k(s)[h_k(s) + g_k(s)]ds \\ &\leq \sup\{|\alpha_i + s\beta_i| : 0 \leq s \leq t\} + \int_0^t a_i(s)ds + [h_i(t) + g_i(t)] \int_0^t b_i(s)ds \\ &\leq h_i(t) + g_i(t) \end{aligned}$$

To show that operator  $\mathcal{F}$  is continuous on the set  $Y_0$ , let us fix arbitrarily  $\epsilon > 0$  and  $y_0 \in Y_0$ . Now, choose  $\delta = \delta(y_0, \epsilon)$  in view of assumption (i) (cf. Remark 1) i.e. for  $y_0 \in Y_0$  such that  $\|y - y_0\| \leq \delta$ , we have  $\|(fy)(t) - (fy_0)(t)\|_{c_0} \leq \epsilon$  for any  $t \in I$ . Then we obtain

$$\begin{aligned} \|(\mathcal{F}y)(t) - (\mathcal{F}y_0)(t)\|_{c_0} &\leq \max\{|(\mathcal{F}y)_i(t) - (\mathcal{F}y_0)_i(t)| : i \geq 1\} \\ &\leq \max\left\{ \int_0^t |f_i(s, y_1, y_2, \dots) - f_i(s, y_1^0, y_2^0, \dots)|ds : i \geq 1 \right\} \\ &\leq T\epsilon. \end{aligned}$$

which proves the continuity of  $\mathcal{F}$ .

Let us consider the set  $Y_1 = \mathcal{F}Y_0$ , clearly  $Y_1$  consists of equicontinuous functions

on  $I$ . If we take arbitrary  $y = (y_i) \in Y_0$  and in view of our assumptions, we obtain

$$\begin{aligned} |(\mathcal{F}y)_i(t) - (\mathcal{F}y)_i(s)| &\leq |(t-s)\beta_i| + \left| \int_0^t f_i(z, y_1(z), y_2(z), \dots) dz \right| \\ &\leq |t-s| \cdot |\beta_i| + \int_0^t [a_i(z) + b_i(z) \sup\{|y_k(z)| : k \geq i\}] dz \\ &\leq |t-s| \cdot |\beta_i| + |t-s| [\sup\{a_i(z) : z \in I\} \\ &\quad + \sup\{b_i(z)(h_i(z) + g_i(z)) : z \in I\}] \end{aligned}$$

Considering our assumptions, it follows that the function sequences  $a_i(t)$ ,  $b_i(t)$ ,  $h_i(t)$  and  $g_i(t)$  are uniformly bounded on  $I$ . We observe that the set  $Y = \mathcal{F}Y_0$  is equicontinuous on  $I$ .

Now let us consider  $Y' = \overline{\text{conv}Y_1}$  (i.e. closed and convex hull of the set  $Y_1$ ). Clearly  $Y'$  is closed, bounded and equicontinuous on  $I$ . Moreover  $Y' \subset Y_0$  and  $\mathcal{F}Y' \subset Y'$ . Also for  $y \in Y_0$

$$|(\mathcal{F}y)_i(t)| \leq h_i(t) + g_i(t), \quad i = 1, 2, 3, \dots$$

As the sequence  $(h_i(t) + g_i(t))$  converges uniformly on  $I$  to the function vanishing identically on  $I$ . We deduce that for each  $\epsilon > 0$  there exists  $n_0 \in \mathbb{N}$  such that

$$|(\mathcal{F}y)_i(t)| \leq \epsilon, \quad \forall i \geq n_0, \forall t \in I$$

Using above inference, we obtain that for any  $t \in I$ , the set  $Y_1(t)$

$$\chi[Y_1(t)] = \lim_{i \rightarrow \infty} \left\{ \sup_{y \in Y_1} \{ \max\{|y_k| : k \geq i\} \} \right\} = 0.$$

Thus  $Y_1(t)$  is relatively compact in the space  $C(I, c_0)$ . Since  $Y'$  is closed and hence it is compact. As  $\mathcal{F}$  maps continuously the set  $Y'$  into itself, by the Schauder fixed point theorem 2, we deduce that the operator  $\mathcal{F}$  has atleast one fixed point in the set  $Y'$  which is a solution of our problem. This completes the proof.  $\square$

**Note:** If  $(E, \|\cdot\|_E)$  denotes a Banach space, and  $\chi_E$  is the Hausdorff MNC on space  $E$ . Then on the space  $C([a, b]; E)$  of continuous functions from  $[a, b]$  with values in  $E$  Hausdorff MNC is defined as follows (for  $X$  a non-empty, bounded subset of  $C([a, b]; E)$ ).

$$\chi_{C([a, b]; E)}(X) = \sup_{t \in [a, b]} \chi_E(X(t))$$

**Example 1.** Consider the system of second second order linear IVP  $y_n''(t) = f_n(t, y(t))$  in sequence space  $c_0$ , where

$$f_n(t, y(t)) = \frac{\sin^2(nt) \exp(-nt)}{(n+1)^2} + \sum_{m=n}^{\infty} \frac{y_m(t)}{\pi^2(m+1)^2},$$

and  $n \in \mathbb{N}, y_n'(0) = \frac{1}{n}, y_n''(0) = \frac{1}{n^2}, t \in [0, T]$ .

In the above problem  $y(t) \in c_0$  for all  $t \in [0, T]$ . We will show that  $f_n(t, y(t)) \in c_0$  for all  $t \in [0, T]$ . Let  $t$  be arbitrarily fixed,

$$\begin{aligned} \lim_{n \rightarrow \infty} f_n(t, y(t)) &= \lim_{n \rightarrow \infty} \frac{\sin^2(nt) \exp(-nt)}{(n+1)^2} + \lim_{n \rightarrow \infty} \sum_{m=n}^{\infty} \frac{y_m(t)}{\pi^2(m+1)^2} \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{(n+1)^2} + \lim_{n \rightarrow \infty} \max_{m \geq n} |y_m(t)| \sum_{m=n}^{\infty} \frac{1}{\pi^2(m+1)^2} = 0. \end{aligned}$$

Thus  $f_n(t, y(t)) \in c_0$ , now we will show that the map  $f$  defined from  $I \times c_0$  to  $c_0$  as  $(t, y) \mapsto (fy)(t) = (f_1(t, y(t)), f_2(t, y(t)), \dots)$  is such that  $\{(fy)(t)_{t \in I}\}$  is equicontinuous at each point of  $c_0$  (cf. Remark 1). Let us choose  $y^0 \in c_0$  and  $\epsilon > 0$  arbitrarily, then for  $\|y(t) - y^0(t)\|_{c_0} < \delta := 4\epsilon$

$$\begin{aligned} \|(fy)(t) - (fy^0)(t)\|_{c_0} &= \max_{n \geq 1} |f_n(t, y(t)) - f_n(t, y^0(t))| \\ &= \max_{n \geq 1} \left| \sum_{m=n}^{\infty} \frac{y_m(t) - y_m^0(t)}{\pi^2(m+1)^2} \right| \\ &\leq \|y(t) - y^0(t)\|_{c_0} \frac{\pi^2}{4\pi^2} \\ &< \epsilon. \end{aligned}$$

Thus  $f(t, y(t))$  satisfies assumption (i) of Theorem 3. Now we will show that assumption (ii) is also satisfied

$$|f_n(t, y(t))| \leq \frac{\sin^2(nt)}{(n+1)^2} + \sup\{|y_m| : m \geq 1\} \sum_{m=n}^{\infty} \frac{1}{\pi^2(m+1)^2}$$

Take  $a_n(t) = \frac{\sin^2(nt)}{(n+1)^2}$  and  $b_n(t) = \sum_{m=n}^{\infty} \frac{1}{\pi^2(m+1)^2}$ . clearly  $a_n(t)$ ,  $b_n(t)$  are bounded and integrable on  $I$ .  $\int_0^T b_n(s) ds = \int_0^T \sum_{m=n}^{\infty} \frac{1}{\pi^2(m+1)^2} ds < \frac{T}{4} < 1$ , also the function sequence  $\left(\int_0^t a_n(s) ds\right) = \left(\int_0^t \frac{\sin^2(ns)}{(n+1)^2} ds\right)$  is monotonically decreasing and converges to zero for each  $t \in I$ . The sequence  $\left(\int_0^t b_n(s) ds\right) = \left(\int_0^t \sum_{m=n}^{\infty} \frac{1}{\pi^2(m+1)^2} ds\right)$  is non-increasing at each  $t \in I$ . As the given problem satisfies the hypotheses of Theorem 3, it has atleast one solution  $y(t) = (y_i(t))$  such that  $y(t) \in c_0$  for each  $t \in I = [0, T]$  where  $T < 1$ .



### 3. SECOND ORDER IVP IN ARBITRARY BANACH SPACE

In this section, we will establish an existence result for the solution of linear second order IVP (4) in a Banach space  $(E, \|\cdot\|)$ , using a variant of Kamke comparison function defined in 4 and an arbitrary measure of noncompactness (satisfying the axioms of MNC as defined in Section 1). Let us observe that the comparison function  $\omega(t, u) = p(t)u$ , where  $p(t)$  is a Lebesgue integrable function and belongs to Kamke comparison class  $\omega$ .

**Theorem 4.** *Assume that  $f : [0, T] \times B(\alpha, r) \rightarrow E$  is uniformly continuous and bounded function,  $\|f(t, y)\| \leq A$  for  $t \in [0, T]$  and  $y \in B(\alpha, r)$ . Moreover assume that  $\mu[\gamma + (t - s)f(s, Y)] \leq p(s)\mu(Y)$  is satisfied for arbitrary  $t \in [0, T]$ , almost all  $s \in [0, T]$  and for arbitrary  $Y \in \mathcal{M}_C$ , where  $\gamma = \alpha + \beta$  and  $\alpha, \beta \in E_\mu$  and  $p(t)$  is Lebesgue integrable function. If  $T(TA + \|\beta\|) \leq r$  and  $T \leq 1$  then IVP (4) has at least one solution  $y(t)$  such that  $y(t) \in E_\mu$  for all  $t \in [0, T]$ .*

*Proof.* Let us denote by  $Y_0$  the subset of the space  $C = C([0, T], E)$  consisting of all functions  $y$  such that  $y(0) = \alpha$ ,  $y'(0) = \beta$  and  $\|y(t) - y(z)\| \leq |t - z|(\|\beta\| + TA)$  for  $t, z \in [0, T]$ . Thus  $Y_0$  is bounded, convex and closed. Also equicontinuity of  $Y_0$  follows from uniform continuity of  $f$ . Let us define the operator

$$(\mathcal{F}y)(t) = \alpha + \int_0^t (\beta + (t - s)f(s, y(s)))ds, \quad t \in [0, T].$$

Clearly the operator  $\mathcal{F}$  maps the set  $Y_0$  into itself. Thus we can proceed to find a fixed point of operator  $\mathcal{F}$  which is a solution of our problem. Now for a fix number  $k > 1$  and for a regular set  $Y \in \mathcal{M}_C$  let us write

$$\mu_k(Y) = \sup \left\{ \mu(Y) \exp \left( -k \int_0^t p(s)ds \right) : t \in [0, T] \right\}$$

The set function  $\mu_k$  is described in (pp. 227 [14]) to solve Cauchy initial value problem, and satisfies the axioms of measure of noncompactness on the family  $\mathcal{M}_C^{eq}$  which is a subfamily of  $\mathcal{M}_C$  comprising of non-empty regular sets. Using Lemma1 we have for any  $Y \in \mathcal{M}_C^{eq}$

$$\begin{aligned} \mu[(\mathcal{F}y)(t)] &= \mu \left( \alpha + \int_0^t [\beta + (t - s)f(s, Y(s))] ds \right) \\ &\leq \int_0^t (\gamma + (t - s)f(s, Y(s)) ds \\ &\leq \int_0^t p(s)\mu(Y(s))ds \\ &\leq \mu_k(Y) \int_0^t p(s) \exp \left( k \int_0^s p(z)dz \right) ds \\ &\leq \exp \left( k \int_0^t p(s)ds \right) \frac{1}{k} \mu_k(Y) \end{aligned}$$

(7)

$$\exp\left(-k \int_0^t p(s) ds\right) \mu(\mathcal{F}Y) \leq \frac{1}{k} \mu_k(X).$$

$$\sup_{t \in I} \exp\left(-k \int_0^t p(s) ds\right) \mu(\mathcal{F}Y) = \mu_k(\mathcal{F}Y) \leq \frac{1}{k} \mu_k(X).$$

The inequality derived above, implies that  $\mathcal{F}$  is a  $\mu_k$ -condensing operator with constant  $1/k < 1$ . By applying Darbo fixed point theorem 1 we get existence of a fixed point of  $\mathcal{F}$  in function space  $C$  which is the solution of our problem. Also the solution  $y(t)$  is such that  $y(t) \in E_\mu$  for all  $t \in [0, T]$ . This completes the proof  $\square$

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