

ON HAMILTONIAN DECOMPOSITIONS OF TENSOR PRODUCTS OF GRAPHS

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Finding a hamiltonian decomposition of G is one of the challenging problems in graph theory. We do not know for what classes of graphs G and H , their tensor product $G \times H$ is hamiltonian decomposable. In this paper, we have proved that, if G is a hamiltonian decomposable circulant graph with certain properties and H is a hamiltonian decomposable multigraph, then $G \times H$ is hamiltonian decomposable. In particular, tensor products of certain sparse hamiltonian decomposable circulant graphs are hamiltonian decomposable.

1. INTRODUCTION

A k -cycle is a subgraph of K_n with k distinct vertices $x_1, x_2, x_3, \dots, x_k$ and k edges $x_1x_2, x_2x_3, \dots, x_{k-1}x_k, x_kx_1$. When $k = n$, a k -cycle is called a *hamiltonian cycle*. For two graphs G and H their *tensor product*, denoted by $G \times H$, has vertex set $V(G) \times V(H)$ in which two vertices (g_1, h_1) and (g_2, h_2) are adjacent whenever $g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)$. The *wreath product* of the graphs G and H , denoted by $G \circ H$, has vertex set $V(G) \times V(H)$ in which $(g_1, h_1)(g_2, h_2)$ is an edge whenever g_1g_2 is an edge in G , or $g_1 = g_2$ and h_1h_2 is an edge in H . Similarly, the *cartesian product* of the graphs G and H , denoted by $G \square H$, has vertex set $V(G) \times V(H)$ in which $(g_1, h_1)(g_2, h_2)$ is an edge whenever $g_1 = g_2$ and h_1h_2 is an edge in H , or $h_1 = h_2$ and g_1g_2 is an edge in G . The subgraph induced by $S \subseteq V(G)$ is denoted by $G[S]$. Similarly, the subgraph induced by $E' \subseteq E(G)$ is denoted by $G[E']$. For

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a graph G , $G(\lambda)$ is the graph obtained from G by replacing each edge of G by λ parallel edges. For a graph G , G^* is the *symmetric digraph* of G .

For two loopless multigraphs $G(\lambda)$ and $H(\mu)$, their tensor product, denoted by $G(\lambda) \times H(\mu)$, has the vertex set $V(G) \times V(H)$ and its edge set is described as follows: if $e = g_1g_2$ is an edge of multiplicity λ in $G(\lambda)$ and $f = h_1h_2$ is an edge of multiplicity μ in $H(\mu)$, then corresponding to these edges there are edges $(g_1, h_1)(g_2, h_2)$ and $(g_1, h_2)(g_2, h_1)$ each of multiplicity $\lambda\mu$ in $G(\lambda) \times H(\mu)$ and $G(\lambda) \times H(\mu)$ is isomorphic to $(G \times H)(\lambda\mu)$. Hence $G(\lambda) \times H \cong G \times H(\lambda) \cong (G \times H)(\lambda)$.

If H_1, H_2, \dots, H_k are edge-disjoint subgraphs of G and $E(G) = E(H_1) \cup E(H_2) \cup \dots \cup E(H_k)$, then we write $G = H_1 \oplus H_2 \oplus \dots \oplus H_k$. For a graph G , if $E(G)$ can be partitioned into E_1, E_2, \dots, E_k such that $\langle E_i \rangle \cong H$, for all i , $1 \leq i \leq k$, then we say that H *decomposes* G , or H -*decomposition* of G exists. If H_1, H_2, \dots, H_k decomposes G , and each H_i is a hamiltonian cycle of G , then we call G is *hamiltonian decomposable* and we call $\{H_1, H_2, \dots, H_k\}$ is a *hamiltonian decomposition of G* . Clearly, the tensor product is commutative and distributive over edge-disjoint union of graphs, that is, if $G = H_1 \oplus H_2 \oplus \dots \oplus H_k$, then $G \times G_1 = (H_1 \times G_1) \oplus (H_2 \times G_1) \oplus \dots \oplus (H_k \times G_1)$.

Let G be a finite abelian group and let S be a symmetric subset of G (that is, $s \in S$ implies $-s \in S$). The vertices of the *Cayley graph*, $Cay(S, G)$, are the elements of G and there is an edge between x and y if and only if $x - y \in S$. Note that $Cay(S, G)$ is connected if and only if S generates the group G . A *circulant* $X = Circ(n; L)$ is a graph with vertex set $V(X) = \{u_0, u_1, \dots, u_{n-1}\}$ and edge set $E(X) = \{u_i u_{i+\ell} \mid i \in \mathbb{Z}_n, \ell \in L\}$, where $L \subseteq \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\}$ and \mathbb{Z}_n is the set of integers modulo n . The elements of L are called *jumps*. Clearly, every circulant graph of order n is a Cayley graph with the underlying group being \mathbb{Z}_n .

The problem of finding hamiltonian decompositions of product graphs is not new. hamiltonian decompositions of various product graphs, including digraphs, have been studied by many authors; see, for example, [1, 6, 9, 10, 11, 13, 14, 15, 16, 18, 17]. It has been conjectured [6] that if both G and H are hamiltonian decomposable graphs, then $G \square H$ is hamiltonian decomposable, where \square denotes the cartesian product of graphs. This conjecture has been verified to be true for a large classes of graphs [18]. Baranyai and Szász [5] proved that if both G and H are even regular hamiltonian decomposable graphs, then $G \circ H$ is hamiltonian decomposable. In [16], Ng has obtained a partial solution to the following conjecture of Alspach et al. [1]: If D_1 and D_2 are directed hamiltonian decomposable digraphs, then $D_1 \circ D_2$ is directed hamiltonian decomposable.

In [3], it had been proved that $G \times H$ is not necessarily hamiltonian decomposable even though G and H are hamiltonian decomposable. Because of this, finding hamiltonian decompositions of the tensor products of hamiltonian decomposable graphs is considered to be difficult. We believe that if the graphs G and H are suitably chosen, that is, with some suitable conditions imposed on them, then $G \times H$ may have hamiltonian decomposition. In [2] and [13] it has been proved that $K_r \times K_s$ and $K_{r,r} \times K_m$ are hamiltonian decomposable; in [14] it is shown

that the tensor product of two regular complete multipartite graphs is hamiltonian decomposable. Hamiltonian decompositions of the tensor products of complete bipartite graphs and complete multipartite graphs are dealt with in [15]. Also in [17], Paulraja and Sivasankar proved that $(K_r \times K_s)^*$, $((K_r \circ \overline{K}_s) \times K_n)^*$, $((K_r \times K_s) \times K_m)^*$, $((K_r \circ \overline{K}_s) \times (K_m \circ \overline{K}_n))^*$ and $(K_{r,r} \times (K_m \circ \overline{K}_n))^*$ are directed hamiltonian decomposable. It can be observed that K_r , $K_{r,r}$, $K_r \circ \overline{K}_s$ are circulant graphs. Based on the results of [10, 11, 13, 14, 15], Manikandan and Paulraja conjectured the following:

Conjecture 1.1. (Manikandan and Paulraja) [12]. *If G and H are hamiltonian decomposable circulant graphs and at least one of them is non bipartite, then $G \times H$ is hamiltonian decomposable.*

Theorem 1.1. [6] *If both G and H have hamiltonian decompositions and at least one of them is of odd order, then $G \times H$ admits a hamiltonian decomposition.*

One may naturally ask when $G \times H$ is hamiltonian decomposable, if both G and H are of even order. In this paper it is answered for some family of sparse graphs.

We say that an even regular circulant graph $X = \text{Circ}(n; L)$ has the *property Q*, if (1) the number of odd jumps and even jumps in L are equal; (2) odd jumps can be paired with even jumps so that the edges of each pair of jumps induce a connected 4-regular graph. There are $\frac{|L|}{2}$ such connected 4-regular graphs. It is known that every 4-regular connected Cayley graph is hamiltonian decomposable; see [7]. Consequently, every circulant graph with property *Q* is hamiltonian decomposable.

Here we prove the following main Theorem.

Theorem 1.2. *Let G be a circulant graph with property Q and let H be any hamiltonian decomposable multigraph, then $G \times H$ is hamiltonian decomposable.*

This theorem has many interesting consequences. In particular, we have the following corollary.

Corollary 1.1. *If G and H are even regular hamiltonian decomposable circulant graphs and at least one of them has the property Q , then $G \times H$ is hamiltonian decomposable.*

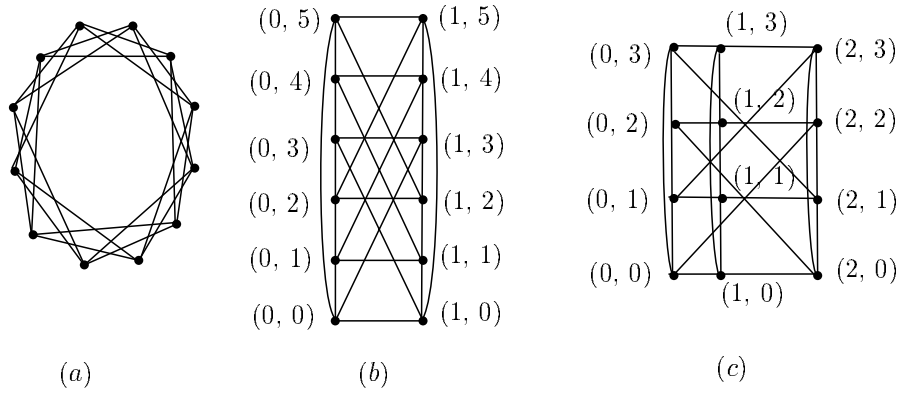
One of the consequences of Corollary 1.1 is that if $G = (K_{4n+2} - F)$ and $H = (K_{2m} - F')$, where F and F' are 1-factors of K_{4n+2} and K_{2m} , respectively, then $G \times H$ is hamiltonian decomposable. In particular, after deleting suitable number of jumps of K_{4n+2} and K_{2m} , the resulting graphs need not be dense but their tensor product is hamiltonian decomposable. This cannot be deduced from the existing results in this direction.

2. NOTATION AND PRELIMINARIES

First we present the necessary definitions here. The notation that we use here are from [7] and for the sake of completeness we give them.

The Cayley graph $\Gamma = Cay(S, G)$, where $S = \{a, b\}$ is a generating set of the finite abelian group with $2a \neq 0, 2b \neq 0$ and $a \neq \pm b$, is a simple connected graph. Let x and $x + a$ be two vertices of Γ , then we call the edge $x(x + a)$ of Γ an a -edge; the subgraph formed by the a -edges is a disjoint union of cycles, called a -cycles, each of length k_a , the order of the element a . Similarly, we define b -edges and b -cycles; each b -cycle is of length k_b , the order of the element b . In Γ , let us denote the number of a -cycles by α and the number of b -cycles by β . Since the length of each a -cycle (respectively b -cycle) is $k_a = \frac{n}{\alpha}$, (respectively $k_b = \frac{n}{\beta}$), $n = \alpha k_a$ (respectively $n = \beta k_b$).

As the graph Γ is connected, the vertices $0, b, \dots, \ell b, \dots, (\alpha - 1)b$ are in different a -cycles, denoted by $C_0, C_1, \dots, C_\ell, \dots, C_{\alpha-1}$, respectively, and ab belongs to C_0 , see [7]. Hence, we have $ab = ca$ for some c with $0 \leq c \leq k_a - 1$. Further, every vertex x of Γ can be uniquely written as $x = ib + ja$ with $i \in \{0, 1, \dots, \alpha - 1\}$ and $j \in \{0, 1, \dots, k_a - 1\}$. Using this uniqueness, label the vertices of Γ as (i, j) with $i \in \{0, 1, \dots, \alpha - 1\}$, and $j \in \{0, 1, \dots, k_a - 1\}$, where the first coordinate indicates the label of the cycle C_i containing the vertex and the second coordinate indicates the position of the vertex on the cycle.



Graphs of Figures 1(a), 1(b) and 1(c) are isomorphic.
 (a) Graph $G = Circ(12, \{2, 3\})$, where the edges of jump 2 give 2 disjoint cycles each of length 6 and the edges of jump 3 give 3 disjoint cycles each of length 4.
 (b) G is an element of $\Gamma(2, 3)$ with $c = 3$.
 (c) G in (b) is drawn as an element of $\Gamma(3, 2)$ with $c = 2$.

Figure 1

The following definition comprises the notation and defines a class of simple graphs, which we denote by $\Gamma(\alpha, \beta)$.

Definition 2.1. [7, Page 144] Let $\Gamma(\alpha, \beta)$ denote the class of simple graphs on αk vertices, where $\alpha \geq 1$, $k \geq 3$, $0 \leq c < k$, and $\beta = \gcd(k, c)$. The αk vertices of the graph can be labeled as (i, j) uniquely with i taken modulo α and j taken modulo k . The edges are classified into (1) First kind: $(i, j)(i, j+1)$ and (2) Second kind: $(i, j)(i+1, j)$ for all $i \in \{0, 1, \dots, \alpha-2\}$, and $(\alpha-1, j)(0, j+c)$. The edges of first kind form α cycles, each of length k and the edges of second kind form β cycles, each of length $\frac{\alpha k}{\beta}$.

Observe that the a -edges, b -edges, a -cycles and b -cycles have a natural orientation (that is, the a -edge $x(x+a)$ is oriented so that x and $(x+a)$ are tail and head, respectively). Hence each of the α “vertical” disjoint cycles C_i , $0 \leq i < \alpha$, of $\Gamma \in \Gamma(\alpha, \beta)$ has a natural orientation. Also Γ has $\alpha - 1$ horizontal *parallel matchings* between the cycles C_i and C_{i+1} for $0 \leq i < \alpha - 1$, and a particular *parallel matching* between $C_{\alpha-1}$ and C_0 (which depends on the value of c). A graph $\Gamma \in \Gamma(2, \beta)$ consists of two cycles plus two perfect matching between them. A graph $\Gamma \in \Gamma(1, \beta)$ consists of a cycle plus the chords joining $(0, j)$ to $(0, j+c)$. Observe that $\Gamma(\alpha, \beta)$ and $\Gamma(\beta, \alpha)$ are isomorphic classes of graphs, that is, an element $\Gamma_1 \in \Gamma(\alpha, \beta)$ is isomorphic to an element $\Gamma_2 \in \Gamma(\beta, \alpha)$ and vice versa, for example see Figure 1.

Definition 2.2. [7] A hamiltonian decomposition of a graph $\Gamma \in \Gamma(\alpha, \beta)$ has the property Q_1 between C_i and C_{i+1} , $0 \leq i \leq \alpha-1$ where $C_\alpha = C_0$, if both hamiltonian cycles use at least one edge of the matching between C_i and C_{i+1} .

Theorem 2.1. [7] The class $\Gamma(\alpha, \beta)$ consists of the 4-regular connected Cayley graphs on a finite abelian group with a generating set $\{a, b\}$, where α is the number of a -cycles and β is the number of b -cycles.

Using the above theorem, Bermond et al. proved the following main theorem

Theorem 2.2. [7] Every 4-regular connected Cayley graph on a finite abelian group can be decomposed into two hamiltonian cycles.

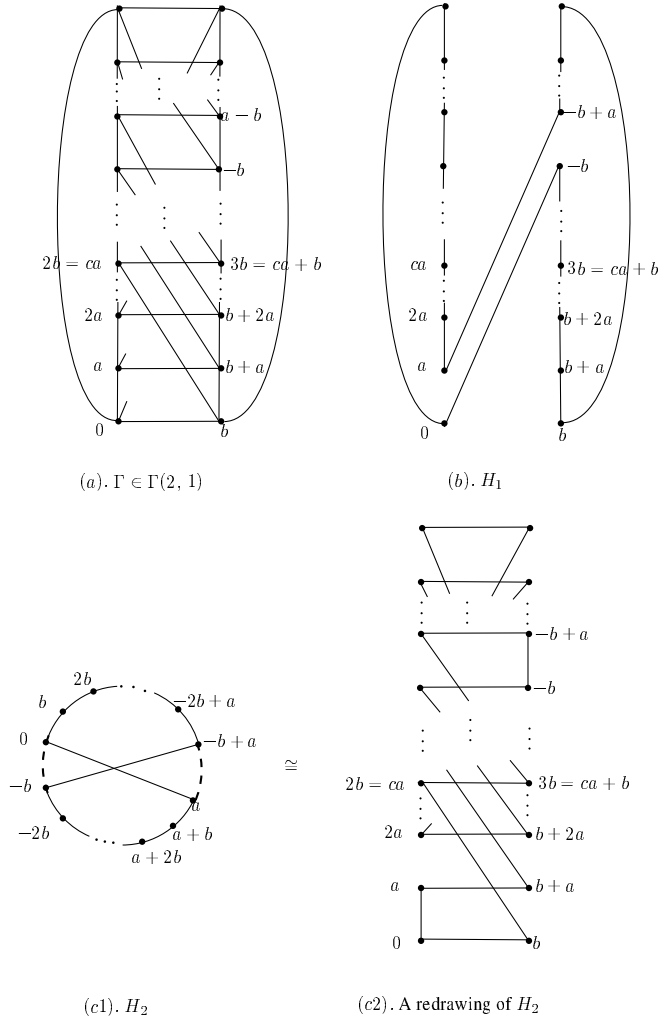
3. MAIN RESULTS

If G and H are hamiltonian decomposable graphs with at least one of them having odd order, then $G \times H$ is hamiltonian decomposable, by Theorem 1.1; hence in what follows, **we assume that both G and H are hamiltonian decomposable graphs each having an even number of vertices.** If G is a connected 4-regular circulant graph of even order with generating set $\{a, b\}$ where a and b are of different parity such that $2a \neq 0$, $2b \neq 0$ and $a \neq \pm b$, then by the Definition 2.1, α and β are of different parity.

Definition 3.1. Let $\{H_1, H_2\}$ be a hamiltonian decomposition of a 4-regular graph G of even order with at least 6 vertices. If G contains a 4-cycle $(abcd)$ such that the edges ab and cd belong to one of the two hamiltonian cycles and the edges bc and

the other hamiltonian cycle, then the 4-cycle is said to be an alternating 4-cycle in G , with respect to H_1 and H_2 . Further, if the distance between the vertices a and c along H_1 and H_2 is odd, or b and d along H_1 and H_2 is odd, then the 4-cycle $(abcd)$ is said to be an odd alternating 4-cycle in G with respect to $\{H_1, H_2\}$. We say that the graph G has property Q_2 with respect to the hamiltonian decomposition $\{H_1, H_2\}$, if G contains an odd alternating 4-cycle with respect to $\{H_1, H_2\}$.

The proof techniques we use here heavily depend on [7].



(a). A member Γ of $\Gamma(2, 1)$; (b). Hamiltonian cycle H_1 of $\Gamma \in \Gamma(2, 1)$; (c1). The solid lines are edges on H_2 . Other than the two a -edges $0a$ and $(-b)(-b+a)$, all edges of H_2 (and thus all edges connecting C_0 and C_1) are b -edges; (c2). Another drawing of the hamiltonian cycle H_2 of $\Gamma \in \Gamma(2, 1)$.

Figure 2

The *reduced graph* $\Gamma' \in \Gamma(\alpha, \beta)$ with respect to $\Gamma \in \Gamma(\alpha + 2, \beta)$ is defined as follows: delete the vertices of the cycles C_α and $C_{\alpha+1}$ of Γ , that is, the vertices (α, j) and $(\alpha + 1, j)$, $0 \leq j \leq k - 1$, and add the edges $(\alpha - 1, j)(0, j + c)$, $0 \leq j \leq k - 1$, joining $C_{\alpha-1}$ and C_0 , where c is given in the definition of $\Gamma(\alpha + 2, \beta)$. In some cases Γ' might be a multigraph and, if it is a simple graph, it is an element of $\Gamma(\alpha, \beta)$ (k and c being unchanged). Successive reductions result in a graph $\Gamma^k \in \Gamma(\alpha + 2 - 2k, \beta)$ for some $k \geq 1$. $\Gamma^k \in \Gamma(\alpha + 2 - 2k, \beta)$ is also called a reduced graph of $\Gamma \in \Gamma(\alpha + 2, \beta)$. In what follows, reduce the graph A to the graph B means that obtain the graph B , where B is the reduced graph of A .

The graph $\Gamma \in \Gamma(\nu + 2, \delta)$ is said to be *lift graph* of $\Gamma' \in \Gamma(\nu, \delta)$, if Γ is obtained as follows: delete the edges $(\nu - 1, j)(0, j + c)$, $0 \leq j \leq k - 1$, joining $C_{\nu-1}$ and C_0 of Γ' ; then add two a -cycles C_ν and $C_{\nu+1}$ with vertices (ν, j) and $(\nu + 1, j)$, $0 \leq j \leq k - 1$, to Γ' , and add the edges $(\nu - 1, j)(\nu, j)$, $(\nu, j)(\nu + 1, j)$, $(\nu + 1, j)(0, j + c)$, $0 \leq j \leq k - 1$, where c is as given in the definition of $\Gamma(\nu, \delta)$. Successive liftings of the resulting graphs result in a graph $\Gamma^k \in \Gamma(\nu + 2k, \delta)$ for some $k \geq 1$. $\Gamma^k \in \Gamma(\nu + 2k, \delta)$ is also called a lift graph of $\Gamma' \in \Gamma(\nu, \delta)$.

By an (x, y) -section, we denote a path from x to y . We use $p_G(x, y)$ to denote the length of an (x, y) -section (not necessarily shortest (x, y) -section) in G .

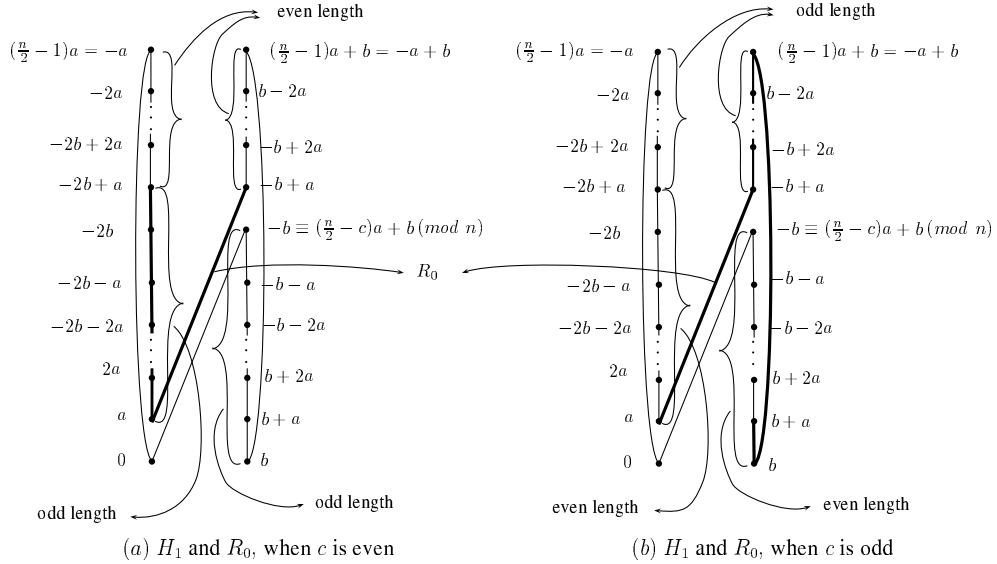
Lemma 3.1. *Every graph $\Gamma \in \Gamma(\alpha, \beta)$, where $1 \leq \alpha, \beta \leq 2$, has a hamiltonian decomposition $\{H_1, H_2\}$ with property Q_2 and with property Q_1 between two a -cycles.*

Proof. Let $\Gamma \in \Gamma(\alpha, \beta)$. Since $1 \leq \alpha, \beta \leq 2$ and $\alpha \neq \beta$, without loss of generality assume that $\alpha = 2$ and $\beta = 1$. Consequently, a is even and b is odd as $2 = \alpha = \gcd(n, a)$ and $1 = \beta = \gcd(n, b)$. Clearly, $c \neq 0$ (otherwise $\beta \neq 1$). The existence of hamiltonian cycles H_1 and H_2 in Γ described below is in [7, Page 149, Lemma 2]. A hamiltonian cycle H_1 is obtained by deleting the edges $0a$ and $(-b)(-b+a)$ from C_0 and C_1 , respectively, and adding the edges $(-b)0$ and $(-b+a)a$ connecting these two a -cycles, that is, $H_1 = \{C_0 - \{0a\}\} \cup \{C_1 - \{(-b)(-b+a)\}\} \cup \{0(-b), (-b+a)a\}$, see Figure 2(b); $H_2 = \Gamma - E(H_1)$ is shown in the solid lines of Figure 2(c1). Clearly, these two hamiltonian cycles have the property Q_1 between C_1 and C_0 .

Claim. The graph $\Gamma \in \Gamma(2, 1)$ satisfies the property Q_2 with respect to the above hamiltonian decomposition $\{H_1, H_2\}$.

Since $\alpha = 2$, there are two a -cycles C_0 and C_1 and there are two parallel matchings between C_0 and C_1 . We denote one of the matchings from C_0 to C_1 as M_0 and the other matching from C_1 to C_0 as M_1 ; let the edges of M_0 be $(0, j)(1, j)$, $0 \leq j \leq \frac{n}{2} - 1$, and let the edges of M_1 be $(1, j)(0, j + c)$, $0 \leq j \leq \frac{n}{2} - 1$, where addition is taken modulo n . We call the vertex $(i, j) (= ib + ja)$ of C_0 as the *corresponding vertex* of $(i + 1, j) (= (i + 1)b + ja)$ of C_1 and vice versa. We prove the existence of the property Q_2 with respect to the hamiltonian decomposition $\{H_1, H_2\}$ of Γ by the parity of c .

Case 1. $n \equiv 2 \pmod{4}$.



H_1 is shown in both Figures 3(a) and 3(b), wherein the bold lines are the required even length path.

Figure 3

First, we shall obtain an even length path along H_1 , so that this path is a section of a suitable odd length path along H_1 having its ends as two “opposite vertices” of an odd alternating 4-cycle in Γ ; the existence of the odd alternating 4-cycle will be proved later.

In **(A)** below, we obtain a required even length path along H_1 and in **(B)** we prove that the path obtained in **(A)** is a section of a path along H_1 joining two “opposite vertices” of an odd alternating 4-cycle.

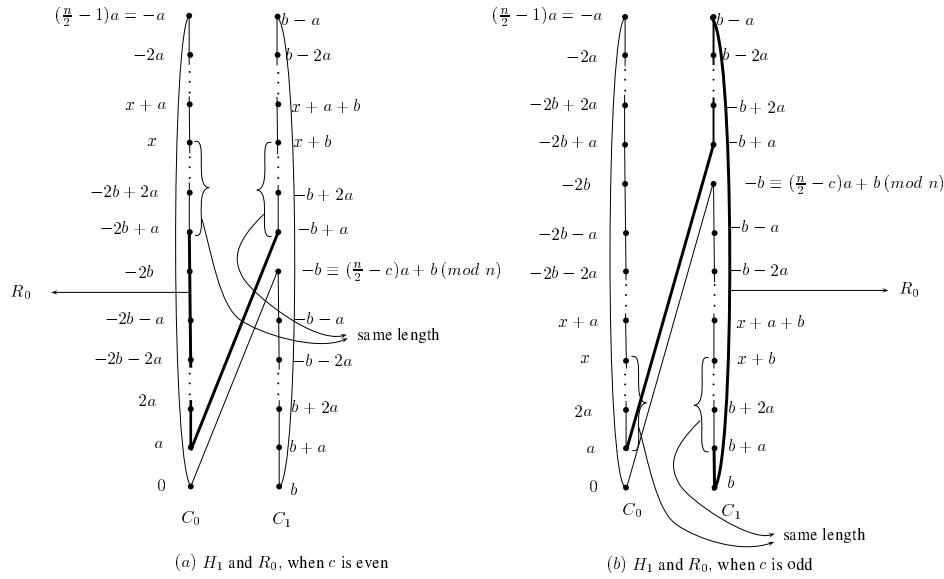
(A). First we consider c to be even and consider the hamiltonian cycle H_1 of Γ obtained above. Since M_1 is a perfect matching of Γ and M_1 matches the vertices in C_1 with vertices in C_0 , $(-b)0$ is an edge of M_1 , which is one of the two M_1 edges in H_1 , see Figure 3(a). Clearly, $-b \equiv (\frac{n}{2} - c)a + b \pmod{n}$, that is, $-b$ is the $(\frac{n}{2} - c + 1)$ th vertex of C_1 , starting from b , see Figure 3(a). As $\frac{n}{2}$ is odd, $(\frac{n}{2} - c + 1)$ is even and hence the length of the section of C_1 from b to $-b$ (containing the vertex $b + a$) is odd.

As the length of the section of C_1 from b to $-b$ (containing the vertex $(b + a)$) is the same as the length of the section of C_0 from a to $(-2b + a)$ (containing the vertex $2a$), the length of the path from a to $-2b + a$ along H_1 , denoted by $p_{H_1}(a, -2b + a)$ is odd and let this path be R . Now R together with the edge $a(-b + a)$ is our required even length path R_0 , see the bold edges of Figure 3(a).

Next we suppose that c is odd. As in the above paragraph, the vertex 0 in C_0 is matched under M_1 with the vertex $-b \equiv (\frac{n}{2} - c)a + b \pmod{n}$; $\frac{n}{2} - c$ is even as $\frac{n}{2}$ is odd. Thus the length of the section of C_1 from b to $-b$ (containing the vertex

$(b + a)$ is even and hence the length of the section R_1 of C_1 from b to $(-b + a)$ (containing the vertex $(-a + b)$) is even (note that $(-b)(-b + a)$ is an edge of C_1 which we have deleted for the construction of H_1). Now R_1 together with the edges $\{(b + a)b, (-b + a)a\}$ is our required even length path R_0 from the vertex a to the vertex $b + a$ along H_1 , see the bold edges of Figure 3(b).

(B). H_1 contains exactly two edges of M_1 as shown in the Figure 3. Hence except the two 4-cycles of Γ , namely, $(0, a, a + b, b)$ and $(-2b, -2b + a, -b + a, -b)$, the other 4-cycles, which have two consecutive vertices of C_0 and the corresponding two vertices of C_1 constitute alternating 4-cycles (with respect to H_1 and H_2 of Γ).



- (a) The lengths of the sections $(-2b + a, x)$ along H_1 (section of C_0) and $(-b + a, x + b)$ along H_1 (section of C_1) are same and $(x + b)(x + a + b)$ is an edge
- (b) the lengths of the sections (a, x) along H_1 (part of C_0) and $(b + a, x + b)$ along H_1 (part of C_1) are same and $(x + a)(x + a + b)$ is an edge.

Figure 4

If c is even, it is clear that every 4-cycle containing two consecutive vertices of the section $(-2b + a)(-2b + 2a)(-2b + 3a) \dots ((\frac{n}{2} - 1)a)0$ of C_0 and the two corresponding consecutive vertices of the section $(-b + a)(-b + 2a)(-b + 3a) \dots (b - a)b$ of C_1 is an alternating 4-cycle of Γ with respect to H_1 and H_2 , see the Figure 3 (note that $a = 2b$ is ruled out, for otherwise, it would imply $c = 1$ for the following reason: $a = 2b$ implies $b = -b + a$ and since $(-b + a)a$ is an edge of H_1 joining C_1 and C_0 implies $c = 1$, which is not the case as we consider c is even). Similarly if c is odd, every 4-cycle containing two consecutive vertices in the section $a(2a)(3a) \dots (-2b)$ of C_0 and the two corresponding consecutive vertices in the

section $(b + a)(b + 2a)(b + 3a) \dots (-b)$ of C_1 is an alternating 4-cycle of Γ with respect to H_1 and H_2 ; note if $a = -2b$, then $a = -ca$ so $-a = ca = (\frac{n}{2} - 1)a$ but then $c \equiv (\frac{n}{2} - 1) \pmod{\frac{n}{2}}$, that is, $c = \frac{n}{2} - 1$, which is even, a contradiction.

Next we prove that the alternating 4-cycles described in the above paragraph using the sections of C_0 and C_1 satisfy the property that the length of the path along H_1 between the opposite vertices of the 4-cycle is of odd length and further this odd length path contains R_0 (described above).

Let $C = (x, x + a, x + a + b, x + b)$ be an alternating 4-cycle so that $x, x + a \in V(C_0)$ and $x + a + b, x + b \in V(C_1)$, see Figure 4. The vertices x and $x + b$ are the corresponding vertices in C_0 and C_1 . If c is even, then the $(-2b + a, x)$ -section of H_1 , contained in C_0 , and the corresponding $(x + b, (-b + a))$ -section of H_1 contained in C_1 have the same length, see Figure 4(a). Hence the $(x, x + a + b)$ -section of H_1 (containing the vertex $-2b$) is of odd length as it contains R_0 , which is of even length, see Figure 4(a); that is $p_{H_1}(x, x + a + b)$ is odd when c is even. Similarly, if c is odd then, the (a, x) -section of H_1 (containing the vertex $2a$), contained in C_0 , and the corresponding $(b + a, x + b)$ -section of H_1 , contained in C_1 (containing the vertex $b + 2a$), have the same length. Thus the $(x, x + a + b)$ -section of H_1 (containing the vertex $-b + a$) is of odd length as it contains R_0 , which is of even length, see Figure 4(b). That is $p_{H_1}(x, x + a + b)$ is odd when c is odd, see Figure 4(b).

Next we shall obtain an appropriate x on C_0 so that $p_{H_2}(x, x + a + b)$ is also odd which gives the property Q_2 with respect to H_1 and H_2 . As H_2 contains exactly two a -edges, namely, $(-b)(-b + a)$ and $0a$ (see Figure 2(c2)), $H_2 - \{0a, (-b)(-b + a)\}$ is a pair of odd length paths (since the vertices of these paths alternate between $V(C_0)$ and $V(C_1)$). Let $S_0 = 0(b)(2b) \dots (-b + a)$ and $S_1 = a(a + b)(a + 2b) \dots (-b)$ denote these paths, namely, the $(0, (-b + a))$ -section and $(-b, a)$ -section of H_2 , respectively, see Figures 2(c1) and 2(c2).

If S_0 (respectively S_1) contains a pair of consecutive vertices x and $x + a$ of C_0 (note that the edge $x(x + a)$ is in H_1 and S_0 is contained in H_2), then $p_{H_2}(x, x + a + b)$ is odd as the vertices of S_0 (respectively S_1) alternate between $V(C_0)$ and $V(C_1)$ and $(x + a)(x + a + b)$ is an edge in H_2 . We shall prove the existence of such a pair of consecutive vertices in the sequence of vertices $L_0 = (-2b + a)(-2b + 2a)(-2b + 3a) \dots (\frac{n}{2} - 1)a0$, or in $L_1 = a(2a)(3a) \dots (-2b)$, see Figure 4.

Let c be even. From the construction of H_1 and H_2 , the vertices 0 and $-2b + a$ are in S_0 , (see Figure 2(c1)). There are even number of vertices in L_0 as R_0 is of even length and C_0 has odd number of vertices (see Figure 3(a)). If there is no pair of consecutive vertices in L_0 of the required type in S_0 or S_1 , then the preceding vertex of 0 , namely, $(\frac{n}{2} - 1)a (= -a)$, and the succeeding vertex of $-2b + a$, namely, $-2b + 2a$, are in S_1 , as the vertices 0 and $-2b + a$ are in S_0 ; consequently, there must be an odd number of vertices from $-2b + 2a$ to $(\frac{n}{2} - 1)a (= -a)$ in L_0 , which is not the case (since in L_0 , $-2b + 2a$ to $-a$ contains even number of vertices). Therefore there must exist a pair of consecutive vertices in L_0 of the required type and hence Γ satisfies property Q_2 with respect to H_1 and H_2 .

Next we assume that c is odd. As above, we shall show the existence of the required pair of consecutive vertices in $L_1 = a(2a)(3a) \dots (-2b)$. L_1 has even number of vertices (see Figure 3(b)). Assume that there is no pair of vertices in L_1 of the required type in S_0 or S_1 and hence alternate vertices of L_1 are in S_0 and S_1 . Then, as the vertices a and $-2b$ are in S_1 , the vertices $2a$ and $-2b - a$ must be in S_0 and hence there must be an odd number of vertices from $2a$ to $-2b - a$ in L_1 , which is not the case (as L_1 contains even number of vertices from $2a$ to $-2b - a$). Thus there must exist a pair of consecutive vertices as required and hence Γ satisfies the property Q_2 .

This completes the proof when $n \equiv 2 \pmod{4}$.

Case 2: $n \equiv 0 \pmod{4}$.

Since $n \equiv 0 \pmod{4}$ and $\Gamma \in \Gamma(2, 1)$, c is always odd, otherwise, the b -edges will not induce a hamiltonian cycle. As $\alpha = 2$ and $n \equiv 0 \pmod{4}$, $a \equiv 2 \pmod{4}$ as $\gcd(a, n) = \alpha = 2$. By assumption b is odd and hence $b \equiv 1 \text{ or } 3 \pmod{4}$; then $-b \equiv 3 \text{ or } 1 \pmod{4}$. For any two vertices x and y on C_1 , the path from x to y along C_1 has even length if and only if $x \equiv y \pmod{4}$. Hence the length of the section of C_1 from b to $(-b)$ (containing the vertex $(b+a)$) is odd and the length of the section of C_1 from the vertex $(-b+a)$ to b (containing the vertex $(b-a)$) is even and let this path be R , (see Figure 4(b); note that the Figure 4(b) is for the case $n \equiv 2 \pmod{4}$; the figure for the case $n \equiv 0 \pmod{4}$ is similar). Now the edges of R together with the edges $\{(b+a)b, (-b+a)a\}$ induce an even length path, say R_1 , from the vertex $b+a$ to a . Again, the length of the section of C_0 from a to $-2b+a$ (containing the vertex $2a$) is odd since it is of same length as the section of C_1 from b to $-b$ (containing the vertex $(b+a)$). Now the edges of the section of C_0 from a to $-2b+a$ together with the edge $a(-b+a)$ induce an even length path, say R_2 .

As in Case 1, except the two 4-cycles, namely, $(0, a, a+b, b)$ and $(-2b, -2b+a, -b+a, -b)$ in Γ , each of the 4-cycles formed by two consecutive vertices of C_0 and their corresponding vertices in C_1 constitute an alternating 4-cycle with respect to H_1 and H_2 . Clearly, one of the two paths along H_1 , joining a pair of opposite vertices of these alternating 4-cycles contains exactly one of the even length paths R_1 or R_2 and hence its length along H_1 is odd; hence if $(x, x+a, x+a+b, x+b)$ is an alternating 4-cycle with $x, x+a \in V(C_0)$ and $x+a+b, x+b \in V(C_1)$, then $p_{H_1}(x, x+a+b)$ is odd.

We now show that in at least one of these alternating 4-cycles, described above, $p_{H_2}(x, x+a+b)$ is odd. Let S_0 and S_1 be as defined in Case 1. Suppose the vertices of C_0 are alternately in S_0 and S_1 , then c is even, but it is not the case. Therefore, there exist two consecutive vertices along C_0 which are in S_0 or S_1 . Thus there exists a pair of opposite vertices of an alternating 4-cycle $(x, x+a, x+a+b, x+b)$, where $x \in V(C_0)$ and $x+a+b \in V(C_1)$ such that $p_{H_2}(x, x+a+b)$ is odd and hence $(x, x+a, x+a+b, x+b)$ is an odd alternating 4-cycle. Thus Γ satisfies property Q_2 , with respect to H_1 and H_2 .

This completes the proof of the lemma.

We use the following remarks in the proof of Lemma 3.3 given below.

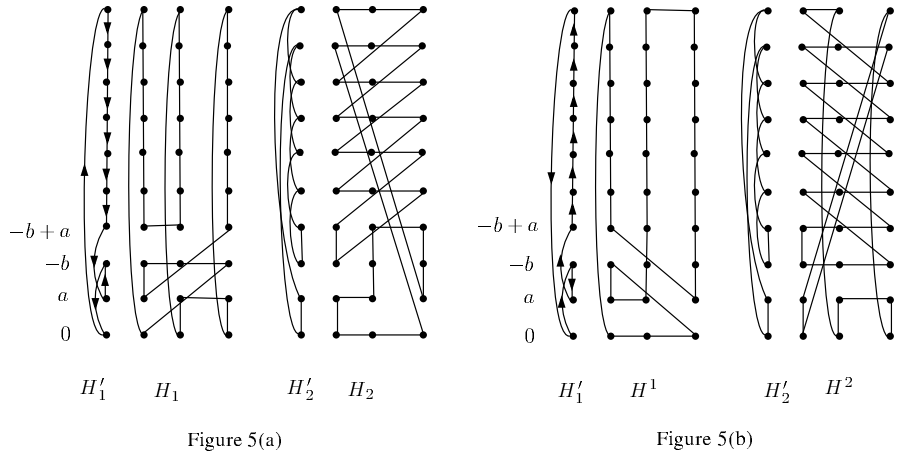
Lemma 3.2. [7] *Let Γ be a graph of $\Gamma(\alpha + 2, \beta)$. If the reduced graph Γ' admits a hamiltonian decomposition having the property Q_1 between $C_{\alpha-1}$ and C_0 , then Γ admits a hamiltonian decomposition having the property Q_1 between $C_{\alpha+1}$ and C_0 .*

Remark 3.1. *In the construction of two edge disjoint hamiltonian cycles H_1 and H_2 of $\Gamma \in \Gamma(\alpha + 2, \beta)$ from the two edge disjoint hamiltonian cycles H'_1 and H'_2 , respectively, of $\Gamma' \in \Gamma(\alpha, \beta)$, every b -edge of H'_i , $i = 1, 2$, connecting the vertices of $C'_{\alpha-1}$ to C'_0 is replaced by a path of odd length, see Lemma 3.2. Hence the parity of length of the path between any pair of vertices in $C_0, C_1, \dots, C_{\alpha-1}$ along H_i remains the same as the parity of length of the path between the respective vertices in $C'_0, C'_1, \dots, C'_{\alpha-1}$ of H'_i , where C'_i are the a -cycles of Γ' .*

By $\overrightarrow{(a, b)}$ we denote a directed arc with tail at a and head at b . In the following Remark 3.2, we show that the lift graph $\Gamma \in \Gamma(3, 2)$ of $\Gamma' \in \Gamma(1, 2)$ is unique up to isomorphism.

Remark 3.2. *Consider the graph $\Gamma \in \Gamma(3, 2)$ and its reduced graph $\Gamma' \in \Gamma(1, 2)$. Let $\{H'_1, H'_2\}$ be the hamiltonian decomposition of Γ' guaranteed by Lemma 3.1. From the construction, H'_1 has a b -edge of Γ' , namely, $0(-b)$. Clearly, this edge has two different orientations. Orient the edge $0(-b)$ of H'_1 as $\overrightarrow{(-b, 0)}$, see Figure 5 (which induces an orientation for the hamiltonian cycle H'_1 to make it as a directed hamiltonian cycle). With respect to this orientation, we obtain a hamiltonian decomposition $\{H_1, H_2\}$ of Γ , using the proof of Lemma 3.2, with property Q_1 , where the arcs $\overrightarrow{(-b, 0)}$ and $\overrightarrow{(-b+a, a)}$ correspond to the $(m_1 = 2)$ edges $\{(\alpha - 1, j_0)(0, j_0 + c)\}$ and $\{(\alpha - 1, j_1)(0, j_1 + c)\}$, respectively, in the proof of Lemma 3.2.*

Next orient the edge $0(-b)$ of H'_1 as $\overrightarrow{(0, -b)}$, see Figure 5(b) (which induces an orientation for the hamiltonian cycle H'_1). With respect to this orientation, we obtain a hamiltonian decomposition $\{H^1, H^2\}$ of Γ , by proof of Lemma 3.2, with the property Q_1 , where the arcs $\overrightarrow{(0, -b)}$ and $\overrightarrow{(a, -b+a)}$ correspond to the $(m_1 = 2)$ edges $\{(\alpha - 1, j_0)(0, j_0 + c)\}$ and $\{(\alpha - 1, j_1)(0, j_1 + c)\}$, respectively, in the proof of Lemma 3.2, see Figure 5. Irrespective of two different orientations of H'_1 , $H_1 \cup H_2 \cong H^1 \cup H^2$, that is, the two circulant graphs obtained from the union of the hamiltonian cycles $H_1 \cup H_2$ and $H^1 \cup H^2$ in two different orientations of the edges of H'_1 are isomorphic to each other. This isomorphism can be described by mapping (i, j) to $(i, n - j)$, $0 \leq i \neq \alpha - 1, 1 \leq j \leq k_a - 1$, where the even integer n is the number of vertices of Γ' , and the vertices $(i, 0), 0 \leq i \leq \alpha - 1$, are the fixed vertices of the isomorphism.



$$H_1 \cup H_2 \cong H^1 \cup H^2$$

The hamiltonian cycles H_1 and H_2 shown in Figure 5(a) arises out of the hamiltonian cycles H'_1 and H'_2 with respect to the orientation of H'_1 induced by the arc $\overrightarrow{(-b, 0)}$ and, the hamiltonian decomposition $\{H^1, H^2\}$ shown in Figure 5(b) arises out of the hamiltonian cycles H'_1 and H'_2 with respect to the orientation of H'_1 , induced by the arc $\overrightarrow{(0, -b)}$, where $|\Gamma'| = n = 10, a = 3, b = 4$.

Figure 5

Lemma 3.3. Let $\Gamma \in \Gamma(\alpha + 2, \beta)$ be a graph with even number of vertices and having the generating set $\{s, t\}$ where s and t are of different parity. If the reduced graph $\Gamma' \in \Gamma(\alpha, \beta)$, of Γ , has the property Q_2 and the property Q_1 between $C_{\alpha-1}$ and C_0 , with respect to a hamiltonian decomposition $\{H'_1, H'_2\}$ of Γ' , then Γ has a hamiltonian decomposition $\{H_1, H_2\}$ with property Q_2 and property Q_1 between $C_{\alpha+1}$ and C_0 .

Proof. Throughout the proof we use $\{s, t\}$ and $\{a, b\}$ as the generating sets for $\Gamma \in \Gamma(\alpha + 2, \beta)$ and $\Gamma' \in \Gamma(\alpha, \beta)$, respectively. Throughout the proof, assume that s is even and t is odd. As Γ is of even order and as s is even and t is odd, α is even and β is odd, since $\alpha + 2 = \gcd(m, s)$ and $\beta = \gcd(m, t)$, where m is the number of vertices of Γ . Let $\{H'_1, H'_2\}$ be a hamiltonian decomposition of Γ' and $\{H_1, H_2\}$ be the corresponding hamiltonian decomposition of Γ .

In the construction of H_1 and H_2 from H'_1 and H'_2 (see [7, Page 148, Lemma 1]), $H_i, i = 1, 2$, is obtained from H'_i by replacing each of the edges of H'_i between $C'_{\alpha-1}$ to C'_0 by an odd length path, whose origin is in $C_{\alpha-1}$ and terminus is in C_0 of Γ and the internal vertices of the paths are in C_α and $C_{\alpha+1}$.

As $\{H'_1, H'_2\}$ has the property Q_1 between $C_{\alpha-1}$ and C_0 , there exists a hamiltonian decomposition $\{H_1, H_2\}$ of $\Gamma \in \Gamma(\alpha + 2, \beta)$ satisfying property Q_1 between $C_{\alpha+1}$ and C_0 , by Lemma 3.2. Next we prove that Γ satisfies the property Q_2 with respect to $\{H_1, H_2\}$. We prove this by induction on α .

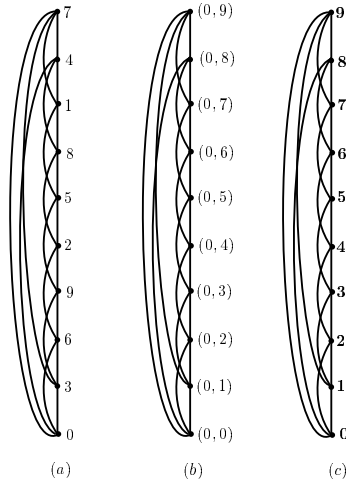
First we explain the idea behind the proof of this lemma. In Claim 1 below, we prove the existence of a hamiltonian decomposition of $\Gamma \in \Gamma(3, 2)$ satisfying Q_2 from a hamiltonian decomposition of $\Gamma' \in \Gamma(1, 2)$ with property Q_2 and in Claim 2 below, we prove the existence of a hamiltonian decomposition of $\Gamma \in \Gamma(4, 1)$ satisfying Q_2 from a hamiltonian decomposition of $\Gamma' \in \Gamma(2, 1)$ with property Q_2 . In Claim 3 below, we obtain a hamiltonian decomposition of $\Gamma \in \Gamma(\alpha + 2, \beta)$ with property Q_2 from a hamiltonian decomposition of $\Gamma' \in \Gamma(\alpha, \beta)$ with property Q_2 . First we assume that $\{H'_1, H'_2\}$ is the hamiltonian decomposition of $\Gamma' \in \Gamma(\alpha, \beta)$, where $\{\alpha, \beta\} = \{1, 2\}$, as described in Lemma 3.1.

Claim 1. For a hamiltonian decomposition $\{H'_1, H'_2\}$ of $\Gamma' \in \Gamma(1, 2)$ with property Q_2 , there is a hamiltonian decomposition $\{H''_1, H''_2\}$ of $\Gamma \in \Gamma(3, 2)$ with property Q_2 .

Let $|V(\Gamma')| = n$; by hypothesis Γ' admits a hamiltonian decomposition $\{H'_1, H'_2\}$ with the property Q_2 . We consider two cases.

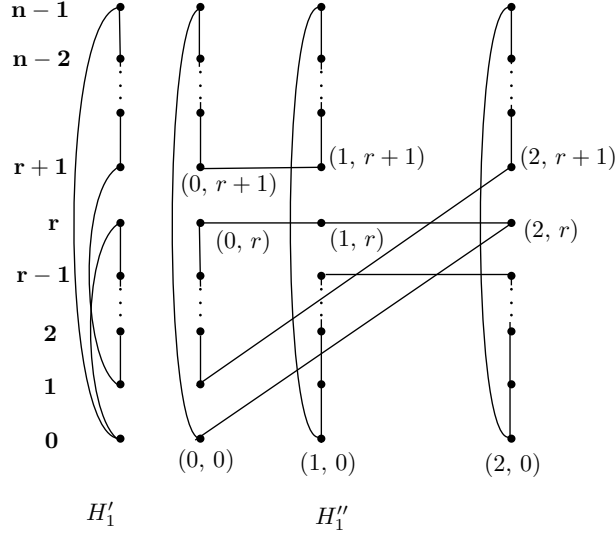
Case 1. $n \equiv 0 \pmod{4}$.

Let $\Gamma' \in \Gamma(1, 2)$ and let its corresponding graph in $\Gamma(3, 2)$ be Γ . From the definition of $\Gamma(\alpha, \beta)$, the labels of the vertices of the i th a -cycle are (i, j) , $0 \leq i \leq \alpha - 1$, $0 \leq j \leq k_a - 1$, where k_a is the length of the i th a -cycle. We know that $\Gamma' \in \Gamma(1, 2)$ has only one a -cycle and hence its vertices are $(0, j)$, $j = 0, 1, \dots, n - 1$. Each vertex $(0, j)$ of Γ' gives rise to three vertices, namely, $(0, j)$, $(1, j)$ and $(2, j)$, in Γ and we call these three vertices of Γ as the corresponding vertices of $(0, j)$ of Γ' and vice versa.



(a) Labeling of the vertices of the circulant graph $\Gamma' \in \Gamma(1, 2)$ on 10 vertices with the group elements, where the generating set is $\{3, 4\} \subset \mathbb{Z}_{10}$ (b). Labeling of the vertices of the graph Γ' with ordered pairs as is described in [7]. (c). Labeling of the vertices of the graph Γ' , with \mathbf{j} if the vertex has the label $(0, j)$ in (b).

Figure 6



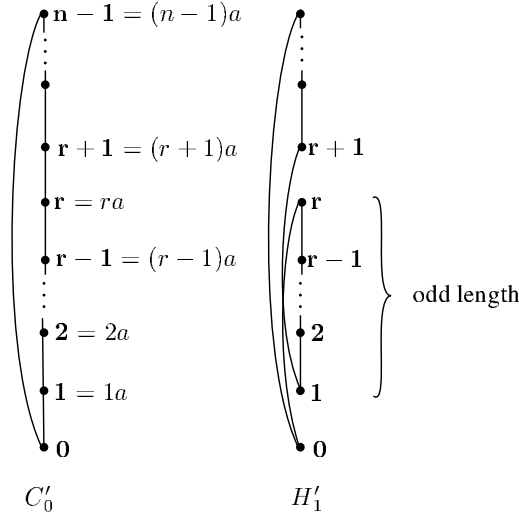
Construction of the hamiltonian cycle H''_1 of $\Gamma \in \Gamma(3, 2)$ from the hamiltonian cycle H'_1 of $\Gamma' \in \Gamma(1, 2)$, as described in [7, Page 148, Lemma 1].

Figure 7

For our convenience we relabel the vertices $(0, j)$, $0 \leq j \leq n - 1$, of $\Gamma' \in \Gamma(1, 2)$ as \mathbf{j} , see Figure 6(c); we call it as *new labeling of Γ'* (throughout this lemma, the new labeling of Γ' is denoted by bold face letters); in fact, in the proof of this lemma each vertex of Γ' will have three different labels, namely, $(0, j)$, \mathbf{j} and the other one the group element and according to our convenience and circumstances we use one of these labels. But, for the vertices of $\Gamma \in \Gamma(3, 2)$ we use the unique label (i, j) as in [7]. In the graph Γ' , let the vertex $-b$ be denoted by the label $(0, r)$ and hence in our new labeling it is denoted by \mathbf{r} , see Figure 6; for our convenience we write $-b = \mathbf{r}$. Clearly, $\mathbf{r} + \mathbf{1}$ ($= -b + a$) is the immediate next vertex of \mathbf{r} ($= -b$) in C'_0 of Γ' in the new labeling. Corresponding to the vertices \mathbf{r} and $\mathbf{r} + \mathbf{1}$ of Γ' , there are two rows, each having three vertices, in Γ ; the vertices of these rows are denoted by, $(0, r)$, $(1, r)$, $(2, r)$ and $(0, r + 1)$, $(1, r + 1)$, $(2, r + 1)$, respectively, see Figure 7.

In the construction of $\Gamma \in \Gamma(3, 2)$ from $\Gamma' \in \Gamma(1, 2)$, the two edges $(0, r)(1, r)$ and $(0, r+1)(1, r+1)$ are in H''_1 and the two edges $(0, r)(0, r+1)$ and $(1, r)(1, r+1)$ are in H''_2 , where $\{H''_1, H''_2\}$ is the hamiltonian decomposition of Γ , corresponding to the hamiltonian decomposition $\{H'_1, H'_2\}$ of Γ' , see Figures 7 and 9. We claim that Γ satisfies the property Q_2 with respect to the hamiltonian decomposition $\{H''_1, H''_2\}$, where the required 4-cycle of Γ is $(0, r) (0, r + 1) (1, r + 1) (1, r)$.

First we show that $p_{H''_1}((0, r), (1, r+1))$ is odd. Clearly, $C'_0 = (\mathbf{0}, \mathbf{1}, \dots, \mathbf{n} - \mathbf{1})$ is a hamiltonian cycle in Γ' , with respect to our new labeling.



Length of the $(\mathbf{1}, \mathbf{r})$ -section along H'_1 is odd, as \mathbf{r} (an even integer) is the length from $\mathbf{0}$ to \mathbf{r} along C'_0 ; this implies that the $(\mathbf{0}, \mathbf{r} + \mathbf{1})$ -section along H'_1 is of odd length.

Figure 8

The hamiltonian cycle H''_1 of Γ contains the four vertices $(0, r)$, $(0, 0)$, $(0, r + 1)$ and $(1, r + 1)$ in the clockwise order as shown in the Figure 7 (the order is guaranteed by the corresponding hamiltonian cycle H'_1 of Γ'), where $(0, r + 1)(1, r + 1)$ is an edge of H''_1 . As C'_0 being an even cycle, it can be thought of as a bipartite graph with bipartition $X = \{\mathbf{0}, \mathbf{2}, \mathbf{4}, \dots, \mathbf{n} - \mathbf{2}\}$ and $Y = \{\mathbf{1}, \mathbf{3}, \dots, \mathbf{n} - \mathbf{1}\}$. Clearly, the vertex $-b$ ($= \mathbf{r}$, the $(\mathbf{r} + \mathbf{1})$ th vertex along C'_0 , the group element $ra \pmod n$); we do not differentiate the labels $-b$, \mathbf{r} and ra and we denote the vertex $-b$ by $-b = \mathbf{r} = ra$ in the three labellings of the vertices of Γ') is in X , because b is even implies $-b$ is even; as $-b = ra$ and a is odd, \mathbf{r} is even. Further, $0 \in X$ and $-b \in X$ implies $p_{C'_0}(\mathbf{0}, \mathbf{r}) = p_{C'_0}(0, -b)$ is even. Now consider the hamiltonian decomposition $\{H'_1, H'_2\}$ of Γ' as in Lemma 3.1. The hamiltonian cycle H'_1 of Γ' is obtained from C'_0 by deleting two edges $0a (= \mathbf{0}\mathbf{1})$ and $(-b)(-b + a)(= \mathbf{r}(\mathbf{r} + \mathbf{1}))$ of C'_0 and adding two edges $0(-b)(= \mathbf{0}\mathbf{r})$ and $a(-b + a)(= \mathbf{1}(\mathbf{r} + \mathbf{1}))$, see Figure 8.

Next we prove that $p_{H'_1}(\mathbf{0}, \mathbf{r} + \mathbf{1}) (= p_{H'_1}(0, -b + a))$ is odd. From the last paragraph, $p_{C'_0}(\mathbf{0}, \mathbf{r})$ is even and hence $p_{C'_0}(\mathbf{1}, \mathbf{r})$ is odd, see Figure 8; the path from $\mathbf{1}$ to \mathbf{r} along C'_0 is also in H'_1 and hence $p_{H'_1}(\mathbf{1}, \mathbf{r})$ is odd; consequently, as n is even,

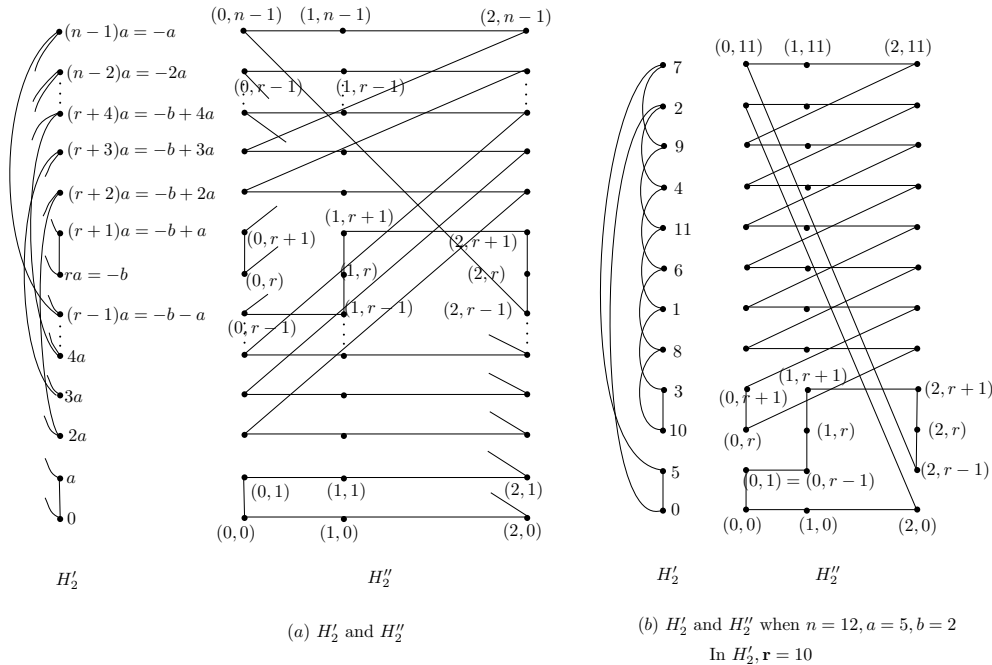
$$(1) \quad p_{H'_1}(\mathbf{0}, \mathbf{r} + \mathbf{1}) \text{ is odd,}$$

(where we consider the $(\mathbf{0}, \mathbf{r} + \mathbf{1})$ -section of H'_1 not containing the vertex \mathbf{r}), see Figure 8.

Next we prove that $p_{H''_1}((0, r), (1, r + 1))$ is odd. We divide the $((0, r), (1, r + 1))$ -section of H''_1 (containing the vertex $(1, r)$) into three subsections and we show

that each one of them is of odd length to conclude $p_{H_1''}((0, r), (1, r + 1))$ is odd; the subsections are $((0, r), (0, 0)), ((0, 0), (0, r + 1))$ and $((0, r + 1), (1, r + 1))$, see Figure 7. From the construction of H_1'' from H_1' , it is clear that as $((0, r), (0, 0))$ -section and $((0, r + 1), (1, r + 1))$ -section of H_1'' are of length 3 and length 1, respectively, it is enough to show that $((0, 0), (0, r + 1))$ -subsection is of odd length. But it is easy to see that $((0, 0), (0, r + 1))$ -section of H_1'' is identical with the $(\mathbf{0}, \mathbf{r} + \mathbf{1})$ -section of the hamiltonian cycle H_1' , see Figure 7. The $(\mathbf{0}, \mathbf{r} + \mathbf{1})$ -section of H_1' is already proved to be of odd length, by (1). Thus $p_{H_1''}((0, 0), (0, r + 1))$ is odd.

Next we prove that $p_{H_2''}((0, r), (1, r + 1))$ is odd. As above, we divide the $((0, r), (1, r + 1))$ -section of H_2'' (not containing the vertex $(0, r + 1)$) into four subsections, namely, $((0, r), (0, 0)), ((0, 0), (0, 1)), ((0, 1), (0, r - 1))$ and $((0, r - 1), (1, r + 1))$ in the cyclic order are guaranteed by H_2' , see Figure 9. From the construction of H_2 , $(0, 0)(0, 1)$ is an edge and the $((0, r - 1), (1, r + 1))$ -section is a path of length 3, namely, $(0, r - 1)(1, r - 1)(1, r)(1, r + 1)$, see Figure 9. Hence we show that the lengths of the other two sections are of different parity. This is achieved by finding the lengths of the corresponding sections in H_2' of Γ' . The sections in H_2' corresponding to the $((0, r), (0, 0))$ -section and $((0, 1), (0, r - 1))$ -section of H_2'' in Γ are $(\mathbf{r}, \mathbf{0})$ -section and $(\mathbf{1}, \mathbf{r} - \mathbf{1})$ -section, respectively, in the new labeling of Γ' .



H_2' in Γ' and H_2'' in Γ

Figure 9

We shall show that $p_{H'_2}(\mathbf{r}, \mathbf{0})$ is odd and $p_{H'_2}(\mathbf{1}, \mathbf{r} - \mathbf{1})$ is even. We prove that $p_{H'_2}(\mathbf{r}, \mathbf{0})$ is odd by observing that the $(\mathbf{0}, \mathbf{r})$ -section (not containing the edge $\mathbf{0r}$) of one of the b -cycles, namely $0b(2b) \dots (-b)0 (= (\mathbf{0} \dots \mathbf{r}\mathbf{0})$ in the new labeling, not containing the edge $\mathbf{0r}$) of Γ' is a section of H'_2 , and it is of odd length as the $(\mathbf{0}, \mathbf{r})$ -section of H'_2 together with the edge $\mathbf{0r}$ is a b -cycle, which is of even length in Γ' , that is, $p_{H'_2}(\mathbf{r}, \mathbf{0})$ is odd.

Next we prove that $p_{H'_2}(\mathbf{1}, \mathbf{r} - \mathbf{1})$ is even. Now consider the $(\mathbf{1}, \mathbf{r} - \mathbf{1})$ -section of H'_2 containing only b -edges. As we move along b -edges of the $(\mathbf{1}, \mathbf{r} - \mathbf{1})$ -section of H'_2 , the alternate labels of the vertices are congruent to $1 \pmod{4}$ or $3 \pmod{4}$. If two nonconsecutive vertices of H'_2 are both congruent to 1 or $3 \pmod{4}$, then their distance along H'_2 is even. Therefore, as $a \equiv 1$ or $3 \pmod{4}$ implies $-b - a \equiv 1$ or $3 \pmod{4}$, $p_{H'_2}(\mathbf{1}, \mathbf{r} - \mathbf{1})$ is even.

As observed earlier, the length of the $((0, r), (1, r + 1))$ -section of H''_2 is sum of the lengths of the sections $((0, r), (0, 0)), ((0, 0), (0, 1)), ((0, 1), (0, r - 1))$ and a path of length 3 from $(0, r - 1)$ to $(1, r + 1)$ in H''_2 of Γ . As $p_{H'_2}(\mathbf{r}, \mathbf{0})$ is odd, $p_{H''_2}((0, r), (0, 0))$ is odd by Remark 3.1, the $((0, 0), (0, 1))$ -section is an edge, the length of $((0, 1), (0, r - 1))$ -section of H''_2 is even, as $p_{H'_2}(\mathbf{1}, \mathbf{r} - \mathbf{1})$ is even and by Remark 3.1 and the last section is a $P_4 = (0, r - 1)(1, r - 1)(1, r)(1, r + 1)$. Thus in Γ , $p_{H''_2}((0, r), (1, r + 1))$ is odd.

Case 2. $n \equiv 2 \pmod{4}$.

Let $\{H'_1, H'_2\}$ be the hamiltonian decomposition of Γ' with property Q_2 obtained in Lemma 3.1. As pointed out in Remark 3.2, the hamiltonian cycle H'_1 has two natural orientations and with respect to each of these two orientations there is a hamiltonian decomposition of $\Gamma \in \Gamma(3, 2)$. We show that in at least one of these two orientations of H'_1 , the corresponding hamiltonian decomposition of Γ has the property Q_2 (Note that both these two hamiltonian decompositions satisfy the property Q_1 as in Remark 3.2). Consider the new labeling of Γ' described in Case 1 above.

Subcase 2.1. $p_{H'_2}(\mathbf{1}, \mathbf{r} - \mathbf{1})$ is odd.

We know that H'_1 has exactly two b -edges. In this case we consider the first orientation of H'_1 (orientation induced by $\rightarrow (-b, 0)$ for the hamiltonian cycle H'_1). Consider the 4-cycle $C = ((0, r)(0, r + 1)(1, r + 1)(1, r))$ in Γ , see Figure 9 (the figure for the case $n \equiv 2 \pmod{4}$ also resembles the same as in the case $n \equiv 0 \pmod{4}$ and note that \mathbf{r} is even).

To prove $p_{H''_1}((0, r), (1, r + 1))$ is odd, we divide the $((0, r), (1, r + 1))$ -section of H''_1 , (containing the vertex $(1, r)$) into three subsections and we show that each of them is of odd length. The subsections are $((0, r), (0, 0)), ((0, 0), (0, r + 1))$ and $((0, r + 1), (1, r + 1))$, see Figure 7 (the figure for the case $n \equiv 2 \pmod{4}$ also resembles the same as in the case $n \equiv 0 \pmod{4}$). From the construction of H''_1 from H'_1 , it is clear that $((0, r), (0, 0))$ -section of H''_1 is of length 3, see Figure 7, and $(0, r + 1)(1, r + 1)$ is an edge, hence it is enough to show that $((0, 0), (0, r + 1))$ -section (containing the vertex $(0, n - 1)$) is of odd length. But it is easy to see (in

fact, it is the same proof as in the case $n \equiv 0 \pmod{4}$) that this section of H_1'' is the same as the $(\mathbf{0}, \mathbf{r} + \mathbf{1})$ -section of the hamiltonian cycle H_1' of Γ' , which is of odd length and hence $p_{H_1''}((0, 0), (0, r + 1))$ is odd. This completes the proof that $p_{H_1''}((0, r), (1, r + 1))$ is odd.

Now we prove that $p_{H_2''}((0, r)(1, r + 1))$ is odd. To prove this, we divide the $((0, r), (1, r + 1))$ -section of H_2'' (not containing the vertex $(0, r + 1)$) into four subsections, namely, $((0, r), (0, 0))$, $((0, 0), (0, 1))$, $((0, 1), (0, r - 1))$ and $((0, r - 1), (1, r + 1))$, see Figure 9. It is clear that the $((0, 0), (0, 1))$ -section is an edge and the $((0, r - 1), (1, r + 1))$ -section is a path of length 3. We shall show that the lengths of the subsections $((0, r), (0, 0))$ and $((0, 1), (0, r - 1))$ are of different parity.

First we prove that the length of the subsection $((0, r), (0, 0))$ of H_2'' is even. Clearly, the section corresponding to $((0, r), (0, 0))$ of H_2'' is $(\mathbf{r}, \mathbf{0})$ -section in H_2' of Γ' , in the new labeling. Observe that the $(\mathbf{0}, \mathbf{r})$ -section of one of the b -cycles, not containing the edge $\mathbf{0r}$, of Γ' is a section of H_2' , and it is of even length as the $(\mathbf{0}, \mathbf{r})$ -section of H_2' together with the edge $\mathbf{0r}$ is a b -cycle of odd length in Γ' (as $n \equiv 2 \pmod{4}$ and $\beta = 2$), that is, $p_{H_2'}(\mathbf{r}, \mathbf{0})$ is even. Hence by Remark 3.1, $p_{H_2''}((0, r), (0, 0))$ is even.

Next consider the $((0, 1), (0, r - 1))$ -section (containing the vertex $(1, 1)$) of H_2'' ; the corresponding $(\mathbf{1}, \mathbf{r} - \mathbf{1})$ -section in H_2' contains only b -edges. As $p_{H_2'}(\mathbf{1}, \mathbf{r} - \mathbf{1})$ is odd, by assumption, and also the section $(\mathbf{1}, \mathbf{r} - \mathbf{1})$ of H_2' containing only b -edges, $p_{H_2''}((0, 1), (0, r - 1))$ is odd, by Remark 3.1. This proves $p_{H_2''}((0, r), (1, r + 1))$ is odd.

Subcase 2.2. $p_{H_2'}(\mathbf{1}, \mathbf{r} - \mathbf{1})$ is even.

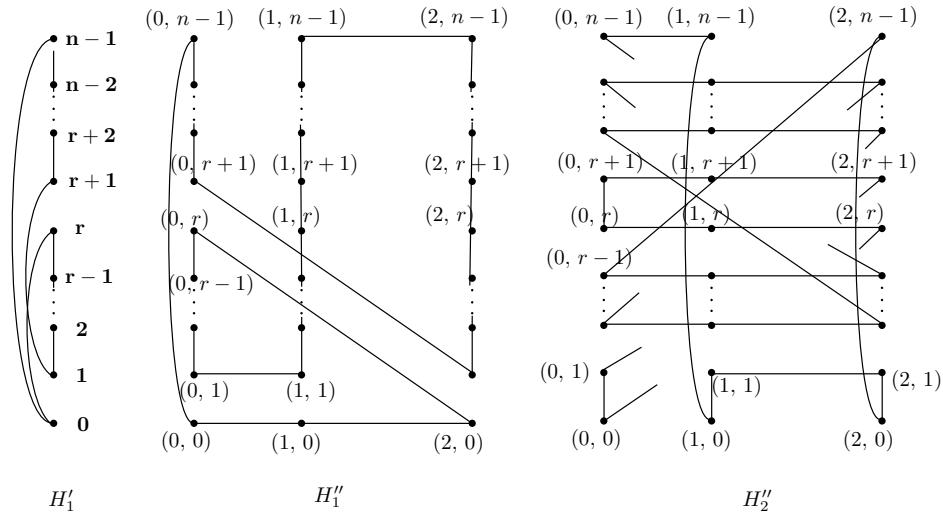
In this subcase, we consider the other orientation of the edges of H_1' (orientation induced by $\rightarrow (0, -b)$ for the hamiltonian cycle H_1'). Then $C = ((0, 0)(0, 1)(1, 1)(1, 0))$ will be proved to be an odd alternating 4-cycle, see Figure 10.

First we prove that $p_{H_1''}((0, 0)(1, 1))$ is odd. For this, we divide the $((0, 0), (1, 1))$ -section of H_1'' (containing the vertex $(1, 0)$) into three subsections, namely, $((0, 0), (0, r))$, $((0, r), (0, 1))$ and $((0, 1), (1, 1))$. Clearly, $((0, 0), (0, r))$ -section is the path $(0, 0)(1, 0)(2, 0)(0, r)$ of length 3 and $((0, 1), (1, 1))$ -section is an edge and hence it is enough to prove that $((0, r), (0, 1))$ -section of H_1'' is of odd length.

In H_1' of Γ' , b is even implies $-b$ is even as n is even. As $-b = ra$ and a is odd implies \mathbf{r} is even. It is an easy observation that the vertices with even numbered labels in the new labeling of H_1' are at odd distance from the vertex $a = \mathbf{1}$, the new label of the vertex a , since the even numbered vertices are $2a = \mathbf{2}$, $4a = \mathbf{4}$, etc. Hence $p_{H_1'}(\mathbf{1}, \mathbf{r})$ is odd as \mathbf{r} is even. As the $((0, r), (0, 1))$ -section of H_1'' has the same length as the $(\mathbf{r}, \mathbf{1})$ -section of H_1' , $p_{H_1''}((0, r), (0, 1))$ is odd.

Finally, we show that $p_{H_2''}((0, 0)(1, 1))$ is odd. To prove this, we divide the $((0, 0), (1, 1))$ -section of H_2'' (containing the vertex $(0, 1)$) into three subsections, namely, $((0, 0), (0, 1))$, $((0, 1), (0, r - 1))$ and $((0, r - 1), (1, 1))$, see Figure 10. It is clear that the $((0, 0), (0, 1))$ -section is an edge and the $((0, r - 1), (1, 1))$ -section is the path $(0, r - 1)(2, n - 1)(2, 0)(2, 1)(1, 1)$ on 5 vertices, see Figure 10. Hence it is enough to show that the $((0, 1), (0, r - 1))$ -section is of even length.

As $p_{H'_2}(\mathbf{1}, \mathbf{r} - \mathbf{1})$ is even, by assumption in this subcase, $(\mathbf{1}, \mathbf{r} - \mathbf{1})$ -section of H'_2 , containing only b -edges, is of even length and hence $p_{H'_2}((0, 1), (0, r - 1))$ is even, by Remark 3.1. Thus $p_{H'_2}((0, 0), (0, 1))$ is odd. Hence the hamiltonian decomposition $\{H''_1, H''_2\}$ of Γ satisfies the property Q_2 .



The hamiltonian cycles H''_1 and H''_2 of $\Gamma \in \Gamma(3, 2)$ corresponding to the orientation of H'_1 , induced by $\rightarrow (0, r)$, of $\Gamma' \in \Gamma(1, 2)$.

Figure 10

Claim 2. For any hamiltonian decomposition $\{H'_1, H'_2\}$ of $\Gamma' \in \Gamma(2, 1)$ with the property Q_2 , there is a hamiltonian decomposition $\{H'''_1, H'''_2\}$ of $\Gamma \in \Gamma(4, 1)$ with the property Q_2 .

Consider the graph $\Gamma' \in \Gamma(2, 1)$. There are two a -cycles C'_0 and C'_1 ; hence there are two parallel matchings between C'_0 and C'_1 . We denote the matching from C'_0 to C'_1 as M_0 and the other matching from C'_1 to C'_0 as M_1 ; the edges of M_0 are $(0, j)(1, j)$ and the edges of M_1 are $(1, j)(0, j + c)$, $0 \leq j \leq \frac{n}{2} - 1$, for some c .

Clearly, from the construction of H'_1 and H'_2 of Γ' , every alternating 4-cycle having two consecutive vertices along C'_0 and two consecutive vertices along C'_1 uses exactly two a -edges and two b -edges and both these two b -edges are in M_0 or in M_1 . Lemma 3.1 guarantees the existence of an odd alternating 4-cycle with respect to the hamiltonian decomposition $\{H'_1, H'_2\}$ of Γ' and this 4-cycle contains two b -edges of M_0 . In the construction of H'''_i , $i = 1, 2$, of Γ (from H'_i , $i = 1, 2$, of Γ') the a -edges and M_0 -edges (that is, b -edges that belong to the matching M_0) of Γ' are retained as it is in the transformation of Γ' to Γ . While obtaining H'''_i from H'_i , each of the b -edges of M_1 in H'_i are replaced by a path of odd length whose internal vertices are in C_2 and C_3 . Hence an odd alternating 4-cycle that exists in Γ' becomes an odd alternating 4-cycle with respect to the hamiltonian decomposition $\{H'''_1, H'''_2\}$ of Γ .

Claim 3. For any hamiltonian decomposition $\{H'_1, H'_2\}$ of $\Gamma' \in \Gamma(\alpha, \beta)$ with property Q_2 , there is a hamiltonian decomposition $\{H_1^{iv}, H_2^{iv}\}$ of $\Gamma \in \Gamma(\alpha + 2, \beta)$ with property Q_2 .

In Claims 1 and 2, we proved that the existence of the property Q_2 in $\Gamma' \in \Gamma(1, 2)$ or $\Gamma(2, 1)$ implies the existence of the property Q_2 in $\Gamma \in \Gamma(3, 2)$ or $\Gamma(4, 1)$, respectively. From the hamiltonian decomposition and construction of the odd alternating 4-cycle of Γ from Γ' , we have shown that the odd alternating 4-cycle lies between C_0 and C_1 , that is having two vertices in C_0 and two vertices in C_1 . Hence, as in Claim 2, the odd alternating 4-cycle in the hamiltonian decomposition $\{H'_1, H'_2\}$ of $\Gamma' \in \Gamma(\alpha, \beta)$ is also the odd alternating 4-cycle in the hamiltonian decomposition $\{H_1^{iv}, H_2^{iv}\}$ of $\Gamma \in \Gamma(\alpha + 2, \beta)$; this completes the proof of existence of the property Q_2 with respect to the hamiltonian decomposition $\{H_1^{iv}, H_2^{iv}\}$ of Γ .

This completes the proof of the lemma by induction on α .

Remark 3.3. Recall that our graphs of $\Gamma(\alpha, \beta)$ always have even order by assumption. As every member of $\Gamma(\alpha, \beta)$ is isomorphic to a member of $\Gamma(\beta, \alpha)$ and vice-versa, to each member of $\Gamma \in \Gamma(\alpha, \beta)$, there corresponds a graph $\Gamma_1 (\cong \Gamma) \in \Gamma(\beta, \alpha)$. If α is odd, then k_a , the length of each of the a -cycles of $\Gamma \in \Gamma(\alpha, 2)$, must be even. Let $\Gamma \in \Gamma(\alpha, 2)$ and let $\Gamma_1 \in \Gamma(2, \alpha)$ be the image of Γ under an isomorphism. Clearly, the a -edges and b -edges of Γ are the b -edges and a -edges of Γ_1 , respectively, and vice versa. As k_a is even, each a -cycle C_i , $0 \leq i \leq \alpha - 1$, in Γ has two 1-factors, say F_1^i and F_2^i in the subgraph induced by C_i ; that is, for $i \in \{0, 1, \dots, \alpha - 1\}$, $F_1^i = \{(i, 0)(i, 1), (i, 2)(i, 3), \dots, (i, n - 2)(i, n - 1)\}$ and $F_2^i = \{(i, 1)(i, 2), (i, 3)(i, 4), \dots, (i, n - 1)(i, 0)\}$. The edges of the 1-factor F_1^i of Γ become the M_0 -edges of $\Gamma_1 \in \Gamma(2, \alpha)$ and the edges of the 1-factor F_2^i of Γ become the M_1 -edges of Γ_1 . In the proof of Lemma 3.3, we constructed an odd alternating 4-cycle with respect to the hamiltonian decomposition of Γ , so that the two a -edges use the edges of F_1^i . Hence, if $\Gamma \in \Gamma(\alpha, 2)$ has an odd alternating 4-cycle, then the image of the 4-cycle under the isomorphism between the corresponding graphs of $\Gamma(\alpha, 2)$ and $\Gamma(2, \alpha)$ gives the odd alternating 4-cycle in $\Gamma_1 \in \Gamma(2, \alpha)$ using two M_0 -edges of Γ_1 .

Remark 3.4. By saying, for $\Gamma \in \Gamma(\alpha, \beta)$ obtain the reduced graph of Γ in $\Gamma(\alpha_1, \beta_1)$, $\alpha_1 \leq \alpha, \beta_1 \leq \beta$, we mean the following: The successive reduced graphs of $\Gamma \in \Gamma(\alpha, \beta)$ in $\Gamma(\alpha - 2, \beta), \Gamma(\alpha - 4, \beta), \dots, \Gamma(\alpha_1, \beta)$, yields a graph $\Gamma_1 \in \Gamma(\alpha_1, \beta)$. But the class of graphs in $\Gamma(\alpha_1, \beta)$ are isomorphic to the class of graphs in $\Gamma(\beta, \alpha_1)$, that is, to each graph of $\Gamma(\alpha_1, \beta)$ there is an isomorphic copy of it in $\Gamma(\beta, \alpha_1)$ and vice versa. Hence Γ_1 can be considered as a graph $\Gamma_2 \in \Gamma(\beta, \alpha_1)$. Then successive reduced graphs of Γ_2 yields a graph $\Gamma_3 \in \Gamma(\beta_1, \alpha_1)$. As the class of graphs in $\Gamma(\beta_1, \alpha_1)$ are isomorphic to the class of graphs in $\Gamma(\alpha_1, \beta_1)$, Γ_3 can be considered as a graph $\Gamma_4 \in \Gamma(\alpha_1, \beta_1)$. We shall call $\Gamma_3 \in \Gamma(\beta_1, \alpha_1)$ or $\Gamma_4 \in \Gamma(\alpha_1, \beta_1)$ (note that $\Gamma_3 \cong \Gamma_4$) as a reduced graph of $\Gamma \in \Gamma(\alpha, \beta)$ according to the circumstances.

Similarly, by saying, for $\Gamma' \in \Gamma(\nu, \delta)$ obtain its lifted graph $\Gamma_{k+l} \in \Gamma(\nu + 2k, \delta + 2\ell)$, where k and ℓ are not simultaneously zero, we mean the following: for $\Gamma' \in \Gamma(\nu, \delta)$, by lifting, we obtain a graph $\Gamma_1 \in \Gamma(\nu + 2, \delta)$. Similarly, for

$\Gamma_1 \in \Gamma(\nu+2, \delta)$, by lifting, we obtain $\Gamma_2 \in \Gamma(\nu+4, \delta)$. Successively, we can get the graph $\Gamma_k \in \Gamma(\nu+2k, \delta)$. But the class of graphs in $\Gamma(\nu+2k, \delta)$ are isomorphic to the class of graphs in $\Gamma(\delta, \nu+2k)$, that is, to each graph of $\Gamma(\nu+2k, \delta)$ there is an isomorphic copy of it in $\Gamma(\delta, \nu+2k)$ and vice versa. Hence Γ_k can be considered as a graph of $\Gamma(\delta, \nu+2k)$. From this, by successive liftings, we obtain $\Gamma_{k+1} \in \Gamma(\delta+2, \nu+2k)$, $\Gamma_{k+2} \in \Gamma(\delta+4, \nu+2k)$, \dots , $\Gamma_{k+\ell} \in \Gamma(\delta+2\ell, \nu+2k)$. We call $\Gamma_{k+\ell}$ as a lifted graph of $\Gamma' \in \Gamma(\nu, \delta)$. As the two classes of graphs $\Gamma(\nu+2k, \delta+2\ell)$ and $\Gamma(\delta+2\ell, \nu+2k)$ have the same set of graphs, upto isomorphism, we can consider $\Gamma_{k+\ell}$ as an element of $\Gamma(\nu+2k, \delta+2\ell)$.

The idea in the next theorem is based on [7, Page 150, Main Theorem].

Theorem 3.1. *Every 4-regular connected circulant graph $\Gamma \in \Gamma(\alpha, \beta)$ of even order with jumps of different parity has the property Q_2 with respect to a hamiltonian decomposition.*

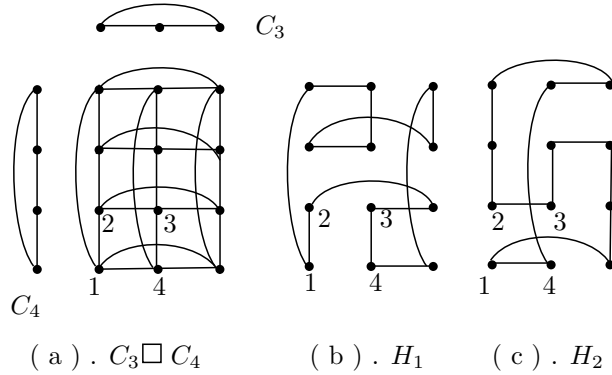
Proof. Let $\Gamma \in \Gamma(\alpha, \beta)$. As the jumps are of different parity, α and β are of different parity. Without loss of generality, we assume that α is even and β is odd. Obtain a reduced graph Γ' of Γ , see Remark 3.4. If Γ' is simple, then it belongs to $\Gamma(1, 2)$ and has a hamiltonian decomposition satisfying properties Q_1 and Q_2 , by Lemma 3.1. Now we “lift” Γ' to $\Gamma'' \in \Gamma(\beta, 2)$. By Lemma 3.3, Γ'' has the properties Q_1 and Q_2 with respect to a hamiltonian decomposition of it. Γ'' can be considered as a graph $\Gamma''' \in \Gamma(2, \beta)$. Again lift Γ''' to a graph in $\Gamma(\alpha, \beta)$, which is precisely Γ , having the properties Q_1 and Q_2 with respect to a hamiltonian decomposition of it, by Lemma 3.3. Hence the result is true if Γ' is simple.

Next we consider the case when the reduced graph Γ' is not simple. If Γ' is not simple, then in this case, stop the reduction process at a stage when the reduced graph is simple but any further reduction of that graph is not simple. Let $\Gamma_1 \in \Gamma(\alpha_1, \beta_1)$ be a reduced graph of $\Gamma \in \Gamma(\alpha, \beta)$ such that Γ_1 is simple but the reduced graph of Γ_1 in $\Gamma(\alpha_1-2, \beta_1)$ is not simple. Then, Γ_1 belongs to one of the following three cases, see [7, Page 150, Main Theorem]:

- (i) $\alpha_1 = 3$ and $c = 0$; in that case we obtain loops in Γ' .
- (ii) $\beta_1 = 4$ and $c = 0$; in this case the edges in Γ' between C'_0 and C'_1 and, between C'_1 and C'_0 are the same, that is, they are multiple edges.
- (iii) $\alpha_1 = 3$ and $c = \frac{k_a}{2}$; where k_a is the order of the element a in the generating set of Γ_1 ; in this case we get multiple edges in Γ' .

In case (i), if $\beta_1 = 4$ then Γ_1 itself is a graph in $\Gamma(3, 4)$. If $\beta_1 \geq 6$ in case (i), then reduce Γ_1 according to β_1 to get a graph in $\Gamma(3, 4)$. Similarly, in case (ii), if $\alpha_1 = 3$ then Γ_1 itself is a graph in $\Gamma(3, 4)$. If $\alpha_1 \geq 5$, in case (ii), then reduce Γ_1 according to α_1 to get a graph in $\Gamma(3, 4)$.

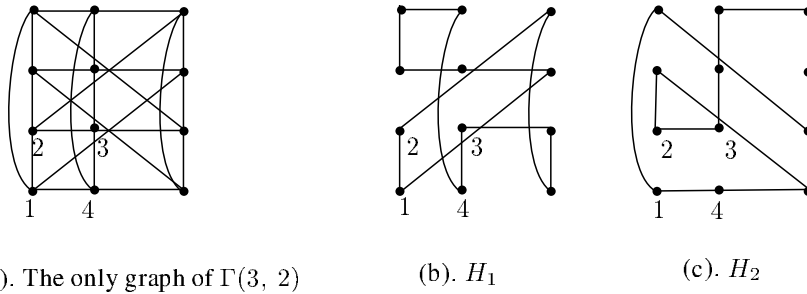
As the jumps of Γ_1 are of different parity, the first two cases correspond to the Cartesian product of cycles with $\alpha_1 = 3$; Hence it is enough to consider $\Gamma(3, 4)$ with $c = 0$. Then the class of graphs $\Gamma(3, 4)$ with $c = 0$ reduces to the single graph $C_3 \square C_4$, where \square is the Cartesian product of graphs. In Figure 11, a hamiltonian decomposition is shown, where $C = (1\ 2\ 3\ 4)$ is an odd alternating 4-cycle.



The graph $C_3 \square C_4 \in \Gamma(3, 4)$ and a hamiltonian decomposition $\{H_1, H_2\}$ with property Q_2 .

Figure 11

Next we consider the last case $\alpha_1 = 3, c = \frac{k_a}{2}$. As Γ_1 is simple, then $k_a \geq 3$ and $\beta_1 \geq 2$, as $c = \beta_1 = \frac{k_a}{2} \geq 2$. (as argued in first two cases, we have $\beta_1 \geq 2$ and if $\beta_1 = 2$, then the graph Γ_1 itself is a graph in the class $\Gamma(3, 2)$). If $\beta_1 \geq 4$, we first reduce the graph Γ_1 to a graph in $\Gamma(3, 2)$. In this case there is only one graph in $\Gamma(3, 2)$, see [7, Page 150, Main Theorem], as shown in Figure 12. In the hamiltonian decomposition shown in Figure 12, $C = (1 2 3 4)$ is an odd alternating 4-cycle.



The only graph of $\Gamma(3, 2)$ and a hamiltonian decomposition with property Q_2 .

Figure 12

To complete the proof, first we lift the graph $C_3 \square C_4 \in \Gamma(3, 4)$ of Figure 11 to the graph $G \in \Gamma(\beta, 4)$ with properties Q_1 and Q_2 with respect to a hamiltonian decomposition, by Lemma 3.3. Let $G_1 \in \Gamma(4, \beta)$ be isomorphic to G . As G has the properties Q_1 and Q_2 , so does G_1 . Now we lift the graph $G_1 \in \Gamma(4, \beta)$ to the graph $\Gamma \in \Gamma(\alpha, \beta)$. Clearly, $\Gamma \in \Gamma(\alpha, \beta)$ has a hamiltonian decomposition, by Lemma 3.3. Similarly, we prove it for the graph in Figure 12.

This completes the proof of theorem.

The following theorem of Jha [11] is used in our proof of Theorem 1.2.

Theorem 3.2. [11] *Let $\{H_1, H_2\}$ be a hamiltonian decomposition of a 4-regular graph G of even order m containing an odd alternating four cycle, that is property Q_2 , with respect to $\{H_1, H_2\}$, then the graph $C_n \times G$, n even, admits a hamiltonian decomposition.*

Proof of Theorem 1.2.

Let $|V(G)| = m$ and $|V(H)| = n$. Since H is hamiltonian decomposable, we have $H = C_n^1 \oplus C_n^2 \oplus \dots \oplus C_n^\ell$, where each C_n^i is a hamiltonian cycle of H . If at least one of G or H has odd order, then $G \times H$ is hamiltonian decomposable, by Theorem 1.1, and hence assume that both G and H are of even order. Since G has the property Q , G can be decomposed into 4-regular connected circulants, that is, $G = G_1 \oplus G_2 \oplus \dots \oplus G_{\ell'}$, where each G_i is a 4-regular circulant graph with jumps of different parity. Thus G_i , $1 \leq i \leq \ell'$, can be decomposed into two hamiltonian cycles with property Q_2 , by Theorem 3.1. Now $G \times H \cong (G_1 \oplus G_2 \oplus \dots \oplus G_{\ell'}) \times (C_n^1 \oplus C_n^2 \oplus \dots \oplus C_n^\ell) = (G_1 \times C_n^1) \oplus \dots \oplus (G_1 \times C_n^\ell) \oplus \dots \oplus (G_{\ell'} \times C_n^1) \oplus \dots \oplus (G_{\ell'} \times C_n^\ell)$. But each $G_i \times C_n^j$, $1 \leq i \leq \ell'$, $1 \leq j \leq \ell$, can be decomposed into hamiltonian cycles, by Theorem 3.2.

This completes the proof of the theorem.

Conclusion. In [2, 10, 13, 14, 15], existence of hamiltonian decompositions of the graphs $K_r \times K_s$, $K_r \times K_{s,s}$, $K_{r(s)} \times K_{m(n)}$, $K_{r,r} \times K_{m(n)}$ are proved; the graphs considered in the product graphs are either complete or complete multipartite graphs, which are dense. However, if we consider one of the graphs in $G \times H$, as circulant, with same number of odd jumps and even jumps, irrespective of the number of jumps, which can be paired so that the resulting set of edges induces connected 4-regular circulants, then $G \times H$ is hamiltonian decomposable. This proves that, with this additional condition on G or H , the product graph $G \times H$ is hamiltonian decomposable.

For example, consider the circulant graph $Circ(4r+2, L_i)$, where $L_i = \{2r+1, 2r, 2r-1, 2r-2, \dots, i, i-1\}$, $i \geq 2$ and H is any hamiltonian decomposable multigraph. Now, the hamiltonian decomposition of $Circ(4r+2, L_i) \times H$, $i \geq 2$, follows from our Theorem 1.2.

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