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# THE FRACTIONAL *K*-METRIC DIMENSION OF GRAPHS

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Let $G$ be a graph with vertex set $V(G)$. For any two distinct vertices $x$ and $y$ of $G$, let $R\{x, y\}$ denote the set of vertices $z$ such that the distance from $x$ to $z$ is not equal to the distance from $y$ to $z$ in $G$. For a function $g$ defined on $V(G)$ and for $U \subseteq V(G)$, let $g(U) = \sum_{s \in U} g(s)$. Let $\kappa(G) = \min\{|R\{x, y\}| : x \neq y \text{ and } x, y \in V(G)\}$. For any real number $k \in [1, \kappa(G)]$, a real-valued function $g : V(G) \to [0, 1]$ is a *$k$-resolving function* of $G$ if $g(R\{x, y\}) \geq k$ for any two distinct vertices $x, y \in V(G)$. The *fractional $k$-metric dimension*, $\dim_f^k(G)$, of $G$ is $\min\{g(V(G)) : g \text{ is a } k\text{-resolving function of } G\}$. In this paper, we initiate the study of the fractional $k$-metric dimension of graphs. For a connected graph $G$ and $k \in [1, \kappa(G)]$, it's easy to see that $k \leq \dim_f^k(G) \leq \frac{k|V(G)|}{\kappa(G)}$; we characterize graphs $G$ satisfying $\dim_f^k(G) = k$ and $\dim_f^k(G) = |V(G)|$, respectively. We show that $\dim_f^k(G) \geq k \dim_f(G)$ for any $k \in [1, \kappa(G)]$, and we give an example showing that $\dim_f^k(G) - k \dim_f(G)$ can be arbitrarily large for some $k \in (1, \kappa(G)]$; we also describe a condition for which $\dim_f^k(G) = k \dim_f(G)$ holds. We determine the fractional $k$-metric dimension for some classes of graphs, and conclude with two open problems, including whether $\phi(k) = \dim_f^k(G)$ is a continuous function of $k$ on every connected graph $G$.

## 1. Introduction

Let $G$ be a finite, simple, undirected, and connected graph with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, the *open neighborhood* of $v$ is $N(v) = \{u \in$

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$V(G) : uv \in E(G)\}$, and the *closed neighborhood* of $v$ is $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V(G)$, denoted by $\deg(v)$, is $|N(v)|$; a *leaf* is a vertex of degree one, and a *major vertex* is a vertex of degree at least three. The *distance* between two vertices $x, y \in V(G)$, denoted by $d(x, y)$, is the length of a shortest path between $x$ and $y$ in $G$. The *diameter*, $\text{diam}(G)$, of a graph $G$ is $\max\{d(x, y) : x, y \in V(G)\}$. The *complement* of $G$, denoted by $\overline{G}$, is the graph whose vertex set is $V(G)$ and $xy \in E(\overline{G})$ if and only if $xy \notin E(G)$ for $x, y \in V(G)$. We denote by $K_n$ and $P_n$ the complete graph and the path on $n$ vertices, respectively.

For two distinct vertices $x, y \in V(G)$, let $R\{x, y\} = \{z \in V(G) : d(x, z) \neq d(y, z)\}$. A subset $S \subseteq V(G)$ is called a *resolving set* of $G$ if $|S \cap R\{x, y\}| \geq 1$ for any two distinct vertices $x$ and $y$ in $G$. The *metric dimension*, $\dim(G)$, of $G$ is the minimum cardinality of $S$ over all resolving sets of $G$. Since metric dimension is suggestive of the dimension of a vector space in linear algebra, sometimes a minimum resolving set of $G$ is called a basis of $G$. The concept of metric dimension was introduced independently by Slater [**28**], and by Harary and Melter [**20**]. Applications of metric dimension can be found in network discovery and verification [**7**], robot navigation [**24**], sonar [**28**], combinatorial optimization [**27**], chemistry [**25**], and strategies for the mastermind game [**9**]. It was noted in [**19**] that determining the metric dimension of a graph is an NP-hard problem. Metric dimension has been extensively studied. For a survey on metric dimension in graphs, see [**4, 8**]. The effect of the deletion of a vertex or of an edge on the metric dimension of a graph was raised as a fundamental question in graph theory in [**8**]; the question is essentially settled in [**11**].

If a minimum number of requisite robots are installed in a network to identify the exact location of an intruder in the network, one malfunctioning robot can lead to failure of detection. Thus, it is natural to build a certain level of redundancy into the detection system. As a generalization of metric dimension, $k$-metric dimension was introduced first by Estrada-Moreno et al. [**12**] and, independently, by Adar and Epstein [**1**] soon afterwards. Let $\kappa(G) = \min\{|R\{x, y\}| : x \neq y \text{ and } x, y \in V(G)\}$. For a positive integer $k \in \{1, 2, \ldots, \kappa(G)\}$, a set $S \subseteq V(G)$ is called a *$k$-resolving set* of $G$ if $|S \cap R\{x, y\}| \geq k$ for any two distinct vertices $x$ and $y$ in $G$. The *$k$-metric dimension*, $\dim^k(G)$[1], of $G$ is the minimum cardinality over all $k$-resolving sets of $G$. It was shown in [**12**] that $k$-metric dimension of a connected graph $G$ exists for every $k \in \{1, 2, \ldots, \kappa(G)\}$, and $G$ is called $\kappa(G)$-metric dimensional. For an application of $k$-metric dimension to error-correcting codes, see [**5**]. For other articles on the $k$-metric dimension of graphs, see [**6, 13, 14**].

The fractionalization of various graph parameters has been extensively studied (see [**26**]). Currie and Oellermann [**10**] defined fractional metric dimension as the optimal solution to a linear programming problem, by relaxing a condition of the integer programming problem for metric dimension. A formulation of fractional metric dimension as a linear programming problem can be found in [**15**]. Arumugam and Mathew [**2**] officially studied the fractional metric dimension of graphs.

[1]In fact, the notation of this parameter has been $\dim_k(G)$ in previous works. However, we rather prefer to use $\dim^k(G)$ here, to facilitate the notation of $\dim_f^k(G)$.

For a function $g$ defined on $V(G)$ and for $U \subseteq V(G)$, let $g(U) = \sum_{s \in U} g(s)$. A real-valued function $g : V(G) \to [0,1]$ is a *resolving function* of $G$ if $g(R\{x,y\}) \geq 1$ for any two distinct vertices $x, y \in V(G)$. The *fractional metric dimension*, $\dim_f(G)$, of $G$ is $\min\{g(V(G)) : g$ is a resolving function of $G\}$. Notice that $\dim_f(G)$ reduces to $\dim(G)$, if the codomain of resolving functions is restricted to $\{0,1\}$. For more articles on the fractional metric dimension, as well as the closely related fractional strong metric dimension, of graphs, see [**3, 16, 17, 18, 21, 22, 23, 29**].

Now, we introduce fractional $k$-metric dimension, which can be viewed as a generalization of $\dim_f(G)$, as well as a fractionalization of $\dim^k(G)$. For any real number $k \in [1, \kappa(G)]$, a real-valued function $h : V(G) \to [0,1]$ is a *$k$-resolving function* of $G$ if $h(R\{x,y\}) \geq k$ for any distinct vertices $x, y \in V(G)$. The *fractional $k$-metric dimension*, $\dim_f^k(G)$, of $G$ is $\min\{h(V(G)) : h$ is a $k$-resolving function of $G\}$. Note $\dim_f^1(G) = \dim_f(G)$ and that $\dim_f^k(G)$ reduces to $\dim^k(G)$ when the codomain of $k$-resolving functions is restricted to $\{0,1\}$ and $k \in [1, \kappa(G)]$ is restricted to positive integers.

Next, we recall some results on the fractional metric dimension of graphs. One can easily see that, for any connected graph $G$ of order at least two, $1 \leq \dim_f(G) \leq \frac{|V(G)|}{2}$ (see [**2**]). For the characterization of graphs $G$ achieving the lower bound, see Theorem 3(a). Regarding the characterization of graphs $G$ achieving the upper bound, the following result is stated in [**2**] and a correct proof is provided in [**21**].

**Theorem 1.** [**2, 21**] *Let $G$ be a connected graph of order at least two. Then $\dim_f(G) = \frac{|V(G)|}{2}$ if and only if there exists a bijection $\alpha : V(G) \to V(G)$ such that $\alpha(v) \neq v$ and $|R\{v, \alpha(v)\}| = 2$ for all $v \in V(G)$.*

An explicit characterization of graphs $G$ satisfying $\dim_f(G) = \frac{|V(G)|}{2}$ is given in [**3**]. We recall the following construction from [**3**]. Let $\mathcal{K} = \{K_n : n \geq 2\}$ and $\overline{\mathcal{K}} = \{\overline{K}_n : n \geq 2\}$. Let $H[\mathcal{K} \cup \overline{\mathcal{K}}]$ be the family of graphs obtained from a connected graph $H$ by (i) replacing each vertex $u_i \in V(H)$ by a graph $H_i \in \mathcal{K} \cup \overline{\mathcal{K}}$, and (ii) each vertex in $H_i$ is adjacent to each vertex in $H_j$ if and only if $u_i u_j \in E(H)$.

**Theorem 2.** [**3**] *Let $G$ be a connected graph of order at least two. Then $\dim_f(G) = \frac{|V(G)|}{2}$ if and only if $G \in H[\mathcal{K} \cup \overline{\mathcal{K}}]$ for some connected graph $H$.*

Now, we recall the fractional metric dimension of some classes of graphs. We begin by recalling some terminologies. Fix a graph $G$. A leaf $u$ is called a *terminal vertex* of a major vertex $v$ if $d(u,v) < d(u,w)$ for every other major vertex $w$. The terminal degree, $ter_G(v)$, of a major vertex $v$ is the number of terminal vertices of $v$. A major vertex $v$ is an *exterior major vertex* if it has positive terminal degree. Let $ex(G)$ denote the number of exterior major vertices of $G$, $ex_a(G)$ the number of exterior major vertices $u$ with $ter_G(u) = a$, and $\sigma(G)$ the number of leaves of $G$.

**Theorem 3.**

*(a)* [**23**] *For any graph $G$ of order $n \geq 2$, $\dim_f(G) = 1$ if and only if $G \cong P_n$.*

*(b)* [**29**] *For a tree $T$, $\dim_f(T) = \frac{1}{2}(\sigma(T) - ex_1(T))$.*

*(c)* [**2**] *For the Petersen graph* $\mathcal{P}$, $\dim_f(\mathcal{P}) = \frac{5}{3}$.

*(d)* [**2**] *For an n-cycle* $C_n$, $\dim_f(C_n) = \begin{cases} \frac{n}{n-2} & \text{if } n \text{ is even,} \\ \frac{n}{n-1} & \text{if } n \text{ is odd.} \end{cases}$

*(e)* [**2**] *For the wheel graph* $W_n$ *of order* $n \geq 5$, $\dim_f(W_n) = \begin{cases} 2 & \text{if } n = 5, \\ \frac{3}{2} & \text{if } n = 6, \\ \frac{n-1}{4} & \text{if } n \geq 7. \end{cases}$

*(f)* [**23**] *If* $B_m$ *is a bouquet of* $m$ *cycles with a cut-vertex (i.e., the vertex sum of* $m$ *cycles at one common vertex), where* $m \geq 2$, *then* $\dim_f(B_m) = m$.

*(g)* [**29**] *For* $m \geq 2$, *let* $G = K_{a_1,a_2,\ldots,a_m}$ *be a complete m-partite graph of order* $n = \sum_{i=1}^{m} a_i$, *and let s be the number of partite sets of G consisting of exactly one element. Then*

$$\dim_f(G) = \begin{cases} \frac{n-1}{2} & \text{if } s = 1, \\ \frac{n}{2} & \text{otherwise.} \end{cases}$$

*(h)* [**2**] *For the grid graph* $G = P_s \square P_t$ $(s, t \geq 2)$, $\dim_f(G) = 2$.

In this paper, we initiate the study of the fractional $k$-metric dimension of graphs. For a connected graph $G$, let $\kappa(G) = \min\{|R\{x,y\}| : x \neq y \text{ and } x, y \in V(G)\}$. The paper is organized as follows. In section 2, we compare $\dim_f^k(G)$ with $\dim^k(G)$ for certain $k$. We prove that $\dim_f^k(G) \geq k \dim_f(G)$ for any $k \in [1, \kappa(G)]$; we describe a condition for which $\dim_f^k(G) = k \dim_f(G)$ holds for all $k \in [1, \kappa(G)]$. For $k \in [1, \kappa(G)]$, we show that $k \leq \dim_f^k(G) \leq \frac{k}{\kappa(G)}|V(G)|$, which implies $k \leq \dim_f^k(G) \leq |V(G)|$; we characterize graphs $G$ satisfying $\dim_f^k(G) = k$ and $\dim_f^k(G) = |V(G)|$, respectively. We also show an example such that two non-isomorphic graphs $H_1$ and $H_2$ satisfy $\dim_f^k(H_1) = \dim_f^k(H_2)$ for all $k \in [1, \kappa]$, where $\kappa(H_1) = \kappa(H_2) = \kappa$. In section 3, for $k \in [1, \kappa(G)]$, we determine the fractional $k$-metric dimension of trees, cycles, wheel graphs, the Petersen graph, a bouquet of cycles (i.e., the vertex sum of cycles at one common vertex), complete multi-partite graphs, and grid graphs (i.e., the Cartesian product of two paths). Along the way, we give an example showing that $\dim_f^k(G) - k \dim_f(G)$ can be arbitrarily large for some $k \in (1, \kappa(G)]$. We conclude with some open problems.

## 2. Some general results on fractional $k$-metric dimension

We begin with some observations. Two distinct vertices $x, y \in V(G)$ are called *twin vertices* if $N(x) - \{y\} = N(y) - \{x\}$.

**Observation 4.** *Let G be a connected graph and let* $k \in [1, \kappa(G)]$. *If two distinct vertices x and y are twin vertices in G, then* $R\{x, y\} = \{x, y\}$, *and thus* $\kappa(G) = 2$ *and* $g(x) + g(y) \geq k$ *for any k-resolving function g of G.*

**Observation 5.** *Let G be a connected graph.*

*(a)* *For* $k \in [1, \kappa(G)]$, $\dim_f(G) \leq \dim_f^k(G)$.

*(b)* *For* $k \in \{1, 2, \ldots, \kappa(G)\}$, $\dim_f^k(G) \le \dim^k(G)$.

*(c)* [**2, 12**] *For* $k \in \{1, 2, \ldots, \kappa(G)\}$, $\dim_f(G) \le \dim(G) \le \dim^k(G)$.

Observation 5 provides inequalities between any two graph parameters among $\dim_f(G)$, $\dim(G)$, $\dim_f^k(G)$ for $k \in [1, \kappa(G)]$, and $\dim^k(G)$ for $k \in \{1, 2, \ldots, \kappa(G)\}$, excluding the relation between $\dim(G)$ and $\dim_f^k(G)$. So, it is natural to compare $\dim(G)$ and $\dim_f^k(G)$ for $k \in [1, \kappa(G)]$.

**Remark 6.** *(a) The value of* $\dim_f^k(G) - \dim(G)$ *can be arbitrarily large, as G varies, for some* $k \in (1, \kappa(G)]$. *For* $n \ge 4$, *note that* $\dim(P_n) = 1 \le \dim_f^k(P_n)$ *for* $k \in [1, n-1]$ *and* $\dim_f^{n-1}(P_n) = n$ *(see Proposition 15). Thus,* $\dim_f^{n-1}(P_n) - \dim(P_n) = n - 1$ *can be arbitrarily large.*

*(b) The value of* $\dim(G) - \dim_f^k(G)$ *can be arbitrarily large, as G varies, for some* $k \in [1, \kappa(G)]$. *For* $n \ge 3$, *note that* $\dim(K_n) = n - 1$ *and* $\dim_f^k(K_n) = \frac{kn}{2}$ *for* $k \in [1, 2]$ *(see Proposition 26). Now, let* $k \in [1, 2)$; *then* $\dim(K_n) - \dim_f^k(K_n) = \frac{(2-k)n}{2} - 1$ *becomes arbitrarily large as* $n \to \infty$.

In light of Observation 5(b), we have the following

**Theorem 7.** *The value of* $\dim^k(G) - \dim_f^k(G)$ *can be arbitrarily large, as G varies, for some* $k \in \{1, 2, \ldots, \kappa(G)\}$.

*Proof.* Let $H$ be a connected graph with vertex set $V(H) = \{u_1, u_2, \ldots, u_n\}$, where $n \ge 2$. Let $G$ be the graph obtained from $H$ as follows:

(i) for each $i \in \{1, 2, \ldots, n\}$, add three vertices $a_{i,1}, b_{i,1}, c_{i,1}$ and three edges $u_i a_{i,1}, u_i b_{i,1}, u_i c_{i,1}$;

(ii) for each $i \in \{1, 2, \ldots, n\}$, subdivide the edge $u_i a_{i,1}$ ($u_i b_{i,1}$ and $u_i c_{i,1}$, respectively) exactly $s - 1$ times so that the edge $u_i a_{i,1}$ ($u_i b_{i,1}$ and $u_i c_{i,1}$, respectively) in (i) becomes the $u_i - a_{i,1}$ path given by $u_i, a_{i,s}, a_{i,s-1}, \ldots, a_{i,1}$ (the $u_i - b_{i,1}$ path given by $u_i, b_{i,s}, b_{i,s-1}, \ldots, b_{i,1}$ and the $u_i - c_{i,1}$ path given by $u_i, c_{i,s}, c_{i,s-1}, \ldots, c_{i,1}$, respectively).

For each $i \in \{1, 2, \ldots, n\}$, let $T_i$ be the subtree of $G$ consisting of the $u_i - a_{i,1}$ path, the $u_i - b_{i,1}$ path, and the $u_i - c_{i,1}$ path; further, let $P^{i,a}$ be the $a_{i,s} - a_{i,1}$ path, $P^{i,b}$ the $b_{i,s} - b_{i,1}$ path, and $P^{i,c}$ the $c_{i,s} - c_{i,1}$ path. Then, for each $i \in \{1, 2, \ldots, n\}$,

$$(1) \qquad \begin{cases} R\{a_{i,s}, b_{i,s}\} = V(P^{i,a}) \cup V(P^{i,b}), \\ R\{a_{i,s}, c_{i,s}\} = V(P^{i,a}) \cup V(P^{i,c}), \\ R\{b_{i,s}, c_{i,s}\} = V(P^{i,b}) \cup V(P^{i,c}). \end{cases}$$

We determine $\kappa(G)$, $\dim^k(G)$ for $k \in \{1, 2, \ldots, \kappa(G)\}$, and $\dim_f^k(G)$ for $k \in [1, \kappa(G)]$.

**Claim 1:** $\kappa(G) = 2s$**.**

Proof of Claim 1. Let $x$ and $y$ be two distinct vertices of $G$. First, let $x, y \in V(T_i)$ for some $i \in \{1, 2, \ldots, n\}$. If $d(u_i, x) \ne d(u_i, y)$, then $R\{x, y\} \supseteq V(T_j)$ with $|R\{x, y\}| \ge$

$|V(T_j)| = 3s+1$, where $j \in \{1,2,\ldots,n\}-\{i\}$. If $d(u_i,x) = d(u_i,y)$, say $x \in V(P^{i,a})$ and $y \in V(P^{i,b})$ without loss of generality, then $R\{x,y\} = V(P^{i,a}) \cup V(P^{i,b})$ with $|R\{x,y\}| = |V(P^{i,a})| + |V(P^{i,b})| = 2s$. Second, let $x \in V(T_i)$ and $y \in V(T_j)$ for distinct $i,j \in \{1,2,\ldots,n\}$, say $x$ lies on the $u_i - a_{i,1}$ path and $y$ lies on the $u_j - a_{j,1}$ path, without loss of generality. Then $|R\{x,y\}| \geq 2s$, since at most one vertex lying on a $a_{i,1} - a_{j,1}$ geodesic is at equal distance from both $x$ and $y$. So, $\kappa(G) = 2s$. □

**Claim 2:** For $k \in \{1,2,\ldots,2s\}$, $\dim^k(G) = \begin{cases} \frac{3kn}{2} & \text{if } k \text{ is even,} \\ \frac{(3k+1)n}{2} & \text{if } k \text{ is odd.} \end{cases}$

Proof of Claim 2. Let $k \in \{1,2,\ldots,2s\}$. First, we show that $\dim^k(G) \geq \frac{3kn}{2}$ for an even $k$, and $\dim^k(G) \geq \frac{(3k+1)n}{2}$ for an odd $k$. Let $S$ be a minimum $k$-resolving set of $G$. For each $i \in \{1,2,\ldots,n\}$, (1) implies

$$(2)\qquad \begin{cases} |S \cap R\{a_{i,s}, b_{i,s}\}| = |S \cap V(P^{i,a})| + |S \cap V(P^{i,b})| \geq k, \\ |S \cap R\{a_{i,s}, c_{i,s}\}| = |S \cap V(P^{i,a})| + |S \cap V(P^{i,c})| \geq k, \\ |S \cap R\{b_{i,s}, c_{i,s}\}| = |S \cap V(P^{i,b})| + |S \cap V(P^{i,c})| \geq k. \end{cases}$$

Suppose $k$ is even. By summing over the three inequalities in (2), we obtain $|S \cap V(P^{i,a})| + |S \cap V(P^{i,b})| + |S \cap V(P^{i,c})| \geq \frac{3k}{2}$, and thus $|S \cap V(T_i)| \geq \frac{3k}{2}$ for each $i \in \{1,2,\ldots,n\}$. So, $|S \cap V(G)| \geq \sum_{i=1}^{n} \frac{3k}{2} = \frac{3kn}{2}$, and hence $\dim^k(G) \geq \frac{3kn}{2}$. Now, suppose $k$ is odd. If $|S \cap V(P^{i,a})| \leq \lfloor \frac{k}{2} \rfloor$ and $|S \cap V(P^{i,b})| \leq \lfloor \frac{k}{2} \rfloor$ for some $i \in \{1,2,\ldots,n\}$, then $|S \cap R\{a_{i,s}, b_{i,s}\}| = |S \cap V(P^{i,a})| + |S \cap V(P^{i,b})| \leq \lfloor \frac{k}{2} \rfloor + \lfloor \frac{k}{2} \rfloor = k-1$, contradicting the assumption that $S$ is a $k$-resolving set of $G$; thus, $|S \cap V(P^{i,t})| \leq \lfloor \frac{k}{2} \rfloor$ for at most one $t \in \{a,b,c\}$ for each $i \in \{1,2,\ldots,n\}$. Let $|S \cap V(P^{i,c})| = \alpha \leq \min\{|S \cap V(P^{i,a})|, |S \cap V(P^{i,b})|\}$ for $i \in \{1,2,\ldots,n\}$. Then $|S \cap V(P^{i,a})| \geq k - \alpha$ and $|S \cap V(P^{i,b})| \geq k - \alpha$ from (2), and thus $|S \cap V(T_i)| \geq 2k - \alpha \geq 2k - \lfloor \frac{k}{2} \rfloor = \frac{3k+1}{2}$. So, $|S \cap V(G)| \geq \sum_{i=1}^{n} \frac{3k+1}{2} = \frac{(3k+1)n}{2}$, and hence $\dim^k(G) \geq \frac{(3k+1)n}{2}$.

Second, we show that $\dim^k(G) \leq \frac{3kn}{2}$ for an even $k$, and $\dim^k(G) \leq \frac{(3k+1)n}{2}$ for an odd $k$. If $k$ is even, let $W_0 = \cup_{i=1}^{n}(\{a_{i,1}, a_{i,2}, \ldots, a_{i,\frac{k}{2}}\} \cup \{b_{i,1}, b_{i,2}, \ldots, b_{i,\frac{k}{2}}\} \cup \{c_{i,1}, c_{i,2}, \ldots, c_{i,\frac{k}{2}}\})$. If $k$ is odd, let $W_1 = \cup_{i=1}^{n}(\{a_{i,1}, \ldots, a_{i,\lceil \frac{k}{2} \rceil}\} \cup \{b_{i,1}, \ldots, b_{i,\lceil \frac{k}{2} \rceil}\} \cup \{c_{i,1}, \ldots, c_{i,\lfloor \frac{k}{2} \rfloor}\})$. Note that $|W_0| = \frac{3kn}{2}$, $|W_1| = \frac{(3k+1)n}{2}$, $|W_0 \cap V(P^{i,a})| = |W_0 \cap V(P^{i,b})| = |W_0 \cap V(P^{i,c})| = \frac{k}{2}$ and $|W_1 \cap V(P^{i,a})| = |W_1 \cap V(P^{i,b})| = \lceil \frac{k}{2} \rceil = 1 + |W_1 \cap V(P^{i,c})|$ for $i \in \{1,2,\ldots,n\}$. It suffices to show that $W_0$ ($W_1$, respectively) is a $k$-resolving set of $G$ when $k$ is even (odd, respectively). Let $x$ and $y$ be distinct vertices of $G$. Suppose $x, y \in V(T_i)$ for some $i \in \{1,2,\ldots,n\}$. If $d(u_i,x) \neq d(u_i,y)$, then $|W_\ell \cap R\{x,y\}| \geq |V(T_j)| \geq \frac{3k}{2} \geq k$ for $j \neq i$ and for $\ell \in \{0,1\}$. If $d(u_i,x) = d(u_i,y)$, say $x \in V(P^{i,a})$ and $y \in V(P^{i,c})$ (other cases can be handled similarly), then $|W_\ell \cap R\{x,y\}| = |W_\ell \cap (V(P^{i,a}) \cup V(P^{i,c}))| \geq k$ for $\ell \in \{0,1\}$. Now, let $x \in V(T_i)$ and $y \in V(T_j)$ for distinct $i,j \in \{1,2\ldots,n\}$; suppose $d(u_i,x) \leq d(u_j,y)$, without loss of generality. Since $R\{x,y\} \supseteq V(T_i)$, $|W_\ell \cap R\{x,y\}| \geq |W_\ell \cap V(T_i)| \geq \frac{3k}{2}$ for $\ell \in \{0,1\}$. So, $W_0$ ($W_1$, respectively) is a $k$-resolving set of $G$ when $k$ is even (odd, respectively). □

**Claim 3:** For $k \in [1, 2s]$, $\dim_f^k(G) = \frac{3kn}{2}$.

Proof of Claim 3. Let $k \in [1, 2s]$. First, we show that $\dim_f^k(G) \geq \frac{3kn}{2}$. Let $g : V(G) \to [0, 1]$ be any $k$-resolving function of $G$. From (1), for each $i \in \{1, 2, \ldots, n\}$, we have $g(R\{a_{i,s}, b_{i,s}\}) = g(V(P^{i,a})) + g(V(P^{i,b})) \geq k$, $g(R\{a_{i,s}, c_{i,s}\}) = g(V(P^{i,a})) + g(V(P^{i,c})) \geq k$, and $g(R\{b_{i,s}, c_{i,s}\}) = g(V(P^{i,b})) + g(V(P^{i,c})) \geq k$. By summing over the three inequalities, we obtain $g(V(P^{i,a})) + g(V(P^{i,b})) + g(V(P^{i,c})) \geq \frac{3k}{2}$, and thus $g(V(T_i)) \geq \frac{3k}{2}$ for each $i \in \{1, 2, \ldots, n\}$. So, $g(V(G)) \geq \sum_{i=1}^{n} \frac{3k}{2} = \frac{3kn}{2}$, and hence $\dim_f^k(G) \geq \frac{3kn}{2}$.

Second, we show that $\dim_f^k(G) \leq \frac{3kn}{2}$. Let $h : V(G) \to [0, 1]$ be a function defined by

$$h(v) = \begin{cases} 0 & \text{if } \deg(v) \geq 3, \\ \frac{k}{2s} & \text{if } \deg(v) \leq 2. \end{cases}$$

Notice that $h(V(P^{i,a})) = h(V(P^{i,b})) = h(V(P^{i,c})) = \frac{k}{2}$ for each $i \in \{1, 2, \ldots, n\}$, and $h(V(G)) = \frac{3kn}{2}$. It suffices to show that $h$ is a $k$-resolving function of $G$. Let $x$ and $y$ be distinct vertices of $G$. Suppose $x, y \in V(T_i)$ for some $i \in \{1, 2, \ldots, n\}$. If $d(u_i, x) \neq d(u_i, y)$, then $h(R\{x, y\}) \geq h(V(T_j)) \geq \frac{3k}{2}$ for $j \neq i$. If $d(u_i, x) = d(u_i, y)$, say $x \in V(P^{i,a})$ and $y \in V(P^{i,b})$ without loss of generality, then $h(R\{x, y\}) = h(V(P^{i,a})) + h(V(P^{i,b})) = k$. Now, let $x \in V(T_i)$ and $y \in V(T_j)$ for distinct $i, j \in \{1, 2 \ldots, n\}$; suppose $d(u_i, x) \leq d(u_j, y)$, without loss of generality. Then $h(R\{x, y\}) \geq h(V(T_i)) \geq \frac{3k}{2}$. So, $h$ is a $k$-resolving function of $G$. □

By Claims 2 and 3, we see that, for each odd $k \in \{1, 2, \ldots, 2s\}$, $\dim^k(G) - \dim_f^k(G) = \frac{(3k+1)n}{2} - \frac{3kn}{2} = \frac{n}{2}$, which can be arbitrarily large. □

Next, we compare the fractional metric dimension and the fractional $k$-metric dimension of graphs.

**Lemma 8.** *For any connected graph $G$ and for any $k \in [1, \kappa(G)]$, $\dim_f^k(G) \geq k \dim_f(G)$.*

*Proof.* Let $g : V(G) \to [0, 1]$ be a minimum $k$-resolving function of $G$. Then $g(R\{x, y\}) \geq k$ for any two distinct vertices $x, y \in V(G)$. Now, let $h : V(G) \to [0, 1]$ be a function defined by $h(u) = \frac{1}{k} g(u)$ for each $u \in V(G)$. Then $h(R\{x, y\}) = \frac{1}{k} g(R\{x, y\}) \geq 1$ for any distinct vertices $x, y \in V(G)$; thus, $h$ is a resolving function of $G$. So, $h(V(G)) = \frac{1}{k} \dim_f^k(G) \geq \dim_f(G)$, i.e., $\dim_f^k(G) \geq k \dim_f(G)$. □

Next, we examine the conditions for which $\dim_f^k(G) = k \dim_f(G)$ holds.

**Lemma 9.** *Let $G$ be a connected graph and let $k \in [1, \kappa(G)]$. If there exists a minimum resolving function $g : V(G) \to [0, 1]$ such that $g(v) \leq \frac{1}{k}$ for each $v \in V(G)$, then $\dim_f^k(G) = k \dim_f(G)$ for any $k \in [1, \kappa(G)]$.*

*Proof.* Let $g : V(G) \to [0, 1]$ be a minimum resolving function of $G$ satisfying $g(v) \leq \frac{1}{k}$ for each $v \in V(G)$. Let $h : V(G) \to [0, 1]$ be a function defined by $h(v) = kg(v)$ for each $v \in V(G)$. Then $h$ is a $k$-resolving function of $G$: (i) for each $v \in V(G)$, $0 \le h(v) = kg(v) \le 1$; (ii) for any two distinct $x, y \in V(G)$, $h(R\{x,y\}) = kg(R\{x,y\}) \ge k$, since $g(R\{x,y\}) \ge 1$ by the assumption that $g$ is a resolving function of $G$. So, $\dim_f^k(G) \le h(V(G)) = kg(V(G)) = k\dim_f(G)$. Since $\dim_f^k(G) \ge k\dim_f(G)$ by Lemma 8, $\dim_f^k(G) = k\dim_f(G)$. □

Next, we obtain the lower and upper bounds of $\dim_f^k(G)$ in terms of $k$, $\kappa(G)$, and the order of $G$.

**Proposition 10.** *Let $G$ be a connected graph of order $n$. For any $k \in [1, \kappa(G)]$, $k \le \dim_f^k(G) \le \dfrac{kn}{\kappa(G)}$, where both bounds are sharp.*

*Proof.* The lower bound is trivial. For the upper bound, let $g : V(G) \to [0,1]$ be a function such that $g(v) = \frac{k}{\kappa(G)}$ for each $v \in V(G)$. Since $\kappa(G) = \min\{|R\{x,y\}| : x \ne y \text{ and } x, y \in V(G)\}$, $g(R\{u,v\}) \ge k$ for any distinct vertices $u, v \in V(G)$. So, $g$ is a $k$-resolving function of $G$, and hence $\dim_f^k(G) \le g(V(G)) = \sum_{i=1}^{n} \frac{k}{\kappa(G)} = \frac{kn}{\kappa(G)}$.

For the sharpness of the lower bound, see Proposition 12(a); for the sharpness of the upper bound, we refer to Proposition 21. □

As an immediate consequence of Proposition 10, we have the following.

**Corollary 11.** *For a connected graph $G$ of order $n$ and for $k \in [1, \kappa(G)]$, $k \le \dim_f^k(G) \le n$.*

Next, we characterize graphs $G$ achieving the lower bound and the upper bound, respectively, of Corollary 11. Let $\mathcal{R}_\kappa(G) = \displaystyle\bigcup_{x,y \in V(G), x \ne y, |R\{x,y\}| = \kappa} R\{x,y\}$, where $\kappa = \kappa(G)$.

**Proposition 12.** *For a connected graph $G$ of order $n \ge 2$ and for $k \in [1, \kappa(G)]$,*

*(a)* $\dim_f^k(G) = k$ *if and only if* $G \cong P_n$ *and* $k \in [1,2]$,

*(b)* $\dim_f^k(G) = n$ *if and only if* $k = \kappa(G) = \kappa$ *and* $V(G) = \mathcal{R}_\kappa(G)$.

*Proof.* **(a)** ($\Leftarrow$) If $G \cong P_n$ and $k \in [1,2]$, then $\dim_f^k(G) = k$ by Proposition 15.

($\Rightarrow$) Suppose that $\dim_f^k(G) = k$. Since $\dim_f^k(G) \ge k\dim_f(G) \ge k$ by Lemma 8, $\dim_f^k(G) = k$ implies $\dim_f(G) = 1$; thus $G \cong P_n$ by Theorem 3(a) and $k \in [1,2]$ from Proposition 15.

**(b)** ($\Leftarrow$) Let $k = \kappa(G) = \kappa$ and $V(G) = \mathcal{R}_\kappa(G)$. Then, for any vertex $v \in V(G)$, there exist two distinct vertices $x, y \in V(G)$ such that $v \in R\{x,y\}$ with $|R\{x,y\}| = \kappa$. Since any $\kappa$-resolving function $g$ of $G$ must satisfy $g(R\{x,y\}) \ge \kappa$ and $g(v) \le 1$ for each $v \in V(G)$, $g(u) = 1$ for each $u \in R\{x,y\}$ with $|R\{x,y\}| = \kappa$. Since $V(G) = \mathcal{R}_\kappa(G)$, $g(v) = 1$ for each $v \in V(G)$. Thus, $g(V(G)) = n$ and hence $\dim_f^k(G) = n$.

($\Rightarrow$) Let $\dim_f^k(G) = n$. By Proposition 10, $k = \kappa$. Suppose that $\mathcal{R}_\kappa(G) \subsetneq V(G)$ and let $w \in V(G) - \mathcal{R}_\kappa(G)$. Then, for any vertex $w' \in V(G) - \{w\}$, $|R\{w, w'\}| \geq \kappa(G) + 1$. If $h : V(G) \to [0,1]$ is a function defined by $h(w) = 0$ and $h(v) = 1$ for each $v \in V(G) - \{w\}$, then $h$ is a $\kappa$-resolving function of $G$ with $h(V(G)) = n - 1$, which contradicts the assumption that $\dim_f^k(G) = n$. □

**Remark 13.** *Let $G \in H[\mathcal{K} \cup \overline{\mathcal{K}}]$ for some connected graph $H$, as described in Theorem 2. Then $\kappa(G) = 2$ and $V(G) = \mathcal{R}_\kappa(G)$; thus $\dim_f^2(G) = |V(G)|$ by Proposition 12(b). More generally, $\dim_f^k(G) = k \dim_f(G) = \frac{k}{2}|V(G)|$ for $k \in [1,2]$ by Lemma 9, since $g : V(G) \to [0,1]$ defined by $g(u) = \frac{1}{2}$, for each $u \in V(G)$, forms a minimum resolving function of $G$.*

We conclude this section with an example showing that two non-isomorphic graphs can have the same $\kappa$ and identical $k$-fractional metric dimension for all $k \in [1, \kappa]$.

**Remark 14.** *There exist non-isomorphic graphs $H_1$ and $H_2$ such that $\dim_f^k(H_1) = \dim_f^k(H_2)$ for all $k \in [1, \kappa]$, where $\kappa = \kappa(H_1) = \kappa(H_2)$. For example, let $H_1 \cong K_{2,2,3}$ and $H_2 \cong K_{3,4}$; then $\dim_f(H_1) = \dim_f(H_2) = \frac{7}{2}$ by Theorem 3(g). Since both $H_1$ and $H_2$ have twin vertices, $\kappa(H_1) = \kappa(H_2) = 2$. Also note that a function $g_i : V(H_i) \to [0,1]$ defined by $g_i(u) = \frac{1}{2}$ for each $u \in V(H_i)$ forms a minimum resolving function for $H_i$, where $i \in \{1,2\}$. By Lemma 9, $\dim_f^k(H_1) = \dim_f^k(H_2) = \frac{7}{2}k$ for every $k \in [1, \kappa]$, whereas $H_1 \not\cong H_2$.*

## 3. The fractional $k$-metric dimension of some graphs

In this section, we determine $\dim_f^k(G)$ for $k \in [1, \kappa(G)]$ when $G$ is a tree, a cycle, a wheel graph, the Petersen graph, a bouquet of cycles, a complete multi-partite graph, or a grid graph (the Cartesian product of two paths). Along the way, we provide an example showing that $\dim_f^k(G) - k \dim_f(G)$ can be arbitrarily large for some $k \in (1, \kappa(G)]$. First, we determine $\dim_f^k(G)$ when $G$ is a path.

**Proposition 15.** *Let $P_n$ be an $n$-path, where $n \geq 2$. Then $\dim_f^k(P_2) = k$ for $k \in [1,2]$ and, for $n \geq 3$,*

$$\dim_f^k(P_n) = \begin{cases} k & \text{if } k \in [1,2], \\ 2 + (k-2)\frac{n-2}{n-3} & \text{if } k \in (2, n-1]. \end{cases}$$

*Proof.* Let $P_n$ be an $n$-path given by $u_1, u_2, \ldots, u_n$, where $n \geq 2$; then $\kappa(P_2) = 2$ and $\kappa(P_n) = n - 1$ for $n \geq 3$. Since a function $h$ defined on $V(P_2)$ by $h(u_1) = h(u_2) = \frac{1}{2}$ is a minimum resolving function of $P_2$, $\dim_f^k(P_2) = k \dim_f(P_2) = k$ for $k \in [1,2]$ by Theorem 3(a) and Lemma 9. So, let $n \geq 3$ and we consider two cases.

**Case 1:** $k \in [1,2]$**.** If $g : V(P_n) \to [0,1]$ is a function defined by $g(u_1) = g(u_n) = \frac{1}{2}$ and $g(u_i) = 0$ for each $i \in \{2, \ldots, n-1\}$, then $g$ is a minimum resolving function of $P_n$: (i) for any two distinct vertices $x, y \in V(P_n)$, $R\{x,y\} \supseteq \{u_1, u_n\}$, and hence $g(R\{x,y\}) \geq g(u_1)+g(u_n) = \frac{1}{2}+\frac{1}{2} = 1$; (ii) $g(V(P_n)) = 1 = \dim_f(P_n)$ by Theorem 3(a). Since $g(u_i) \leq \frac{1}{2} \leq \frac{1}{k}$ for each $i \in \{1, 2, \ldots, n\}$, we have $\dim_f^k(P_n) = k\dim_f(P_n) = k$ for any $k \in [1, 2]$ by Lemma 9 and Theorem 3(a).

**Case 2:** $k \in (2, n-1]$. Note that $n \geq 4$ in this case since $\kappa(P_3) = 2$. Let $h : V(P_n) \to [0, 1]$ be a $k$-resolving function of $P_n$. Let $h(u_1) + h(u_n) = b$; then $0 \leq b \leq 2$. Since $R\{u_i, u_{i+2}\} = V(P_n) - \{u_{i+1}\}$ for $i \in \{1, 2, \ldots, n-2\}$, $h(R\{u_i, u_{i+2}\}) = h(V(P_n))-h(u_{i+1}) \geq k$ for each $i \in \{1, 2, \ldots, n-2\}$. By summing over the $(n-2)$ inequalities, we have $(n-2)h(V(P_n)) - \sum_{j=2}^{n-1} h(u_j) \geq (n-2)k$, i.e., $(n-3)h(V(P_n)) + h(u_1) + h(u_n) \geq (n-2)k$. So, $h(V(P_n)) \geq \frac{(n-2)k-b}{n-3}$ since $n > 3$; note that the minimum of $h(V(P_n))$ is $\frac{(n-2)k-2}{n-3}$ when $b = h(u_1) + h(u_n)$ takes the maximum value 2. Thus, $\dim_f^k(P_n) \geq \frac{(n-2)k-2}{n-3} = 2 + (k-2)\frac{n-2}{n-3}$.

Now, let $g : V(P_n) \to [0, 1]$ be a function defined by

$$g(u_i) = \begin{cases} 1 & \text{if } i \in \{1, n\}, \\ \frac{k-2}{n-3} & \text{otherwise.} \end{cases}$$

Then $g$ is a $k$-resolving function of $P_n$: (i) $0 < \frac{k-2}{n-3} \leq 1$ for any $k \in (2, n-1]$; (ii) for any distinct $i, j \in \{1, 2, \ldots, n\}$, $g(R\{u_i, u_j\}) \geq 2+(n-3)\frac{k-2}{n-3} = k$ since $R\{u_i, u_j\} \supseteq \{u_1, u_n\}$ and $|R\{u_i, u_j\}| \geq n-1$. So, $\dim_f^k(P_n) \leq g(V(P_n)) = 2 + (n-2)\frac{k-2}{n-3}$.

Therefore, $\dim_f^k(P_n) = 2 + (k-2)\frac{n-2}{n-3}$ for $k \in (2, n-1]$. □

Second, we determine $\dim_f^k(T)$ when $T$ is a tree that is not a path. Let $M(T)$ be the set of exterior major vertices of a tree $T$. Let $M_1(T) = \{w \in M(T) : ter_T(w) = 1\}$, $M_2(T) = \{w \in M(T) : ter_T(w) = 2\}$, and $M_3(T) = \{w \in M(T) : ter_T(w) \geq 3\}$; note that $M(T) = M_1(T) \cup M_2(T) \cup M_3(T)$. For any vertex $v \in M(T)$, let $T_v$ be the subtree of $T$ induced by $v$ and all vertices belonging to the paths joining $v$ with its terminal vertices. Let $M^*(T) = M_2(T) \cup M_3(T)$. We recall the following result on $\kappa(T)$.

**Theorem 16.** [**12**] *For a tree $T$ that is not a path,*

$$\kappa(T) = \min \bigcup_{w \in M^*(T)} \{d(\ell_i, \ell_j) : \ell_i \text{ and } \ell_j \text{ are two distinct terminal vertices of } w\}.$$

We begin by examining $\dim_f^k(T)$ when $T$ is a tree with exactly one exterior major vertex.

**Proposition 17.** *Let $T$ be a tree with $ex(T) = 1$. Let $v$ be the exterior major vertex of $T$ and let $\ell_1, \ell_2, \ldots, \ell_a$ be the terminal vertices of $v$ in $T$ (note that $a \geq 3$). Suppose that $d(v, \ell_1) \leq d(v, \ell_2) \leq \ldots \leq d(v, \ell_a)$. Then $\kappa(T) = d(\ell_1, \ell_2)$, and*

*(a) if $d(v, \ell_1) = d(v, \ell_2)$, then $\dim_f^k(T) = k\dim_f(T) = \frac{ka}{2}$ for $k \in [1, \kappa(T)]$;*

*(b) if $d(v,\ell_1) < d(v,\ell_2)$, then*

$$\dim_f^k(T) = \begin{cases} \frac{ka}{2} & \text{for } k \in [1, 2d(v,\ell_1)], \\ (a-1)k - (a-2)d(v,\ell_1) & \text{for } k \in (2d(v,\ell_1), \kappa(T)]. \end{cases}$$

*Proof.* For each $i \in \{1,2,\ldots,a\}$, let $s_i$ be the neighbor of $v$ lying on the $v-\ell_i$ path, and let $P^i$ denote the $s_i - \ell_i$ path in $T$. By Theorem 16, $\kappa(T) = d(\ell_1,\ell_2)$. Let $k \in [1,\kappa(T)]$.

**(a)** Let $d(v,\ell_1) = d(v,\ell_2)$. By Lemma 8, $\dim_f^k(T) \geq k\dim_f(T)$. We will show that $\dim_f^k(T) \leq k\dim_f(T)$. Let $g : V(T) \to [0,1]$ be a function defined by

$$(3) \qquad g(u) = \begin{cases} 0 & \text{if } u = v, \\ \frac{k}{2d(v,\ell_i)} & \text{for each vertex } u \in V(P^i), \text{ where } i \in \{1,2,\ldots,a\}. \end{cases}$$

Note that, for each $i \in \{1,2,\ldots,a\}$, (i) $g(V(P^i)) = \frac{k}{2}$; (ii) $0 \leq \frac{k}{2d(v,\ell_i)} \leq 1$ since $k \leq \kappa(T) = d(\ell_1,\ell_2) = 2d(v,\ell_1) \leq 2d(v,\ell_i)$. If two distinct vertices $x$ and $y$ lie on the $v-\ell_i$ path for some $i \in \{1,2,\ldots,a\}$, then $g(R\{x,y\}) \geq g(V(T)-V(P^i)) \geq k$ since $a \geq 3$. For distinct $i,j \in \{1,2,\ldots,a\}$, if $x \in V(P^i)$ and $y \in V(P^j)$ with $d(v,x) \neq d(v,y)$, say $d(v,x) < d(v,y)$ without loss of generality, then $g(R\{x,y\}) \geq g(V(T)-V(P^j)) \geq k$; if $x \in V(P^i)$ and $y \in V(P^j)$ with $d(v,x) = d(v,y)$, then $g(R\{x,y\}) = g(V(P^i) \cup V(P^j)) = k$. So, $g$ is a $k$-resolving function of $T$ with $g(V(T)) = \frac{ka}{2}$. Thus $\dim_f^k(T) \leq \frac{ka}{2} = k\dim_f(T)$ by Theorem 3(b). Therefore, $\dim_f^k(T) = k\dim_f(T) = \frac{ka}{2}$ for $k \in [1,\kappa(T)]$.

**(b)** Let $d(v,\ell_1) < d(v,\ell_2)$, and we consider two cases.

**Case 1:** $k \in [1, 2d(v,\ell_1)]$. In this case, the function $g$ in (3) is a $k$-resolving function of $T$ as shown in the proof for (a); thus, $\dim_f^k(T) \leq g(V(T)) = \frac{ka}{2} = k\dim_f(T)$. Since $\dim_f^k(T) \geq k\dim_f(T)$ by Lemma 8, $\dim_f^k(T) = k\dim_f(T) = \frac{ka}{2}$.

**Case 2:** $k \in (2d(v,\ell_1), \kappa(T)]$. Let $h : V(T) \to [0,1]$ be a minimum $k$-resolving function of $T$. Note that (i) $h(V(P^1)) \leq d(v,\ell_1)$ since $h(u) \leq 1$ for each $u \in V(T)$; (ii) for distinct $i,j \in \{1,2,\ldots,a\}$, $h(V(P^i)) + h(V(P^j)) \geq k$ since $R\{s_i,s_j\} = V(P^i) \cup V(P^j)$. Let $h(V(P^1)) = \beta$. From $h(V(P^1)) + h(V(P^j)) \geq k$ for each $j \in \{2,3,\ldots,a\}$, $h(V(P^j)) \geq k-\beta$. So, $h(V(T)) \geq h(V(P^1)) + \sum_{i=2}^{a} h(V(P^i)) \geq \beta + (a-1)(k-\beta) = (a-1)k-(a-2)\beta \geq (a-1)k-(a-2)d(v,\ell_1)$ since $\beta \leq d(v,\ell_1)$; thus $\dim_f^k(T) \geq (a-1)k-(a-2)d(v,\ell_1)$.

Next, we show that $\dim_f^k(T) \leq (a-1)k-(a-2)d(v,\ell_1)$. Let $g : V(T) \to [0,1]$ be a function defined by

$$g(u) = \begin{cases} 0 & \text{if } u = v, \\ 1 & \text{for each vertex } u \in V(P^1), \\ \frac{k-d(v,\ell_1)}{d(v,\ell_j)} & \text{for each vertex } u \in V(P^j), \text{ where } j \in \{2,\ldots,a\}. \end{cases}$$

Note that (i) $g(V(P^1)) = d(v,\ell_1)$; (ii) for $j \in \{2,3,\ldots,a\}$, $g(V(P^j)) = k - d(v,\ell_1) > k - \frac{k}{2} = \frac{k}{2}$ since $k > 2d(v,\ell_1)$; (iii) $g(V(T)) = d(v,\ell_1) + (a-1)(k-d(v,\ell_1)) = (a-1)k-(a-2)d(v,\ell_1)$. Also note that $g$ is a $k$-resolving function of

$T$: (i) $0 \le \frac{d(v,\ell_1)}{d(v,\ell_j)} \le \frac{k-d(v,\ell_1)}{d(v,\ell_j)} \le \frac{d(\ell_1,\ell_2)-d(v,\ell_1)}{d(v,\ell_j)} = \frac{d(v,\ell_2)}{d(v,\ell_j)} \le 1$ for $j \in \{2,\dots,a\}$; (ii) if two distinct vertices $x$ and $y$ lie on the $v-\ell_i$ path for some $i \in \{1,2,\dots,a\}$, then $g(R\{x,y\}) \ge g(V(T)) - g(V(P^i)) \ge \min\{k, 2(k-d(v,\ell_1))\} = k$ since $a \ge 3$ and $k - d(v,\ell_1) > \frac{k}{2}$; (iii) if $x \in V(P^i)$ and $y \in V(P^j)$ with $d(v,x) \ne d(v,y)$ for distinct $i,j \in \{1,2,\dots,a\}$, then $g(R\{x,y\}) \ge \min\{g(V(T)-V(P^i)), g(V(T)-V(P^j))\} \ge k$, since at most one vertex in the $\ell_i - \ell_j$ path can be at equal distance from both $x$ and $y$ in $T$; (iv) if $x \in V(P^i)$ and $y \in V(P^j)$ with $d(v,x) = d(v,y)$ for distinct $i,j \in \{1,2,\dots,a\}$, then $g(R\{x,y\}) = g(V(P^i)) + g(V(P^j)) \ge \min\{k, 2(k-d(v,\ell_1))\} = k$. Thus, $\dim_f^k(T) \le g(V(T)) = (a-1)k - (a-2)d(v,\ell_1)$.

Therefore, $\dim_f^k(T) = (a-1)k - (a-2)d(v,\ell_1)$. □

Next, we determine $\dim_f^k(T)$ for a tree $T$ with $ex(T) \ge 1$. We begin with the following lemma, which, besides being useful for Theorem 19, bears independent interest.

**Lemma 18.** *Let $T$ be a tree with $ex(T) \ge 2$. For $w \in M^*(T)$, let $x \in V(T_w)$ and $y \in V(T) - V(T_w)$. Then either $R\{x,y\} \supseteq V(T_w)$ or $R\{x,y\} \supseteq V(T_{w'})$ for some $w' \in M^*(T) - \{w\}$.*

*Proof.* Since $ex(T) \ge 2$, $|M^*(T)| \ge 2$. Let $x \in V(T_w)$ for some $w \in M^*(T)$. First, suppose $y \in V(T_{w'})$ for some $w' \in M^*(T) - \{w\}$. Assume, for contradiction, that there exist $u \in V(T_w)$ and $v \in V(T_{w'})$ such that

$$d(x,u) = d(y,u) \text{ and } d(x,v) = d(y,v). \tag{4}$$

Put $a = d(y,v), b = d(v,w'), c = d(w',w), d = d(w,u), e = d(u,x)$. Let us call a path leading from $w$ to any of its leaves a "$w$-terminal path". We may assume that $u$ and $x$ lie in the same $w$-terminal path, since $d(x,u) = d(y,u)$ implies $d(x,w) = d(y,w)$ if $u$ and $x$ lie in distinct $w$-terminal paths. Likewise, we assume $v$ and $y$ lie in the same $w'$-terminal path. After writing the two equations (4) in terms of components $a,b,c,d,e$ and simplifying, we obtain $b+c+d = 0$. This means that $b = c = d = 0$, since all variables denote (nonnegative) distances. In particular, the distinctness of $w$ and $w'$ is contradicted by $c = 0$.

Now, suppose $y \notin V(T_z)$ for any $z \in M^*(T)$; then either $y \in V(T_{y'})$ for some $y' \in M_1(T)$ or $y \notin V(T_{z'})$ for any $z' \in M(T)$. Note that there exists a vertex $w' \in M^*(T) - \{w\}$ such that either $y$ or $y'$ lies in the $w - w'$ path in $T$. Since $d(s,x) = d(s,y)$ for $s \in V(T_{w'})$ is equivalent to $d(w',x) = d(w',y)$, we may assume, for contradiction, that

$$d(x,w') = d(y,w') \text{ and } d(x,t) = d(y,t), \text{ where } t \in V(T_w). \tag{5}$$

Put $d(y,w') = a+b$ and $d(w,w') = c+b$, where $b \ge 0$ denotes the length of the path shared between the $w-w'$ path and the $y-w'$ path. Similarly, put $d(w,t) = d$ and $d(x,t) = e$. As before, we may assume that $x$ and $t$ lie in the same $w$-terminal path. After simplifying the two equations (5) in terms of $a,b,c,d,e$, we obtain $c = d = 0$. This implies that either $y \in V(T_w)$ (when $b > 0$) or $w = w'$ (when $b = 0$); both possibilities contradict the present assumptions. □

**Theorem 19.** *Let $T$ be a tree with $ex(T) \geq 1$. Then $\kappa(T) = \min\{\kappa(T_v) : v \in M^*(T)\}$ and, for $k \in [1, \kappa(T)]$,*

$$\dim_f^k(T) = k|M_2(T)| + \sum_{v \in M_3(T)} \dim_f^k(T_v). \tag{6}$$

*Proof.* If $ex(T) = 1$, then $|M_2(T)| = 0$ and $|M_3(T)| = 1$, and so (6) trivially holds; see Propostion 17 for explicit formulas for $\kappa(T)$ and $\dim_f^k(T)$. So, let $ex(T) \geq 2$; then $|M^*(T)| \geq 2$. By Theorem 16, $\kappa(T) = \min\{\kappa(T_v) : v \in M^*(T)\}$. Let $k \in [1, \kappa(T)]$; notice that $\kappa(T) \leq \kappa(T_z)$ for any $z \in M^*(T)$.

First, we show that $\dim_f^k(T) \geq k|M_2(T)| + \sum_{v \in M_3(T)} \dim_f^k(T_v)$. For $v \in M_2(T)$, let $N(v) \cap V(T_v) = \{r_1, r_2\}$. For $w \in M_3(T)$, let $\ell_1, \ell_2, \ldots, \ell_\sigma$ be the terminal vertices of $w$ with $ter_T(w) = \sigma$, and let $s_i$ be the neighbor of $w$ lying on the $w - \ell_i$ path, where $i \in \{1, 2, \ldots, \sigma\}$; further, let $P^i$ denote the $s_i - \ell_i$ path. Let $g : V(T) \to [0, 1]$ be a minimum $k$-resolving function of $T$. If $M_2(T) \neq \emptyset$, then, for any $v \in M_2(T)$, $R\{r_1, r_2\} = V(T_v) - \{v\}$ and $g(R\{r_1, r_2\}) = g(V(T_v) - \{v\}) \geq k$; thus $\sum_{v \in M_2(T)} g(V(T_v)) \geq \sum_{v \in M_2(T)} k = k|M_2(T)|$. If $M_3(T) \neq \emptyset$, then, for any $w \in M_3(T)$, notice $R\{s_i, s_j\} = V(P^i) \cup V(P^j)$ for any distinct $i, j \in \{1, 2, \ldots, \sigma\}$. This, together with the argument used in the proof of Proposition 17, we have $\sum_{w \in M_3(T)} g(V(T_w)) \geq \sum_{w \in M_3(T)} \dim_f^k(T_w)$. Thus, we have

$$g(V(T)) \geq \sum_{v \in M_2(T)} g(V(T_v)) + \sum_{w \in M_3(T)} g(V(T_w)) \geq k|M_2(T)| + \sum_{w \in M_3(T)} \dim_f^k(T_w).$$

Next, we show that $\dim_f^k(T) \leq k|M_2(T)| + \sum_{v \in M_3(T)} \dim_f^k(T_v)$. For each $w \in M_3(T)$, let $g_w$ be a minimum $k$-resolving function on $V(T_w)$. For each $w \in M_2(T)$, define a function $h_w$ on $V(T_w)$ such that $h_w(w) = 0$ and $h_w(u) = \frac{k}{|V(T_w)|-1}$ if $u \neq w$. For $k \in [1, \kappa(T)]$, let $g : V(T) \to [0, 1]$ be the function defined by

$$g(u) = \begin{cases} g_w(u) & \text{if } u \in V(T_w) \text{ for } w \in M_3(T), \\ h_w(u) & \text{if } u \in V(T_w) \text{ for } w \in M_2(T), \\ 0 & \text{otherwise.} \end{cases}$$

Note that (i) for each $w \in M_2(T)$, $g(V(T_w)) = h_w(V(T_w) - \{w\}) = k$; (ii) for each $w \in M_3(T)$, $g(V(T_w)) = g_w(V(T_w)) = \dim_f^k(T_w) \geq k$; (iii) $g(V(T)) = k|M_2(T)| + \sum_{w \in M_3(T)} \dim_f^k(T)$. It suffices to show $g$ is a $k$-resolving function of $T$. Obviously, $0 \leq g(u) \leq 1$ for each $u \in V(T)$. So, let $x$ and $y$ be distinct vertices of $T$; we will show that $g(R\{x, y\}) \geq k$. Consider three cases: (1) there is a $w \in M^*(T)$ such that $\{x, y\} \subseteq V(T_w)$; (2) there is a $w \in M^*(T)$ such that $x \in V(T_w)$ and $y \notin V(T_w)$; (3) $\{x, y\} \subseteq V(T) - \cup_{w \in M^*(T)} V(T_w)$. In case (1), if $w \in M_3(T)$, then $g(R\{x, y\}) \geq g_w(R\{x, y\} \cap V(T_w)) \geq k$, since $g_w$ is a $k$-resolving function on $V(T_w)$; if $w \in M_2(T)$ and $d(x, w) \neq d(y, w)$, then there is a $z \in M^*(T) - \{w\}$ such that $g(R\{x, y\}) \geq g(V(T_z)) \geq k$; if $w \in M_2(T)$ and $d(x, w) = d(y, w)$, then $g(R\{x, y\}) = h_w(V(T_w) - \{w\}) = k$. In case (2), by Lemma 18, either $R\{x, y\} \supseteq V(T_w)$ or $R\{x, y\} \supseteq V(T_{w'})$ for some $w' \in M^*(T) - \{w\}$; thus $g(R\{x, y\}) \geq k$. So, we consider case (3). Note that $x \in V(T_{x'})$ for some $x' \in M_1(T)$ or $x \notin V(T_z)$ for any $z \in M(T)$; similarly, $y \in V(T_{y'})$ for some $y' \in M_1(T)$ or $y \notin V(T_{z'})$ for any $z' \in M(T)$. If $\{x, y\} \subseteq V(T_v)$ for some $v \in M_1(T)$, then $d(v, x) \neq d(v, y)$ and there exist distinct $v', v'' \in M^*(T)$ such that $v$ lies on the $v' - v''$ path in $T$; thus $g(R\{x, y\}) \geq g(V(T_{v'})) + g(V(T_{v''})) \geq 2k$. If $\{x, y\} \not\subseteq V(T_v)$ for any $v \in M(T)$, there exist distinct $w_1, w_2 \in M^*(T)$ such that both $x$ (or $x'$) and $y$ (or $y'$) lie on the $w_1 - w_2$ path in $T$; then $d(w_1, x) = d(w_1, y)$ and $d(w_2, x) = d(w_2, y)$ imply either $x = y$ or $\{x, y\} \subseteq V(T_s)$ for some $s \in M_1(T)$, where both possibilities contradict the present assumptions. Thus $R\{x, y\} \supseteq V(T_{w_i})$ for at least one $i \in \{1, 2\}$, and $g(R\{x, y\}) \geq g(V(T_{w_i})) \geq k$. □

Next, we provide an example showing that $\dim_f^k(G) - k \dim_f(G)$ can be arbitrarily large for some $k \in (1, \kappa(G)]$.

**Remark 20.** *The value of* $\dim_f^k(G) - k \dim_f(G)$ *can arbitrarily large, as* $G$ *varies, for some* $k \in (1, \kappa(G)]$. *Let* $T$ *be a tree with* $ex(T) = 1$. *Let* $v$ *be the exterior major vertex of* $T$ *and let* $\ell_1, \ell_2, \ldots, \ell_\alpha$ *be the terminal vertices of* $T$ *such that* $d(v, \ell_1) = 1 < \beta = d(v, \ell_j)$ *for each* $j \in \{2, 3, \ldots, \alpha\}$, *where* $\alpha \geq 3$. *By Proposition 17,* $\kappa(T) = \beta + 1$ *and* $\dim_f^{\beta+1}(T) = (\alpha - 1)(\beta + 1) - (\alpha - 2) = (\alpha - 1)\beta + 1$. *Since* $\dim_f(T) = \frac{\alpha}{2}$ *by Theorem 3(b),* $\dim_f^{\beta+1}(T) - (\beta + 1) \dim_f(T) = (\alpha - 1)\beta + 1 - (\beta + 1)(\frac{\alpha}{2}) = (\frac{\alpha}{2} - 1)(\beta - 1)$ *can be arbitrarily large, as* $\alpha$ *or* $\beta$ *gets big enough.*

Next, we determine the fractional $k$-metric dimension of cycles.

**Proposition 21.** *Let* $C_n$ *be an* $n$*-cycle, where* $n \geq 3$. *Then*

$$\text{(7)} \qquad \dim_f^k(C_n) = k \dim_f(C_n) = \begin{cases} \frac{kn}{n-2} & \text{if } n \text{ is even and } k \in [1, n-2], \\ \frac{kn}{n-1} & \text{if } n \text{ is odd and } k \in [1, n-1]. \end{cases}$$

*Proof.* Note that $\kappa(C_n) = n - 2$ for an even $n$, and $\kappa(C_n) = n - 1$ for an odd $n$. Let $k \in [1, \kappa(C_n)]$. For an even $n \geq 4$, a function $g : V(C_n) \to [0, 1]$ defined by $g(u) = \frac{1}{n-2}$, for each $u \in V(C_n)$, is a minimum resolving function of $C_n$: (i) $0 < g(u) = \frac{1}{n-2} \leq \frac{1}{k} \leq 1$ since $n \geq 4$; (ii) for distinct $x, y \in V(C_n)$, $|R\{x, y\}| \geq n - 2$, and thus $g(R\{x, y\}) \geq (n - 2)(\frac{1}{n-2}) = 1$; (iii) $g(V(C_n)) = \frac{n}{n-2} = \dim_f(C_n)$ by Theorem 3(d). Similarly, for an odd $n \geq 3$, one can easily check that a function $h : V(C_n) \to [0, 1]$ defined by $h(u) = \frac{1}{n-1}$, for each $u \in V(C_n)$, is a minimum resolving function of $C_n$ satisfying $h(u) \leq \frac{1}{k}$. By Lemma 9 and Theorem 3(d), (7) follows. □

**Remark 22.** *Note that, for any fixed* $k \in [1, \kappa(C_n)]$, $\lim_{n\to\infty} \dim_f^k(C_n) = k$ *by Proposition 21 (c.f. Proposition 12(a)).*

Next, we determine the fractional $k$-metric dimension of wheel graphs.

**Proposition 23.** *For the wheel graph* $W_n$ *of order* $n \geq 5$,

$$\text{(8)} \qquad \dim_f^k(W_n) = k \dim_f(W_n) = \begin{cases} 2k & \text{if } n = 5 \text{ and } k \in [1, 2], \\ \frac{3k}{2} & \text{if } n = 6 \text{ and } k \in [1, 4], \\ \frac{k(n-1)}{4} & \text{if } n \geq 7 \text{ and } k \in [1, 4]. \end{cases}$$

*Proof.* For $n \ge 5$, the wheel graph $W_n = C_{n-1} + K_1$ is obtained from an $(n-1)$-cycle $C_{n-1}$ by joining an edge from each vertex of $C_{n-1}$ to a new vertex, say $v$; let the $C_{n-1}$ be given by $u_1, u_2, \ldots, u_{n-1}, u_1$. Note that $\text{diam}(W_n) = 2$ for $n \ge 5$.

**Case 1:** $n = 5$**.** Note that $\kappa(W_5) = 2$ since $R\{u_1, u_3\} = \{u_1, u_3\}$. Let $k \in [1, 2]$. Let $g : V(W_5) \to [0, 1]$ be a function defined by $g(v) = 0$ and $g(u_i) = \frac{1}{2}$ for each $i \in \{1, 2, 3, 4\}$. Then $g$ is a minimum resolving function of $W_5$: (i) $0 \le g(x) \le \frac{1}{k} \le 1$ for each $x \in V(W_5)$; (ii) for distinct $i, j \in \{1, 2, 3, 4\}$, $g(R\{u_i, u_j\}) \ge g(u_i) + g(u_j) = 1$; (iii) for $i \in \{1, 2, 3, 4\}$, $g(R\{v, u_i\}) \ge g(u_i) + g(u_\ell) = 1$, where $u_\ell \in V(W_5) - N[u_i]$; (iv) $g(V(W_5)) = 2 = \dim_f(W_5)$ by Theorem 3(e). By Lemma 9 and Theorem 3(e), $\dim_f^k(W_5) = k \dim_f(W_5) = 2k$ for $k \in [1, 2]$.

**Case 2:** $n \ge 6$**.** First, we show that $\kappa(W_n) = 4$ in this case. For each $i \in \{1, 2, \ldots, n-1\}$, $R\{v, u_i\} = (V(W_n) - N(u_i)) \cup \{v\}$ with $|R\{v, u_i\}| = n-2 \ge 4$. For distinct $i, j \in \{1, 2, \ldots, n-1\}$, (i) if $u_i u_j \in E(W_n)$, then $R\{u_i, u_j\} = (N(u_i) \cup N(u_j)) - \{v\}$ with $|R\{u_i, u_j\}| = 4$; (ii) if $u_i u_j \notin E(W_n)$ and $|N(u_i) \cap N(u_j)| = 1$, then $R\{u_i, u_j\} = (N[u_i] \cup N[u_j]) - \{v\}$ with $|R\{u_i, u_j\}| = 6$; (iii) if $u_i u_j \notin E(W_n)$ and $|N(u_i) \cap N(u_j)| = 2$, then $R\{u_i, u_j\} = (N[u_i] \cup N[u_j]) - (N(u_i) \cap N(u_j))$ with $|R\{u_i, u_j\}| = 4$. So, $\kappa(W_n) = 4$ for $n \ge 6$.

Second, we determine $\dim_f^k(W_n)$ for $n \ge 6$. Let $k \in [1, 4]$. For $n = 6$, a function $g : V(W_6) \to [0, 1]$ defined by $g(x) = \frac{1}{4}$, for each $x \in V(W_6)$, is a minimum resolving function of $W_6$: (i) $0 < g(x) \le \frac{1}{k} \le 1$ for each $x \in V(W_6)$; (ii) for any distinct $x, y \in V(W_6)$, $g(R\{x, y\}) \ge 4(\frac{1}{4}) = 1$; (iii) $g(V(W_6)) = \frac{3}{2} = \dim_f(W_6)$ by Theorem 3(e). For $n \ge 7$, let $h : V(W_n) \to [0, 1]$ be a function defined by $h(v) = 0$ and $h(u_i) = \frac{1}{4}$ for each $i \in \{1, 2, \ldots, n-1\}$. Then $h$ is a minimum resolving function of $W_n$: (i) $0 \le h(x) \le \frac{1}{k} \le 1$ for each $x \in V(W_n)$; (ii) for each $i \in \{1, 2, \ldots, n-1\}$, $|R\{v, u_i\}| \ge n - 2 \ge 5$ since $n \ge 7$, and hence $h(R\{v, u_i\}) \ge 4(\frac{1}{4}) = 1$; (iii) for distinct $i, j \in \{1, 2, \ldots, n-1\}$, $|R\{u_i, u_j\}| \ge 4$ and $v \notin R\{u_i, u_j\}$, and thus $h(R\{u_i, u_j\}) \ge 4(\frac{1}{4}) = 1$; (iv) $h(V(W_n)) = \frac{n-1}{4} = \dim_f(W_n)$ by Theorem 3(e). Therefore, by Lemma 9 and Theorem 3(e), (8) holds for $n \ge 6$ and $k \in [1, 4]$. □

Next, we determine the fractional $k$-metric dimension of the Petersen graph.

**Proposition 24.** *For the Petersen graph $\mathcal{P}$, $\dim_f^k(\mathcal{P}) = k \dim_f(\mathcal{P}) = \frac{5}{3}k$ for $k \in [1, 6]$.*

*Proof.* Note that $\mathcal{P}$ is 3-regular and vertex-transitive. Since $\text{diam}(\mathcal{P}) = 2$, any two distinct vertices in $\mathcal{P}$ are either adjacent or at distance two apart.

We first show that $\kappa(\mathcal{P}) = 6$. For any distinct $x, y \in V(\mathcal{P})$, $R\{x, y\} = N[x] \cup N[y] - (N(x) \cap N(y))$ and $|R\{x, y\}| = 6$: (i) if $xy \in E(\mathcal{P})$, then $N(x) \cap N(y) = \emptyset$ and $x \in N[y]$ and $y \in N[x]$; (ii) if $xy \notin E(\mathcal{P})$, then $|N(x) \cap N(y)| = 1$. So, $\kappa(\mathcal{P}) = 6$.

Now, let $k \in [1, 6]$. Since $\dim_f^k(\mathcal{P}) \ge k \dim_f(\mathcal{P})$ by Lemma 8, it suffices to show that $\dim_f^k(\mathcal{P}) \le k \dim_f(\mathcal{P})$. Let $g : V(\mathcal{P}) \to [0, 1]$ be a function defined by $g(v) = \frac{k}{6}$ for each $v \in V(\mathcal{P})$. Since $0 \le g(v) \le 1$ for each vertex $v \in V(\mathcal{P})$ and $g(R\{x, y\}) = 6(\frac{k}{6}) \ge k$ for any two distinct $x, y \in V(\mathcal{P})$, $g$ is a $k$-resolving function of $\mathcal{P}$. So, $\dim_f^k(\mathcal{P}) \le |V(\mathcal{P})|(\frac{k}{6}) = \frac{10k}{6} = k(\frac{5}{3}) = k \dim_f(\mathcal{P})$ by Theorem 3(c). □

Next, we determine the fractional $k$-metric dimension of a bouquet of cycles.

**Proposition 25.** *Let $B_m$ be a bouquet of $m$ cycles $C^1, C^2, \ldots, C^m$ with a cut-vertex (i.e., the vertex sum of $m$ cycles at one common vertex), where $m \geq 2$; further, let $C^1$ be the cycle of the minimum length among the $m$ cycles of $B_m$. Then* $\kappa(B_m) = \begin{cases} |V(C^1)|-1 & \text{if } C^1 \text{ is an odd cycle,} \\ |V(C^1)|-2 & \text{if } C^1 \text{ is an even cycle,} \end{cases}$ *and, for $k \in [1, \kappa(B_m)]$, $\dim_f^k(B_m) = k \dim_f(B_m) = km$.*

*Proof.* Let $v$ be the cut-vertex of $B_m$. For each $i \in \{1, 2, \ldots, m\}$, let $C^i$ be given by $v, u_{i,1}, u_{i,2}, \ldots, u_{i,r_i}, v$ and let $P^i = C^i - v$; further, let $P^{i,1}$ be the $u_{i,1} - u_{i,\lceil \frac{r_i}{2} \rceil}$ geodesic. Without loss of generality, let $r_1 \leq r_2 \leq \ldots \leq r_m$.

**Claim 1:** If $C^1$ is an odd cycle, then $\kappa(B_m) = r_1 = |V(C^1)| - 1$; if $C^1$ is an even cycle, $\kappa(B_m) = r_1 - 1 = |V(C^1)| - 2$.

Proof of Claim 1. Let $x$ and $y$ be distinct vertices of $B_m$. First, let $x, y \in V(C^i)$ for some $i \in \{1, 2, \ldots, m\}$. If $d(v,x) \neq d(v,y)$, then $R\{x,y\} \supseteq V(C^j)$ with $|R\{x,y\}| \geq |V(C^j)| = r_j + 1 \geq r_1 + 1$ for $j \neq i$. If $d(v,x) = d(v,y)$ and $C^i$ is an odd cycle, then $R\{x,y\} = V(P^i)$ with $|R\{x,y\}| = r_i \geq r_1$; notice, for an odd cycle $C^1$, $|R\{u_{1,1}, u_{1,r_1}\}| = r_1$. If $d(v,x) = d(v,y)$ and $C^i$ is an even cycle, then $R\{x,y\} = V(P^i) - \{u_{i,\lceil \frac{r_i}{2} \rceil}\}$ with $|R\{x,y\}| = r_i - 1$, where $r_i - 1 \geq r_1$ if $C^1$ is an odd cycle, and $r_i - 1 \geq r_1 - 1$ if $C^1$ is an even cycle; notice, for an even cycle $C^1$, $|R\{u_{1,1}, u_{1,r_1}\}| = r_1 - 1$.

Second, let $x \in V(P^i)$ and $y \in V(P^j)$ for distinct $i, j \in \{1, 2, \ldots, m\}$; let $x \in V(P^{i,1})$ and $y \in V(P^{j,1})$, without loss of generality. If $d(v,x) = d(v,y)$, then $R\{x,y\} \supseteq V(P^{i,1}) \cup V(P^{j,1})$ with $|R\{x,y\}| \geq \lceil \frac{r_i}{2} \rceil + \lceil \frac{r_j}{2} \rceil \geq r_1$. So, let $d(v,x) \neq d(v,y)$, say $d(v,x) < d(v,y)$ without loss of generality; then $d(u,x) \neq d(u,y)$ for each $u \in V(P^{i,1})$, and $r_j \geq 3$. If $r_j = 3$, then $R\{x,y\} \supseteq V(P^{i,1}) \cup V(P^j)$. If $r_j \geq 4$, then at most two vertices in $C^j$ are at equal distance from $x$ and $y$; thus, $R\{x,y\} \supseteq V(P^{i,1}) \cup (V(P^j) - \{w_1, w_2\})$ such that $w_t \in V(P^j)$ with $d(w_t, x) = d(w_t, y)$, where $t \in \{1, 2\}$. In each case, $|R\{x,y\}| \geq \lceil \frac{r_i}{2} \rceil + \lceil \frac{r_j}{2} \rceil \geq r_1$. $\square$

**Claim 2:** For $k \in [1, \kappa(B_m)]$, $\dim_f^k(B_m) = k \dim_f(B_m) = km$.

Proof of Claim 2. Let $k \in [1, \kappa(B_m)]$, and let $h : V(B_m) \to [0,1]$ be a function defined by

$$h(u) = \begin{cases} \frac{1}{r_i} & \text{for each } u \in V(P^i) \text{ if } C^i \text{ is an odd cycle,} \\ \frac{1}{r_j - 1} & \text{for each } u \in V(P^j) - \{u_{j,\lceil \frac{r_j}{2} \rceil}\} \text{ if } C^j \text{ is an even cycle,} \\ 0 & \text{otherwise.} \end{cases}$$

Note that, for each $i \in \{1, 2, \ldots, m\}$, $h(V(P^i)) = 1$ if $C^i$ is an odd cycle, and $h(V(P^i)) = h(V(P^i)) - h(u_{i,\lceil \frac{r_i}{2} \rceil}) = 1$ if $C^i$ is an even cycle. We also note that $h$ is a minimum resolving function of $B_m$: (i) $0 \leq h(u) \leq \frac{1}{k} \leq 1$ for each $u \in V(B_m)$; (ii) if $x, y \in V(C^i)$ with $d(v,x) \neq d(v,y)$, for some $i \in \{1, 2, \ldots, m\}$, then $h(R\{x,y\}) \geq h(V(C^j)) \geq 1$ for $j \neq i$; (iii) if $x, y \in V(C^i)$ with $d(v,x) = d(v,y)$ and $x \neq y$, for some $i \in \{1, 2, \ldots, m\}$, then $h(R\{x,y\}) = h(V(P^i)) = 1$ when

$C^i$ is an odd cycle, and $h(R\{x,y\}) = h(V(P^i)) - h(u_{i,\lceil \frac{r_i}{2} \rceil}) = 1$ when $C^i$ is an even cycle; (iv) if $x \in V(P^i)$ and $y \in V(P^j)$ for distinct $i,j \in \{1,2,\ldots,m\}$, then $h(R\{x,y\}) \geq \frac{1}{2}h(V(P^i)) + \frac{1}{2}h(V(P^j)) \geq 1$ using a similar argument used in the proof of Claim 1; (v) $h(V(B_m)) = m = \dim_f(B_m)$ by Theorem 3(f). So, by Lemma 9 and Theorem 3(f), $\dim_f^k(B_m) = k\dim_f(B_m) = km$ for $k \in [1, \kappa(B_m)]$. □ □

Next, we determine the fractional $k$-metric dimension of complete multipartite graphs.

**Proposition 26.** *For $m \geq 2$, let $G = K_{a_1,a_2,\ldots,a_m}$ be a complete $m$-partite graph of order $n = \sum_{i=1}^{m} a_i \geq 3$, and let $s$ be the number of partite sets of $G$ consisting of exactly one element. Then, for $k \in [1,2]$,*

$$\dim_f^k(G) = k\dim_f(G) = \begin{cases} \frac{k(n-1)}{2} & \text{if } s = 1, \\ \frac{kn}{2} & \text{otherwise.} \end{cases}$$

*Proof.* Let $V(G)$ be partitioned into $m$-partite sets $V_1, V_2, \ldots, V_m$ with $|V_i| = a_i$, where $i \in \{1,2,\ldots,m\}$. Without loss of generality, let $a_1 \leq a_2 \leq \ldots \leq a_m$. Note that $\kappa(G) = 2$: (i) if $a_m \geq 2$, then, for two distinct $x, y \in V_m$, $R\{x,y\} = \{x,y\}$; (ii) if $a_m = 1$, then, for $x \in V_1$ and $y \in V_2$, $R\{x,y\} = \{x,y\}$. Let $k \in [1,2]$.

First, let $s \neq 1$. A function $g : V(G) \to [0,1]$ defined by $g(v) = \frac{1}{2}$, for each $v \in V(G)$, is a minimum resolving function of $G$: (i) $0 < g(v) \leq \frac{1}{k} \leq 1$ for each $v \in V(G)$; (ii) for any distinct vertices $x, y \in V(G)$, $g(R\{x,y\}) \geq g(x) + g(y) = 1$; (iii) $g(V(G)) = \frac{n}{2} = \dim_f(G)$ by Theorem 3(g). So, $\dim_f^k(G) = k\dim_f(G) = \frac{kn}{2}$ by Lemma 9 and Theorem 3(g).

Second, let $s = 1$. Let $h : V(G) \to [0,1]$ be a function defined by $h(u) = 0$ for $u \in V_1$ and $h(v) = \frac{1}{2}$ for each $v \in V(G) - V_1$. Then $h$ is a minimum resolving function of $G$ : (i) $0 \leq h(v) \leq \frac{1}{k} \leq 1$ for each $v \in V(G)$; (ii) for any two distinct vertices $x, y \in V(G) - V_1$, $h(R\{x,y\}) \geq h(x) + h(y) = 1$; (iii) for $x \in V_1$ and $y \in V_i \subseteq V(G) - V_1$, $h(R\{x,y\}) \geq h(V_i) \geq 1$, where $i \in \{2,\ldots,m\}$; (iv) $h(V(G)) = \frac{n-1}{2} = \dim_f(G)$ by Theorem 3(g). So, $\dim_f^k(G) = k\dim_f(G) = \frac{k(n-1)}{2}$ for $k \in [1,2]$ by Lemma 9 and Theorem 3(g). □

Now, we consider the fractional $k$-metric dimension of grid graphs (i.e., the Cartesian product of two paths). The *Cartesian product* of two graphs $G$ and $H$, denoted by $G \square H$, is the graph with the vertex set $V(G) \times V(H)$ such that $(u,w)$ is adjacent to $(u',w')$ if and only if either $u = u'$ and $ww' \in E(H)$, or $w = w'$ and $uu' \in E(G)$. See Figure 1 for the labeling of $P_6 \square P_4$.

We recall the following result.

**Theorem 27.** [**5**] *For $s,t \geq 2$, $\kappa(P_s \square P_t) = s + t - 2$ and $\dim^k(P_s \square P_t) = 2k$, where $k \in \{1,2,\ldots,s+t-2\}$.*

**Proposition 28.** *For $k \in [1, s+t-2]$, $\dim_f^k(P_s \square P_t) = k\dim_f(P_s \square P_t) = 2k$, where $s,t \geq 2$.*

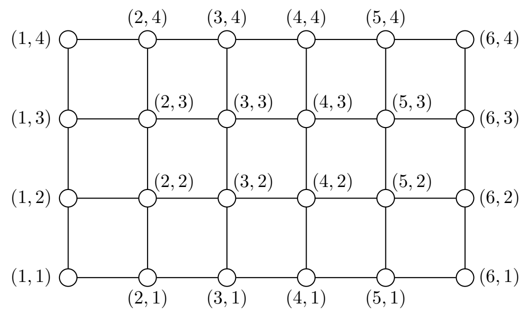


Figure 1: Labeling of $P_6\Box P_4$.

*Proof.* Let $s \geq t \geq 2$, and let $G = P_s\Box P_t$ and $L = \{v \in V(G) : 2 \leq \deg(v) \leq 3\}$. By Theorem 27, $\kappa(G) = s+t-2$. Let $k \in [1, s+t-2]$. Since $\dim_f^k(G) \geq k\dim_f(G) = 2k$ by Lemma 8 and Theorem 3(h), it suffices to show that $\dim_f^k(G) \leq 2k$. Let $g : V(G) \to [0, 1]$ be a function defined by

$$g(v) = \begin{cases} \frac{k}{s+t-2} & \text{if } v \in L, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $g(V(G)) = 2k$. We will show that $g$ is a $k$-resolving function of $G$. Clearly, $0 \leq g(v) \leq 1$ for each $v \in V(G)$. Let $x = (a, b)$ and $y = (c, d)$ be two distinct vertices of $G$. We consider two cases.

**Case 1:** $a = c$ **or** $b = d$**.** If $a = c$, then $R\{x, y\} \cap L \supseteq \cup_{i=1}^{s}\{(i, 1), (i, t)\}$, and thus $g(R\{x, y\}) \geq (2s)(\frac{k}{s+t-2}) \geq k$ since $s \geq t \geq 2$. So, let $b = d$; without loss of generality, let $a < c$. Let $z = (\alpha, \beta) \in L$. Note that (i) if $\alpha \leq a$, then $d(z, x) < d(z, y)$ and thus $R\{x, y\} \cap L \supseteq \cup_{j=1}^{t}\{(1, j)\} \cup (\cup_{i=2}^{a}\{(i, 1), (i, t)\})$; (ii) if $\alpha \geq c$, then $d(z, x) > d(z, y)$ and thus $R\{x, y\} \cap L \supseteq \cup_{j=1}^{t}\{(s, j)\} \cup (\cup_{i=c}^{s-1}\{(i, 1), (i, t)\})$; (iii) if $a < \alpha < c$ and $\beta = 1$, then there exists at most one such $z \in L$ satisfying $d(z, x) = d(z, y)$, since $d(z, x) = d(z, y)$ implies $\alpha - a + b - 1 = c - \alpha + b - 1$, i.e., $2\alpha = a + c$; similarly, if $a < \alpha < c$ and $\beta = t$, then there exists at most one such $z \in L$ satisfying $d(z, x) = d(z, y)$. Thus, $|R\{x, y\} \cap L| \geq 2s + 2t - 6$, and hence $g(R\{x, y\}) \geq (2s + 2t - 6)(\frac{k}{s+t-2}) = 2k(\frac{s+t-3}{s+t-2}) \geq k$, since $s + t \geq 4$.

**Case 2:** $a \neq c$ **and** $b \neq d$**.** Without loss of generality, let $a < c$; further, assume that $b < d$ (the case for $b > d$ can be handled similarly). Let $z' = (\alpha', \beta') \in L$. Note that (i) if $\alpha' \leq a$ and $\beta' = 1$, then $d(z', x) < d(z', y)$ and thus $R\{x, y\} \cap L \supseteq \cup_{i=1}^{a}\{(i, 1)\}$; (ii) if $\alpha' \geq c$ and $\beta' = t$, then $d(z', x) > d(z', y)$ and thus $R\{x, y\} \cap L \supseteq \cup_{i=c}^{s}\{(i, t)\}$; (iii) if $a < \alpha' < c$ (i.e., $c \neq a+1$) and $\beta' = 1$, then there exists at most one such $z' \in L$ satisfying $d(z', x) = d(z', y)$, since $d(z', x) = d(z', y)$ implies $\alpha' - a + b - 1 = c - \alpha' + d - 1$, i.e., $2\alpha' = a - b + c + d$; similarly, if $a < \alpha' < c$ and $\beta' = t$, there exists at most one such $z' \in L$ satisfying $d(z', x) = d(z', y)$. Likewise, we note that (i) if $\alpha' = 1$ and $\beta' \leq b$, then $d(z', x) < d(z', y)$ and thus $R\{x, y\} \cap L \supseteq \cup_{j=1}^{b}\{(1, j)\}$; (ii) if $\alpha' = s$ and $\beta' \geq d$, then $d(z', x) > d(z', y)$ and thus $R\{x, y\} \cap L \supseteq \cup_{j=d}^{t}\{(s, j)\}$; (iii) if $\alpha' = 1$ and $b < \beta' < d$ (i.e., $d \neq b+1$), then there exists at most one such $z' \in L$ satisfying $d(z',x) = d(z',y)$, since $d(z',x) = d(z',y)$ implies $a-1+\beta'-b = c-1+d-\beta'$, i.e., $2\beta' = -a+b+c+d$; similarly, if $\alpha' = s$ and $b < \beta' < d$, then there exists at most one such $z' \in L$ satisfying $d(z',x) = d(z',y)$. So, if $c = a+1$ or $d = b+1$, then $|R\{x,y\} \cap L| \geq s+t-2$; if $c \geq a+2$ and $d \geq b+2$, then $|R\{x,y\} \cap L| \geq a+(s-c+1)+2(c-a-2)+b+(t-d+1)+2(d-b-2) \geq s+t-2$. In each case, $g(R\{x,y\}) \geq (s+t-2)(\frac{k}{s+t-2}) = k$.

Thus, in each case, $g$ is a $k$-resolving function of $G$, and hence $\dim_f^k(G) \leq g(V(G)) = 2k$. Therefore, $\dim_f^k(G) = k \dim_f(G) = 2k$ for $k \in [1, s+t-2]$ for $s \geq t \geq 2$. $\square$

## 4. Open Problems

We conclude this paper with two open problems.

**Problem 1.** Let $\phi(k) = \dim_f^k(G)$ be a function of $k$, for a fixed $G$, on domain $[1, \kappa(G)]$. Is $\phi$ a continuous function of $k$ on every connected graph $G$?

**Problem 2.** Suppose $\dim_f^k(G)$ is given by $\psi(k)$ for integral values of $k$. When and how can we interpolate $\psi$ and deduce $\dim_f^k(G)$ for any real number $k \in [1, \kappa(G)]$?

For example, let $G = P_s \square P_t$, where $s, t \geq 2$. Then $\dim_f^k(G) = 2k$ for integers $k \in \{1, 2, \ldots, \kappa(G)\}$ by Theorems 3(h), Lemma 8, Observation 5(b), and Theorem 27. In Proposition 28, we proved that $\dim_f^k(G) = 2k$ for any real number $k \in [1, \kappa(G)]$, by using Lemma 8 and constructing a $k$-resolving function $g$ on $V(G)$ with $g(V(G)) = 2k$ for $k \in [1, \kappa(G)]$. The construction of $k$-resolving function for any real number $k \in [1, \kappa(G)]$ in determining $\dim_f^k(G)$ in Proposition 28 does not appear to carry to the construction of $k$-resolving set for any integral values $k \in \{1, 2, \ldots, \kappa(G)\}$ in determining $\dim^k(G)$, and vice versa.

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