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ON THE SOLUTIONS OF A CLASS OF MAX-MIN-TYPE DIFFERENCE EQUATION SYSTEM

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We mainly investigate the general solutions and periodic solutions to the following system of max-type difference equations

$$x_{n+1} = \max\left\{y_{n-1}^2, \frac{A_n}{y_{n-1}}\right\},$$

$$y_{n+1} = \min\left\{x_{n-1}^2, \frac{B_n}{x_{n-1}}\right\},$$

where $n \in N$, $(A_n)_{n\in N}$ and $(B_n)_{n\in N}$ are positive real sequences, and the initial values $x_{-1} = \alpha, x_0 = \lambda, y_{-1} = \beta, y_0 = \mu$ are real numbers.

1. Introduction

Recently, there has been great interest in studying difference equations and systems which do not stem from differential ones, see [1-3]. A class of difference equations that has attracted recent attentions is the, so called, max-type difference equations, see [4-7]. This type of difference equations arise naturally from certain models in automatic control theory[8]. There appeared a lot of research involving the boundedness character, the global attractivity and the periodic nature of max-type difference equations, for example [9-11]. Particularly, in [12], Taixiang Sun, Jing Liu et al. have studied the eventually periodic solutions of the following max-type difference equation

$$x_n = \max\left\{\frac{A_n}{x_{n-r}}, x_{n-k}\right\}, \tag{1}$$

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where $\{A_n\}_{n=1}^{+\infty}$ is a periodic sequence with period p, $k, r \in \{1, 2, \cdots\}$ with $gcd(k, r) = 1$, $k \neq r$, and the initial conditions$x_{1-d}, x_{2-d}, \cdots, x_0$ are real numbers with $d = max\{r, k\}$. Also in [13], Tarek F.Ibrahim examine the periodicity and formularization of the solutions for a system of semi-type difference equations of second order in the form

$$x_{n+1} = \max\left\{\frac{A_n}{y_{n-1}}, x_{n-1}\right\},$$

$$y_{n+1} = \max\left\{\frac{B_n}{x_{n-1}}, y_{n-1}\right\}, \tag{2}$$

$n \in N_0$, where $N_0 = N \bigcup \{0\}$,$(A_n)_{n\in N_0}$ and $(B_n)_{n\in N_0}$ are two-periodic positive sequences, and initial values $x_o, x_{-1}, y_o, y_{-1} \in (0, +\infty)$. Motivated by above mentioned work, our main objective is to investigate the general solutions and periodic solutions of the following system of max-type difference equations

$$x_{n+1} = \max\left\{y_{n-1}^2, \frac{A_n}{y_{n-1}}\right\},$$

$$v_{n+1} = \min\left\{x_{n-1}^2, \frac{B_n}{x_{n-1}}\right\}, \tag{3}$$

where $n \in N$, $(A_n)_{n\in N}$ and $(B_n)_{n\in N}$ are positive real sequences, and the initial values $x_{-1} = \alpha, x_0 = \lambda, y_{-1} = \beta, y_0 = \mu$ are real numbers.

2. Main results

In this section, we study the general solutions and the periodic solutions of (3). We will give four theorems and eight corollaries to describe the solutions of (3).

Definition A solution (x_n, y_n) of (3) is said to be eventually periodic with period $p \in N$ if there is an $n_0 \geq -1$, such that $x_{n+p} = x_n, y_{n+p} = y_n$ for $n \geq n_0$. If $n_0 = -1$, then we say that the sequence $(x_n), (y_n)_{n=-1}^{\infty}$ is periodical with period p. Period p is said to be a prime period if there is no $p_1 < p$ which is a period to the sequence $(x_n), (y_n)_{n=-1}^{\infty}$.

Theorem 1 Suppose that (x_n, y_n) is a solution of (3) such that $0 < \alpha, \beta, \lambda, \mu < B_n \leq 1 \leq A_n$. Then the general solution of (3) has the following form.

$$x_1 = \frac{A_0}{\beta}; x_2 = \frac{A_1}{\mu}; x_3 = \frac{A_2}{\alpha^2}; x_4 = \frac{A_3}{\lambda^2};$$

$$y_1 = \alpha^2; y_2 = \lambda^2; y_3 = \frac{B_2}{A_0}\beta; y_4 = \frac{B_3}{A_1}\mu;$$

$$x_{4n+1}=\frac{\prod_{i=1}^{n}A_{4i}}{\prod_{i=1}^{n}B_{4i-2}}\frac{A_0}{\beta};y_{4n+1}=\frac{\prod_{i=1}^{n}B_{4i}}{\prod_{i=1}^{n}A_{4i-2}}\frac{A_0}{\alpha^2};$$

$$x_{4n+2}=\frac{\prod_{i=1}^{n}A_{4i+1}}{\prod_{i=1}^{n}B_{4i-1}}\frac{A_1}{\mu};y_{4n+2}=\frac{\prod_{i=1}^{n}B_{4i+1}}{\prod_{i=1}^{n}A_{4i-1}}\frac{A_1}{\lambda^2};$$

$$x_{4n+3}=\frac{\prod_{i=1}^{n}A_{4i+2}}{\prod_{i=1}^{n}B_{4i}}\frac{A_2}{\alpha^2};y_{4n+3}=\frac{\prod_{i=1}^{n}B_{4i+2}}{\prod_{i=1}^{n}A_{4i}}\frac{B_2\beta}{A_0};$$

$$x_{4n+4}=\frac{\prod_{i=1}^{n}A_{4i+3}}{\prod_{i=1}^{n}B_{4i+1}}\frac{A_3}{\lambda^2};y_{4n+4}=\frac{\prod_{i=1}^{n}B_{4i+3}}{\prod_{i=1}^{n}A_{4i+1}}\frac{B_3\mu}{A_1},$$

where $i=1,2,3,\cdots$.

proof We proof the theorem by mathematical induction. For $n=0$, we have

$$x_1=\max\left\{y_{-1}^2,\frac{A_0}{y_{-1}}\right\}=\max\left\{\beta^2,\frac{A_0}{\beta}\right\}=\frac{A_0}{\beta};x_2=\max\left\{y_0^2,\frac{A_1}{y_0}\right\}=\max\left\{\mu^2,\frac{A_1}{\mu}\right\}=\frac{A_1}{\mu};$$

$$y_1=\min\left\{x_{-1}^2,\frac{B_0}{x_{-1}}\right\}=\max\left\{\alpha^2,\frac{B_0}{\alpha}\right\}=\alpha^2;y_2=\min\left\{x_0^2,\frac{B_1}{x_0}\right\}=\max\left\{\lambda^2,\frac{B_1}{\lambda}\right\}=\lambda^2;$$

$$x_3=\max\left\{y_1^2,\frac{A_2}{y_1}\right\}=\max\left\{\alpha^4,\frac{A_2}{\alpha^2}\right\}=\frac{A_2}{\alpha^2};x_4=\max\left\{y_2^2,\frac{A_3}{y_2}\right\}=\max\left\{\lambda^4,\frac{A_3}{\lambda^2}\right\}=\frac{A_3}{\lambda^2};$$

$$y_3=\min\left\{x_1^2,\frac{B_2}{x_1}\right\}=\max\left\{(\frac{A_0}{\beta})^2,\frac{B_2\beta}{A_0}\right\}=\frac{B_2\beta}{A_0};y_4=\min\left\{x_2^2,\frac{B_3}{x_2}\right\}=\max\left\{(\frac{A_1}{\mu})^2,\frac{B_3\mu}{A_1}\right\}=\frac{B_3\mu}{A_1},$$

so the results hold. Now we suppose that $k>0$ and the result is true for $n=k$. For $n=k+1$, we have the following results.

$$x_{4(k+1)+1}=\max\left\{y_{4(k+1)-1}^2,\frac{A_{4(k+1)}}{y_{4(k+1)-1}}\right\}=\max\left\{y_{4k+3}^2,\frac{A_{4k+4}}{y_{4k+3}}\right\}$$

$$=\max\left\{(\frac{\prod_{i=1}^{k}B_{4i+2}}{\prod_{i=1}^{k}A_{4i}}\frac{B_2\beta}{A_0})^2,\frac{\prod_{i=1}^{k}A_{4i}A_{4(k+1)}}{\prod_{i=1}^{k}B_{4i+2}}\frac{A_0}{B_2\beta}\right\}=\frac{\prod_{i=1}^{k+1}A_{4i}}{\prod_{i=1}^{k+1}B_{4i-2}}\frac{A_0}{\beta};$$

$$y_{4(k+1)+1}=\min\left\{x_{4(k+1)-1}^2,\frac{B_{4(k+1)}}{x_{4(k+1)-1}}\right\}=\min\left\{x_{4k+3}^2,\frac{B_{4k+4}}{x_{4k+3}}\right\}$$

$$= \min\left\{ (\frac{\prod_{i=1}^{k} A_{4i+2}}{\prod_{i=1}^{k} B_{4i}} \frac{A_2}{\alpha^2})^2, \frac{\prod_{i=1}^{k} B_{4i} B_{4(k+1)}}{\prod_{i=1}^{k} A_{4i+2}} \frac{\alpha^2}{A_2} \right\} = \frac{\prod_{i=1}^{k+1} B_{4i}}{\prod_{i=1}^{k+1} A_{4i-2}} \alpha^2;$$

$$x_{4(k+1)+2} = \max\left\{ y^2_{4(k+1)}, \frac{A_{4(k+1)+1}}{y_{4(k+1)}} \right\} = \max\left\{ y^2_{4k+4}, \frac{A_{4k+5}}{y_{4k+4}} \right\}$$

$$= \max\left\{ (\frac{\prod_{i=1}^{k} B_{4i+3}}{\prod_{i=1}^{k} A_{4i+1}} \frac{B_3\mu}{A_1})^2, \frac{\prod_{i=1}^{k} A_{4i+1} A_{4(k+1)+1}}{\prod_{i=1}^{k} B_{4i+3}} \frac{A_1}{B_3\mu} \right\} = \frac{\prod_{i=1}^{k+1} A_{4i+1}}{\prod_{i=1}^{k+1} B_{4i-1}} \frac{A_1}{\mu};$$

$$y_{4(k+1)+2} = \min\left\{ x^2_{4(k+1)}, \frac{B_{4(k+1)+1}}{x_{4(k+1)}} \right\} = \min\left\{ x^2_{4k+4}, \frac{B_{4k+5}}{x_{4k+4}} \right\}$$

$$= \min\left\{ (\frac{\prod_{i=1}^{k} A_{4i+3}}{\prod_{i=1}^{k} B_{4i+1}} \frac{A_3}{\lambda^2})^2, \frac{\prod_{i=1}^{k} B_{4i+1} B_{4(k+1)+1}}{\prod_{i=1}^{k} A_{4i+3}} \frac{\lambda^2}{A_3} \right\} = \frac{\prod_{i=1}^{k+1} B_{4i+1}}{\prod_{i=1}^{k+1} A_{4i-1}} \lambda^2;$$

$$x_{4(k+1)+3} = \max\left\{ y^2_{4(k+1)+1}, \frac{A_{4(k+1)+2}}{y_{4(k+1)+1}} \right\} = \max\left\{ y^2_{4k+5}, \frac{A_{4k+6}}{y_{4k+5}} \right\}$$

$$= \max\left\{ (\frac{\prod_{i=1}^{k+1} B_{4i}}{\prod_{i=1}^{k+1} A_{4i-2}} \alpha^2)^2, \frac{\prod_{i=1}^{k+1} A_{4i-2} A_{4(k+1)+2}}{\prod_{i=1}^{k+1} B_{4i}\alpha^2} \right\} = \frac{\prod_{i=1}^{k+1} A_{4i+2}}{\prod_{i=1}^{k+1} B_{4i}} \frac{A_2}{\alpha^2};$$

$$y_{4(k+1)+3} = \min\left\{ x^2_{4(k+1)+1}, \frac{B_{4(k+1)+2}}{x_{4(k+1)+1}} \right\} = \min\left\{ x^2_{4k+5}, \frac{B_{4k+6}}{x_{4k+5}} \right\}$$

$$= \min\left\{ (\frac{\prod_{i=1}^{k+1} A_{4i}}{\prod_{i=1}^{k+1} B_{4i-2}} \frac{A_0}{\beta})^2, \frac{\prod_{i=1}^{k+1} B_{4i-2} B_{4(k+1)+2}}{\prod_{i=1}^{k+1} A_{4i}} \frac{\beta}{A_0} \right\} = \frac{\prod_{i=1}^{k+1} B_{4i+2}}{\prod_{i=1}^{k+1} A_{4i}} \frac{B_2\beta}{A_0};$$

$$x_{4(k+1)+4} = \max\left\{ y^2_{4(k+1)+2}, \frac{A_{4(k+1)+3}}{y_{4(k+1)+2}} \right\} = \max\left\{ y^2_{4k+6}, \frac{A_{4k+7}}{y_{4k+6}} \right\}$$

$$= \max\left\{ (\frac{\prod_{i=1}^{k+1} B_{4i+1}}{\prod_{i=1}^{k+1} A_{4i-1}} \lambda^2)^2, \frac{\prod_{i=1}^{k+1} A_{4i-1} A_{4(k+1)+3}}{\prod_{i=1}^{k+1} B_{4i+1}\lambda^2} \right\} = \frac{\prod_{i=1}^{k+1} A_{4i+3}}{\prod_{i=1}^{k+1} B_{4i+1}} \frac{A_3}{\lambda^2};$$

$$y_{4(k+1)+4} = \min\left\{ x^2_{4(k+1)+2}, \frac{B_{4(k+1)+3}}{x_{4(k+1)+2}} \right\} = \min\left\{ x^2_{4k+6}, \frac{B_{4k+7}}{x_{4k+6}} \right\}$$

$$= \min\left\{ \left(\frac{\prod_{i=1}^{k+1} A_{4i+1}}{\prod_{i=1}^{k+1} B_{4i-1}} \frac{A_1}{\mu}\right)^2, \frac{\prod_{i=1}^{k+1} B_{4i-1}B_{4(k+1)+3}}{\prod_{i=1}^{k+1} A_{4i+1}} \frac{\mu}{A_1} \right\} = \frac{\prod_{i=1}^{k+1} B_{4i+3}}{\prod_{i=1}^{k+1} A_{4i+1}} \frac{B_3\mu}{A_1},$$

where $i = 1, 2, 3, \cdots$, therefore the result is true for every $k \in N$.

When $A_n = A, B_n = B$, the following corollary holds.

Corollary 1 Suppose that (x_n, y_n) is a solution of (3) such that $0 < \alpha, \beta, \lambda, \mu < B \leq 1 \leq A$. Then the general solution of (3) are as the following.

$$x_{4n-3} = \frac{A^n}{B^{n-1}\beta}; x_{4n-2} = \frac{A^n}{B^{n-1}\mu}; x_{4n-1} = \frac{A^n}{B^{n-1}\alpha^2}; x_{4n} = \frac{A^n}{B^{n-1}\lambda^2};$$

$$y_{4n-3} = \frac{B^{n-1}}{A^{n-1}}\alpha^2; y_{4n-2} = \frac{B^{n-1}}{A^{n-1}}\lambda^2; y_{4n-1} = \frac{B^n}{A^n}\beta; y_{4n} = \frac{B^n}{A^n}\mu,$$

where $n = 1, 2, 3, \cdots$.

proof Similar to Theorem (1) by mathematical induction.

For a special case, when A_n, B_n are same two-periodic positive sequences, (3) could has four-periodic solution as the following corollary.

Corollary 2 Suppose that (x_n, y_n) is a solution of (3) such that $0 < \alpha, \beta, \lambda, \mu < 1$.If $A_n = B_n = A_{n+2} = B_{n+2}, n = 0, 1, 2, \cdots$. Then (3) has four-periodic solution as the following.

$$x_{4n-3} = \frac{A_0}{\beta}; x_{4n-2} = \frac{A_1}{\mu}; x_{4n-1} = \frac{A_0}{\alpha^2}; x_{4n} = \frac{A_1}{\lambda^2};$$

$$y_{4n-3} = \alpha^2; y_{4n-2} = \lambda^2; y_{4n-1} = \beta; y_{4n} = \mu,$$

where $n = 1, 2, 3, \cdots$.

proof Similar to Theorem (1) by mathematical induction.

Theorem 2 Suppose that (x_n, y_n) is a solution of (3) such that $\alpha, \beta, \lambda, \mu \leq -1, 0 < B_n \leq 1 \leq A_n$. Then the general solution of (3) has the following form.

$$x_1 = \beta^2; x_2 = \mu^2; x_3 = \frac{B_0^2}{\alpha^2}; x_4 = \frac{B_1^2}{\lambda^2};$$

$$y_1 = \frac{B_0}{\alpha}; y_2 = \frac{B_1}{\lambda}; y_3 = \frac{B_2}{\beta^2}; y_4 = \frac{B_3}{\mu^2};$$

$$x_5 = \frac{A_4}{B_2}\beta^2; x_6 = \frac{A_5}{B_3}\mu^2; x_7 = \frac{A_6}{B_0^4}\alpha^4; x_8 = \frac{A_7}{B_1^4}\lambda^4;$$

$$y_5 = \frac{B_0^4}{\alpha^4}; y_6 = \frac{B_1^4}{\lambda^4}; y_7 = \frac{B_2B_6}{A_4\beta^2}; y_8 = \frac{B_3B_7}{A_5\mu^2};$$

$$x_{4n+1}=\frac{\prod_{i=1}^{n}A_{4i}}{\prod_{i=1}^{n}B_{4i-2}}\beta^2;y_{4n+1}=\frac{\prod_{i=2}^{n}B_{4i}}{\prod_{i=2}^{n}A_{4i-2}}\frac{B_0^4}{\alpha^4};$$

$$x_{4n+2}=\frac{\prod_{i=1}^{n}A_{4i+1}}{\prod_{i=1}^{n}B_{4i-1}}\mu^2;y_{4n+2}=\frac{\prod_{i=2}^{n}B_{4i+1}}{\prod_{i=2}^{n}A_{4i-1}}\frac{B_1^4}{\lambda^4};$$

$$x_{4n+3}=\frac{\prod_{i=1}^{n}A_{4i+2}}{\prod_{i=2}^{n}B_{4i}}\frac{\alpha^4}{B_0^4};y_{4n+3}=\frac{\prod_{i=1}^{n}B_{4i+2}}{\prod_{i=1}^{n}A_{4i}}\frac{B_2}{\beta^2};$$

$$x_{4n+4}=\frac{\prod_{i=1}^{n}A_{4i+3}}{\prod_{i=2}^{n}B_{4i+1}}\frac{\lambda^4}{B_1^4};y_{4n+4}=\frac{\prod_{i=1}^{n}B_{4i+3}}{\prod_{i=1}^{n}A_{4i+1}}\frac{B_3}{\mu^2},$$

where $i=2,3,4,\cdots$.

proof The proof is similar to Theorem (1).

When $A_n=A,B_n=B$, the following corollary holds.

Corollary 3 Suppose that (x_n,y_n) is a solution of (3) such that $\alpha,\beta,\lambda,\mu\le-1, 0<B\le 1\le A$. Then the general solution of (3) are as the following.

$$x_1=\beta^2;x_2=\mu^2;x_3=\frac{B^2}{\alpha^2};x_4=\frac{B^2}{\lambda^2};$$

$$y_1=\frac{B}{\alpha};y_2=\frac{B}{\lambda};y_3=\frac{B}{\beta^2};y_4=\frac{B}{\mu^2};$$

$$x_{4n+1}=\frac{A^n}{B^n}\beta^2;x_{4n+2}=\frac{A^n}{B^n}\mu^2;x_{4n+3}=\frac{A^n}{B^{n+3}}\alpha^4;x_{4n+4}=\frac{A^n}{B^{n+3}}\lambda^4;$$

$$y_{4n+1}=\frac{B^{n+3}}{A^{n-1}\alpha^4};y_{4n+2}=\frac{B^{n+3}}{A^{n-1}\lambda^4};y_{4n+3}=\frac{B^{n+1}}{A^n\beta^2};y_{4n+4}=\frac{B^{n+1}}{A^n\mu^2},$$

where $n=1,2,3,\cdots$.

proof Similar to Theorem (1) by mathematical induction.

For a special case, when A_n,B_n are same two-periodic positive sequences, (3) could has eventually four-periodic solution as the following corollary.

Corollary 4 Suppose that (x_n,y_n) is a solution of (3) such that $\alpha,\beta,\lambda,\mu\le-1$. If $A_n=B_n=A_{n+2}=B_{n+2}, n=0,1,2,\cdots$. Then (3) has eventually four-periodic solution as the following.

$$x_1=\beta^2;x_2=\mu^2;x_3=\frac{B_0^2}{\alpha^2};x_4=\frac{B_1^2}{\lambda^2};$$

$$y_1 = \frac{B_0}{\alpha}; y_2 = \frac{B_1}{\lambda}; y_3 = \frac{B_0}{\beta^2}; y_4 = \frac{B_1}{\mu^2};$$

$$x_{4n+1} = \beta^2; x_{4n+2} = \mu^2; x_{4n+3} = \alpha^4; x_{4n+4} = \lambda^4;$$

$$y_{4n+1} = \frac{B_0^4}{\alpha^4}; y_{4n+2} = \frac{B_1^4}{\lambda^4}; y_{4n+3} = \frac{B_0}{\beta^2}; y_{4n+4} = \frac{B_1}{\mu^2},$$

where $n = 1, 2, 3, \cdots$.

proof Similar to Theorem (1) by mathematical induction.

Theorem 3 Suppose that (x_n, y_n) is a solution of (3) such that $-1 \leq \alpha, \beta, \lambda, \mu < 0, A_n \geq B_n \geq 1$. Then the general solution of (3) has the following form.

$$x_1 = \beta^2; x_2 = \mu^2; x_3 = \frac{B_0^2}{\alpha^2}; x_4 = \frac{B_1^2}{\lambda^2};$$

$$y_1 = \frac{B_0}{\alpha}; y_2 = \frac{B_1}{\lambda}; y_3 = \beta^4; y_4 = \mu^4;$$

$$x_5 = \frac{A_4}{\beta^4}; x_6 = \frac{A_5}{\mu^4}; x_7 = \frac{A_6}{B_4}\frac{B_0^2}{\alpha^2}; x_8 = \frac{A_7}{B_5}\frac{B_1^2}{\lambda^2};$$

$$y_5 = B_4\frac{\alpha^2}{B_0^2}; y_6 = B_5\frac{\lambda^2}{B_1^2}; y_7 = \frac{B_6}{A_4}\beta^4; y_8 = \frac{B_7}{A_5}\mu^4;$$

$$x_{4n+1} = \frac{\prod_{i=1}^{n} A_{4i}}{\prod_{i=2}^{n} B_{4i-2}\beta^4}; y_{4n+1} = \frac{\prod_{i=1}^{n} B_{4i}}{\prod_{i=2}^{n} A_{4i-2}}\frac{\alpha^2}{B_0^2};$$

$$x_{4n+2} = \frac{\prod_{i=1}^{n} A_{4i+1}}{\prod_{i=2}^{n} B_{4i-1}\mu^4}; y_{4n+2} = \frac{\prod_{i=1}^{n} B_{4i+1}}{\prod_{i=2}^{n} A_{4i-1}}\frac{\lambda^2}{B_1^2};$$

$$x_{4n+3} = \frac{\prod_{i=1}^{n} A_{4i+2}}{\prod_{i=1}^{n} B_{4i}}\frac{B_0^2}{\alpha^2}; y_{4n+3} = \frac{\prod_{i=1}^{n} B_{4i+2}}{\prod_{i=1}^{n} A_{4i}}\beta^4;$$

$$x_{4n+4} = \frac{\prod_{i=1}^{n} A_{4i+3}}{\prod_{i=1}^{n} B_{4i+1}}\frac{B_1^2}{\lambda^2}; y_{4n+4} = \frac{\prod_{i=1}^{n} B_{4i+3}}{\prod_{i=1}^{n} A_{4i+1}}\mu^4,$$

where $i = 2, 3, 4, \cdots$.

proof The proof is similar to Theorem (1).

When $A_n = A, B_n = B$, the following corollary holds.

Corollary 5 Suppose that (x_n, y_n) is a solution of (3) such that $-1 \leq \alpha, \beta, \lambda, \mu < 0, A \geq B \geq 1$. Then the general solution of (3) are as the following.

$$x_1 = \beta^2; x_2 = \mu^2; x_3 = \frac{B^2}{\alpha^2}; x_4 = \frac{B^2}{\lambda^2};$$

$$y_1 = \frac{B}{\alpha}; y_2 = \frac{B}{\lambda}; y_3 = \beta^4; y_4 = \mu^4;$$

$$x_{4n+1} = \frac{A^n}{B^{n-1}\beta^4}; x_{4n+2} = \frac{A^n}{B^{n-1}\mu^4}; x_{4n+3} = \frac{A^n}{B^{n-2}\alpha^2}; x_{4n+4} = \frac{A^n}{B^{n-2}\lambda^2};$$

$$y_{4n+1} = \frac{B^{n-2}}{A^{n-1}}\alpha^2; y_{4n+2} = \frac{B^{n-2}}{A^{n-1}}\lambda^2; y_{4n+3} = \frac{B^n}{A^n}\beta^4; y_{4n+4} = \frac{B^n}{A^n}\mu^4,$$

where $n = 1, 2, 3, \cdots$.

proof Similar to Theorem (1) by mathematical induction.

For a special case, when A_n, B_n are same two-periodic positive sequences, (3) could has eventually four-periodic solution as the following corollary.

Corollary 6 Suppose that (x_n, y_n) is a solution of (3) such that $-1 \leq \alpha, \beta, \lambda, \mu < 0$. If $A_n = B_n = A_{n+2} = B_{n+2}, n = 0, 1, 2, \cdots$. Then (3) has eventually four-periodic solution as the following.

$$x_1 = \beta^2; x_2 = \mu^2; x_3 = \frac{B_0^2}{\alpha^2}; x_4 = \frac{B_1^2}{\lambda^2};$$

$$y_1 = \frac{B_0}{\alpha}; y_2 = \frac{B_1}{\lambda}; y_3 = \beta^4; y_4 = \mu^4;$$

$$x_{4n+1} = \frac{1}{\beta^4}; x_{4n+2} = \frac{1}{\mu^4}; x_{4n+3} = \frac{B_0^2}{\alpha^2}; x_{4n+4} = \frac{B_1^2}{\lambda^2};$$

$$y_{4n+1} = \frac{\alpha^2}{B_0^2}; y_{4n+2} = \frac{\lambda^2}{B_1^2}; y_{4n+3} = \beta^4; y_{4n+4} = \mu^4,$$

where $n = 1, 2, 3, \cdots$.

proof Similar to Theorem (1) by mathematical induction.

Theorem 4 Suppose that (x_n, y_n) is a solution of (3) such that $\alpha, \beta, \lambda, \mu \geq 1, 0 < B_n \leq A_n \leq 1$. Then the general solution of (3) has the following form.

$$x_1 = \beta^2; x_2 = \mu^2; x_3 = \frac{A_2}{B_0}\alpha; x_4 = \frac{A_3}{B_1}\lambda;$$

$$y_1 = \frac{B_0}{\alpha}; y_2 = \frac{B_1}{\lambda}; y_3 = \frac{B_2}{\beta^2}; y_4 = \frac{B_3}{\mu^2};$$

$$x_{4n+1} = \frac{\prod_{i=1}^{n} A_{4i}}{\prod_{i=1}^{n} B_{4i-2}}\beta^2; y_{4n+1} = \frac{\prod_{i=1}^{n} B_{4i}}{\prod_{i=1}^{n} A_{4i-2}}\frac{B_0}{\alpha};$$

$$x_{4n+2} = \frac{\prod\limits_{i=1}^{n} A_{4i+1}}{\prod\limits_{i=1}^{n} B_{4i-1}} \mu^2; y_{4n+2} = \frac{\prod\limits_{i=1}^{n} B_{4i+1}}{\prod\limits_{i=1}^{n} A_{4i-1}} \frac{B_1}{\lambda};$$

$$x_{4n+3} = \frac{\prod\limits_{i=1}^{n} A_{4i+2}}{\prod\limits_{i=1}^{n} B_{4i}} \frac{A_2}{B_0} \alpha; y_{4n+3} = \frac{\prod\limits_{i=1}^{n} B_{4i+2}}{\prod\limits_{i=1}^{n} A_{4i}} \frac{B_2}{\beta^2};$$

$$x_{4n+4} = \frac{\prod\limits_{i=1}^{n} A_{4i+3}}{\prod\limits_{i=1}^{n} B_{4i+1}} \frac{A_3}{B_1} \lambda; y_{4n+4} = \frac{\prod\limits_{i=1}^{n} B_{4i+3}}{\prod\limits_{i=1}^{n} A_{4i+1}} \frac{B_3}{\mu^2},$$

where $i = 1, 2, 3, \cdots$.

proof The proof is similar to Theorem (1).

When $A_n = A, B_n = B$, the following corollary holds.

Corollary 7 Suppose that (x_n, y_n) is a solution of (3) such that $\alpha, \beta, \lambda, \mu \geq 1, 0 < B \leq A \leq 1$. Then the general solution of (3) are as the following.

$$x_{4n+1} = \frac{A^n}{B^n}\beta^2; x_{4n+2} = \frac{A^n}{B^n}\mu^2; x_{4n+3} = \frac{A^{n+1}}{B^{n+1}}\alpha; x_{4n+4} = \frac{A^{n+1}}{B^{n+1}}\lambda;$$

$$y_{4n+1} = \frac{B^{n+1}}{A^n\alpha}; y_{4n+2} = \frac{B^{n+1}}{A^n\lambda}; y_{4n+3} = \frac{B^{n+1}}{A^n\beta^2}; y_{4n+4} = \frac{B^{n+1}}{A^n\mu^2},$$

where $n = 1, 2, 3, \cdots$.

proof Similar to Theorem (1) by mathematical induction.

For a special case, when A_n, B_n are same two-periodic positive sequences, (3) could has four-periodic solution as the following corollary.

Corollary 8 Suppose that (x_n, y_n) is a solution of (3) such that $\alpha, \beta, \lambda, \mu \geq 1$. If $A_n = B_n = A_{n+2} = B_{n+2}, n = 0, 1, 2, \cdots$. Then (3) has four-periodic solution as the following.

$$x_{4n+1} = \beta^2; x_{4n+2} = \mu^2; x_{4n+3} = \alpha; x_{4n+4} = \lambda;$$

$$y_{4n+1} = \frac{B_0}{\alpha}; y_{4n+2} = \frac{B_1}{\lambda}; y_{4n+3} = \frac{B_0}{\beta^2}; y_{4n+4} = \frac{B_1}{\mu^2},$$

where $n = 1, 2, 3, \cdots$.

proof Similar to Theorem (1) by mathematical induction.

3. Examples

example 1 Let $\alpha = 1/2, \beta = 1/3, \lambda = 2/3, \mu = 3/4, A_0 = B_0 = 1, A_1 = B_1 = 2$, then by Corollary 2, (3) has 4-periodic solution $\{x_n, y_n\}$ as the following simulation.

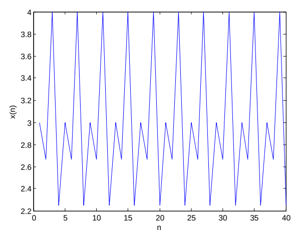

Figure 1: plot of $x(n)$

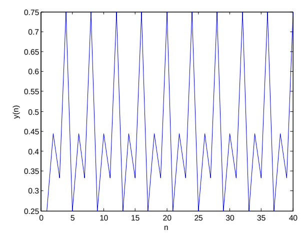

Figure 2: plot of $y(n)$

example 2 Let $\alpha = -5/3, \beta = -1/3, \lambda = -4/3, \mu = -5/4, A_0 = B_0 = 3, A_1 = B_1 = 2$, then by Corollary 4, (3) has eventually 4-periodic solution $\{x_n, y_n\}$ as the following simulation.

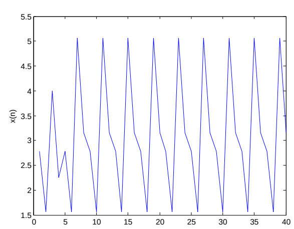

Figure 3: plot of $x(n)$

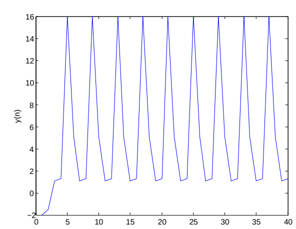

Figure 4: plot of $y(n)$

example 3 Let $\alpha = -1/2, \beta = -2/3, \lambda = -1/2, \mu = -1/3, A_0 = B_0 = 1/2, A_1 = B_1 = 1$, then by Corollary 6, (3) has eventually 4-periodic solution $\{x_n, y_n\}$ as the following simulation.

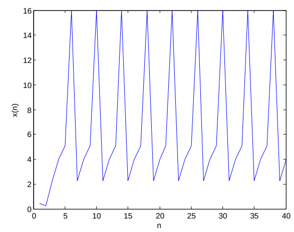

Figure 5: plot of $x(n)$

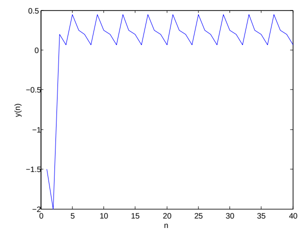

Figure 6: plot of $y(n)$

example 4 Let $\alpha = 1, \beta = 3/2, \lambda = 2, \mu = 4/3, A_0 = B_0 = 1/3, A_1 = B_1 = 1$, then by Corollary 8, (3) has 4-periodic solution $\{x_n, y_n\}$ as the following simulation.

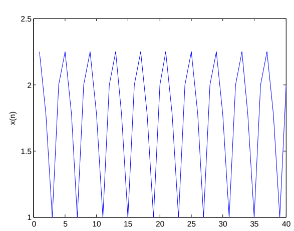

Figure 7: plot of $x(n)$

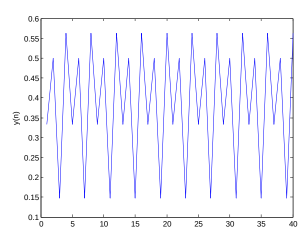

Figure 8: plot of $y(n)$

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