

MAXIMALLY AND SUPER CONNECTED MULTISPLIT GRAPHS AND DIGRAPHS

Litao Guo, Guifu Su, Lutz Volkmann, Xingke Zhao*

Fault tolerance is especially important for interconnection networks, since the growing size of networks increases their vulnerability to component failures. A classical measure for the fault tolerance of a network in the case of vertex failures is its connectivities. A network based on a graph $G = (X_1 \cup X_2 \cup \dots \cup X_k, I, E)$ is called a k -multisplit network, if its vertex set V can be partitioned into $k + 1$ stable sets I, X_1, X_2, \dots, X_k such that $X_1 \cup X_2 \cup \dots \cup X_k$ induces a complete k -partite graph and I is an independent set.

In this note, we first show that: for any non-complete connected k -multisplit graph $G = (X_1 \cup X_2 \cup \dots \cup X_k, I, E)$ with $k \geq 3$ and $|X_1| \leq |X_2| \leq \dots \leq |X_k|$, each of the following holds

- (1) If $|X_1 \cup X_2 \cup \dots \cup X_{k-1}| \geq \delta$, then $\kappa(G) = \delta(G)$.
- (2) If $|X_1 \cup X_2 \cup \dots \cup X_{k-1}| < \delta$, then $\kappa(G) \geq |X_1 \cup X_2 \cup \dots \cup X_{k-1}|$.
- (3) $\lambda(G) = \delta(G)$.
- (4) If $|X_1 \cup X_2 \cup \dots \cup X_{k-1}| > \delta$ with respect to $|X_1| \geq 2$ and $\delta \geq 2$, then G is super- κ .
- (5) G is super- λ .

In addition, we present sufficient conditions for digraphs to be maximally edge-connected and super-edge connected in terms of the zeroth-order general Randić index of digraphs.

1. INTRODUCTION

Interconnection networks is an important research area for parallel and distributed computer systems. Network reliability is one of the most significant factors

*Corresponding author. Guifu Su

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in designing the topology of an interconnection network. Fault tolerance is an important property of network performance. A classic measure of the fault tolerance of a network is the connectivity and edge-connectivity of the corresponding graph. In general, the larger connectivity/edge-connectivity is, the more reliable the network is.

A network can be conveniently modelled as a graph or digraph. In the note, we mainly explore sufficient conditions for k -multisplit graphs and digraphs to be super-edge-connected and maximally edge-connected, respectively. We begin with some notations and concepts for graphs and digraphs.

1.1 NOTATIONS FOR GRAPHS

Let $G = (V, E)$ be a connected graph. For any vertex $x \in V$, we use $N(x)$ to denote the set of neighbors of x in G and the number $|N(x)|$ is said to be the **degree** of x . The minimum value of these numbers is the **minimum degree** of G , denoted by $\delta = \delta(G)$. Let $E(x : G)$ denote the set of edges incident with x in G . For $X, Y \subseteq V$, we use $[X, Y]$ to denote the set of edges with one end in X and the other in Y . The **connectivity**, denoted by $\kappa = \kappa(G)$, of a connected graph G is the least positive integer k such that there is a vertex subset $S \subset V$ with $|S| = k$ and $G - S$ is disconnected or reduces to the trivial graph K_1 . We call that a connected graph G with at least two vertices have **edge-connectivity** $\lambda = \lambda(G)$ if the removal of some set of λ edges disconnects G , but there is no smaller set of edges whose removal disconnects G .

In 1932, Whitney presented a basic inequality on connectivity, edge-connectivity and minimum degree of a graph.

Theorem 1.1. ([15]) *Let G be a connected graph, then $\kappa(G) \leq \lambda(G) \leq \delta(G)$.*

It was shown in [3] that this result is best possible in the sense that if a, b and c are positive integers such that $a \leq b \leq c$, then there must exist a graph G with $\kappa(G) = a$, $\lambda(G) = b$ and $\delta(G) = c$.

Because of Theorem 1.1, it is reasonable to say that a graph G is **maximally connected** if $\kappa(G) = \delta(G)$ and write $\max\text{-}\kappa$ for simplicity; a graph G is **maximally-edge connected** if $\lambda(G) = \delta(G)$ and write $\max\text{-}\lambda$ for short. Furthermore, a graph G is **super-connected** or simply $\text{super-}\kappa$ if every minimum vertex cut is the set of the neighbors of a vertex of G , that is every minimum vertex cut isolates a vertex. Similarly, a graph G is called **super-edge-connected** or $\text{super-}\lambda$ if every minimum edge-cut is trivial; that is, if every minimum edge-cut consists of edges incident with a vertex of minimum degree. Thus every super connected (resp. super-edge-connected) graph is maximally connected (resp. maximally edge-connected). We encourage the interested reader to consult [8, 12] for more information on connectivities of graphs.

A graph $G = (V, E)$ is called a **split graph** if its vertex set V can be partitioned into a clique C and an independent set I , more details refer to [4]. Usually, the split graph G is denoted by $G = (C, I, E)$. In what follows, we always assume

that $N(I) = C$. Otherwise, if $N(I) \neq C$, then by choosing a vertex $v \in C \setminus N(I)$, and replacing C by $C \setminus \{v\}$ and I by $I \cup \{v\}$, there would produce a new split graph $G = (C \setminus \{v\}, I \cup \{v\}, E)$ such that $N(I \cup \{v\}) = C \setminus \{v\}$. It immediately follows that $\delta(G) \leq |C|$. Analogously, a graph $G = (Y \cup Z, I, E)$ is called a **bisplit graph** if its vertex set V can be partitioned into three stable sets I, X_1 and X_2 such that $X_1 \cup X_2$ induces a complete bipartite graph and I is an independent set, where $|X_2| \geq |X_1|$. We encourage the interested reader to consult [2] for more details. It is similar to split graphs, we always assume that $N(I) = X_1 \cup X_2$ for any bisplit graph $G = (X_1 \cup X_2, I, E)$.

Inspired by the definitions of split graphs and bisplit graphs, a natural generalization could be presented as follows: a graph $G = (X_1 \cup X_2 \cup \dots \cup X_k, I, E)$ is called a **k -multisplit graph**, if its vertex set V can be partitioned into $k+1$ stable sets I, X_1, X_2, \dots, X_k such that $X_1 \cup X_2 \cup \dots \cup X_k$ induces a complete k -partite graph and I is an independent set, where $|X_1| \leq |X_2| \leq \dots \leq |X_k|$. As usual, in what follows we always assume that $N(I) = X_1 \cup X_2 \cup \dots \cup X_k$.

1.2 NOTATIONS FOR DIGRAPHS

Let D be a finite and simple digraph with vertex set $V(D)$. For any vertex v of a digraph D , we denote the set of **out-neighbors** and **in-neighbors** of v be $N^+(v) = N_D^+(v)$ and $N^-(v) = N_D^-(v)$, respectively. For a vertex $v \in V(D)$, the degree of v , denoted by $d(v)$, is defined as the minimum value of its out-degree $d^+(v) = |N^+(v)|$ and its in-degree $d^-(v) = |N^-(v)|$. The **minimum out-degree** and **minimum in-degree** of a digraph D are denoted by $\delta^+(D)$ and $\delta^-(D)$. In addition, let $\delta = \delta(D) = \min\{\delta^+(D), \delta^-(D)\}$ be the **minimum degree** of D .

If X and Y are two subsets of $V(D)$, then we denote by (X, Y) the set of arcs with tail in X and head in Y . A digraph is **strongly connected** if for every pair u, v of distinct vertices there exists a directed path from u to v , D is **k -edge-connected** if for any set S of at most $k-1$ arcs the subdigraph $D - S$ is strongly connected. The **edge-connectivity** $\lambda = \lambda(D)$ of a digraph D is defined as the largest value k such that D is k -edge-connected. The inequality $\lambda(D) \leq \delta(D)$ is immediate. We call a digraph D **maximally edge-connected**, if $\lambda(D) = \delta(D)$. A digraph is called **super-edge-connected** or **super- λ** , if every minimum edge cut is trivial, that means, every minimum edge cut consists of edges adjacent to or from a vertex of minimum degree. Clearly, if D is super- λ , then $\lambda(D) = \delta(D)$.

1.3 MOTIVATION

Split graphs have many interesting properties such as simple induced subgraph characterization, linear time recognition as well as polynomial time algorithms for some NP-hard problems. A series of excellent contributions have been obtained on this topic up to now. For example, Woeginger [14] and Zhang et al. [16] studied the toughness and the scattering number of split graphs in terms of algorithms, respectively. In 2008, Metsidik and Vumar [10] considered the connectivity and edge toughness of bisplit graphs. In 2010, Guo and co-workers investigated super connectivity of Kronecker product of graphs, see more information from [5].

Three years later, Guo et al. explored the super connectivity of Kronecker product of complete graph and split graphs, see [6].

More recently, the first author of the presented paper [7] showed that split graphs and bisplit graphs are super-connected and super-edge-connected. Motivated by these results, we now turn our attention to the connectivity and super connectivity of k -multisplit graphs with $k \geq 3$. For convenience, in the following discussions, we always assume that $I \neq \emptyset$.

In this note, we first prove that: Let $G = (X_1 \cup X_2 \cdots \cup X_k, I, E)$ be a non-complete connected k -multisplit graph with $k \geq 3$ and $|X_1| \leq |X_2| \leq \cdots \leq |X_k|$, each of the following holds

- (1) If $|X_1 \cup X_2 \cdots \cup X_{k-1}| \geq \delta(G)$, then G is maximally connected.
- (2) If $|X_1 \cup X_2 \cdots \cup X_{k-1}| < \delta(G)$, then $\kappa(G) \geq |X_1 \cup X_2 \cdots \cup X_{k-1}|$.
- (3) G is maximally edge-connected.
- (4) If $|X_1 \cup X_2 \cdots \cup X_{k-1}| > \delta(G)$ with respect to $|X_1| \geq 2$ and $\delta(G) \geq 2$, then G is super-connected.
- (5) G is super-edge connected.

Sufficient conditions for graphs to be maximally edge-connected or super-edge-connected were given by several authors, see for example some recent papers [11], [12] and the survey paper by Hellwig and Volkmann [8]. To our knowledge, there are no results on this topic for digraphs. In Section 3, we shall explore sufficient conditions for digraphs to be maximally edge-connected in terms of the zeroth-order general Randić index of digraphs.

All (di)-graphs considered in this paper are simple and finite. For notation and terminology not defined here, we refer to Bondy [1].

2. MAXIMALLY EDGE-CONNECTED AND SUPER-EDGE-CONNECTED SPLIT GRAPHS

Problems concerning on relations between the (edge)-connectivity and minimum degree attracted mathematicians and theoretical computer scientists for a long time. We begin with some results for (bi)-split graphs which will be used in later proofs.

Lemma 2.2. ([17]) *Let $G = (C, I, E)$ be a non-complete connected split graph, then $\kappa(G) = \lambda(G) = \delta(G)$.*

Lemma 2.3. ([10]) *Let $G = (X_1 \cup X_2, I, E)$ be a non-complete connected bisplit graph, each of the following holds:*

- (1) *If $|X_1| \geq \delta(G)$, then $\kappa(G) = \delta(G)$.*
- (2) *If $|X_1| < \delta(G)$, then $\kappa(G) \geq |X_1|$.*

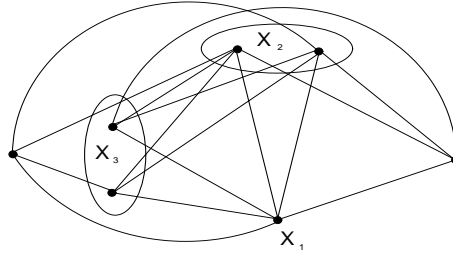


Fig.1

Illustration of Remark 1.

(3) $\lambda(G) = \delta(G)$.

In what follows, we shall focus on investigating sufficient conditions for k -multisplit graphs to be maximally connected. We begin with the following result.

Theorem 2.4. *Let $G = (X_1 \cup X_2 \cup \dots \cup X_k, I, E)$ be a non-complete multipartite connected k -multisplit graph, each of the following holds:*

- (1) *If $|X_1 \cup X_2 \cup \dots \cup X_{k-1}| \geq \delta(G)$, then $\kappa(G) = \delta(G)$ and each minimum vertex cut $S \subseteq X_1 \cup X_2 \cup \dots \cup X_k$.*
- (2) *If $|X_1 \cup X_2 \cup \dots \cup X_{k-1}| < \delta(G)$, then $\kappa(G) \geq |X_1 \cup X_2 \cup \dots \cup X_{k-1}|$.*

Proof. Suppose that $X = X_1 \cup X_2 \cup \dots \cup X_k$ and $Y = X_1 \cup X_2 \cup \dots \cup X_{k-1}$. Let S be a minimum vertex cut with $|S| \leq \delta$.

Case 1. $G - S$ does not contain the isolated vertex of I . Then $S \cap I = \emptyset$.

Otherwise, $|S \cap X| = |S| - |S \cap I| \leq \delta - 1$. Hence $G[X] - S$ is connected. Note that $N(I) = X$. We obtain $G - S$ is connected, a contradiction.

So $S \cap I = \emptyset$ and $S \subseteq X$. Furthermore, $|S| \geq \delta$. Then $\kappa(G) = \delta(G)$.

Case 2. $G - S$ contains the isolated vertex v of I . Then $N(v) \subseteq S$. We know $\delta \leq d(v) \leq |S| \leq \delta$. That is $\delta(G) = d(v) = |S| = \kappa(G)$ and $S \subseteq X$.

Let $|Y| < \delta$. Hence $G - S$ is connected for any S with $|S| < |Y|$. That is $\kappa \geq |Y|$. □

Remark 1. It follows from Fig.1 that $X_1 \cup X_2 \cup X_3$ induces a $K_{1,2,2}$ and $|I| = 2$. But $\delta = 4$, and $X_1 \cup X_2$ is a vertex cut. Hence, Theorem 2.4 is the best possible.

Remark 2. In Fig.2, $X_1 \cup X_2 \cup X_3$ induces a $K_{1,2,2}$ and $|I| = 2$. But $\delta = 3$ and $X_1 \cup X_2$ is a vertex cut, and G is not super-connected.

Theorem 2.5. *Let $G = (X_1 \cup X_2 \cup \dots \cup X_k, I, E)$ be a non-complete multipartite connected k -multisplit graph with $|X_1| \geq 2$, $\delta \geq 2$. If $|X_1 \cup X_2 \cup \dots \cup X_{k-1}| > \delta$, then G is super-connected.*

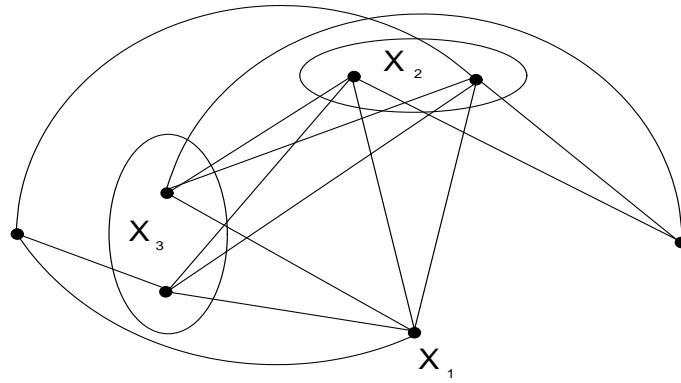


Fig.2

Illustration of Remark 2.

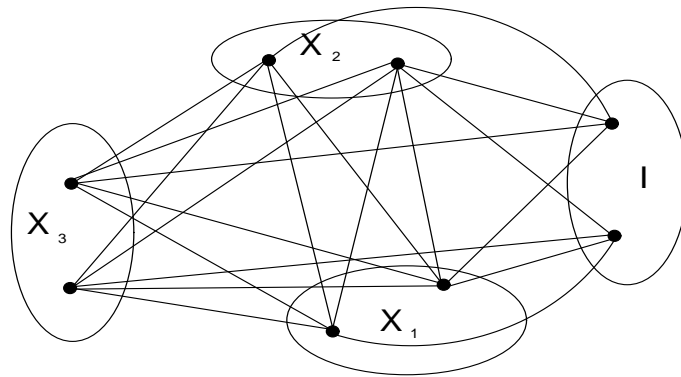


Fig.3

Illustration of Remark 3.

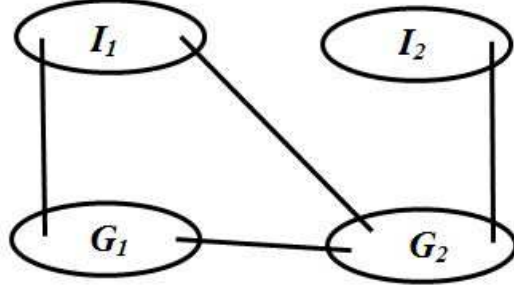


Fig.4 Illustration of Case 1.

Proof. By contradiction. Let $X = X_1 \cup X_2 \cup \dots \cup X_k$, and S be a minimum vertex cut of G with $|S| = \kappa = \delta$. Then $G - S$ is not connected and has no isolated vertices. Note that $|X_1 \cup X_2 \cup \dots \cup X_{k-1}| > \delta$, it follows from Theorem 2.4 that $S \subseteq X$. Hence $G[X - S]$ is connected. Consequently, each vertex of I is either connected to $G[X - S]$ or not connected to $G[X - S]$. This implies that $G - S$ is connected or has isolated vertices, a contradiction. \square

Remark 3. In Fig.3, $X_1 \cup X_2 \cup X_3$ induces a $K_{2,2,2}$ and $|I| = 2$. But $\delta = 4$ and $X_1 \cup X_2$ is a vertex cut, and G is not super-connected. If $k \geq 3$, $\delta = 1$, then G is super-connected. Hence, Theorem 2.5 is the best possible.

We will give a useful lemma for the edge cut of k -multisplit graphs.

Lemma 2.6. *Let $G = (X_1 \cup X_2 \cup \dots \cup X_k, I, E)$ be a non-complete multipartite connected k -split graph and $X = X_1 \cup X_2 \cup \dots \cup X_k$. And F is any minimum edge cut of G such that $F \cap [X, I] \neq \emptyset, F \cap E(G[X]) \neq \emptyset$. Set G_1 is the connected component of $G[X] - F$. Then*

- (1) *If $|G_1| = 1$, then G is super edge connected.*
- (2) *$F|_{G[X]}$ is the minimum edge cut of $G[X]$.*

Proof. (1) If $|G_1| = 1$ for any F , then it is obviously right.

(2) Suppose F is the minimum edge cut of G such that $F \cap [X, I] \neq \emptyset, F \cap E(G[X]) \neq \emptyset$. And assume that $G[X] - F$ is not connected. Assume that $F|_{G[X]}$ is not the minimum edge cut of $G[X]$ and each component of $G[X] - F$ has at least two vertices. Set $G_2 = G[X] - F - G_1$, and $I_1 = N_I(G_1), I_2 = I - I_1$.

Notice that $|G_1| \geq 2, |G_2| \geq 2$. It is depicted in Fig.4. Clearly, $[G_1, G_2]$ must be in F .

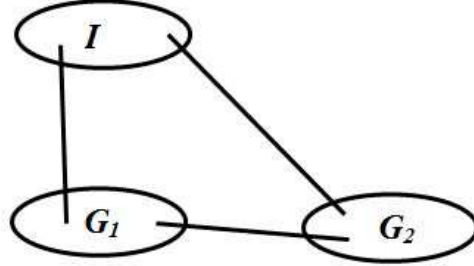


Fig.5 Illustration of Case 2.

Case 1. $I_2 \neq \emptyset$. We can see Fig.4.

Since $N(I) = X$. In G , each vertex of G_1 has a neighbor of I_1 , each vertex of I_1 has a neighbor of G_1 , and each vertex of I_2 has a neighbor of G_2 .

Note that $G[X]$ is a complete k -partite graph. If there exist two vertices $u, v \in G_1$ such that u and v are not in the same X_i , then for any vertex $y \in G_2$, we have $uy \in E$ or $vy \in E$. Hence $||G_1, G_2|| \geq |G_2|$.

Therefore, if all of vertices of G_1 are in the same X_i , then G_1 is an independent set, a contradiction.

For any $u \in I_2, d(u) \leq |G_2|$. Since $F \cap [X, I] \neq \emptyset, |F| \geq 1 + |G_2|$. Then $E(u : G)$ is another edge cut of G smaller than F , a contradiction.

Case 2. $I_2 = \emptyset$. We can see Fig.5.

There is not a vertex $u \in I$ such that $N(u) \subseteq G_1$ or $N(u) \subseteq G_2$. Otherwise we can find another edge cut smaller than F . Hence for any vertex $u \in I$, we have $N(u) \cap G_1 \neq \emptyset$ and $N(u) \cap G_2 \neq \emptyset$.

Notice that $||G_1, G_2|| \geq |G_2|, ||G_1, I|| \geq |G_1|$, and $||G_2, I|| \geq |G_2|$. We obtain $|F| \geq |G_1| + |G_2| = |X|$.

For any $u \in I, d(u) \leq |X|$. If for any $u \in I, d(u) = |X|$, then G is a complete $k + 1$ -partite graph, a contradiction. Hence there is a vertex $v \in I$ such that $d(v) < |X|$. Then $E(v : G)$ is another edge cut smaller than F , a contradiction.

□

Theorem 2.7. Let $G = (X_1 \cup X_2 \cup \dots \cup X_k, I, E)$ be a non-complete connected multipartite k -multisplit graph. Then G is super-edge connected.

Proof. For convenience, let $X = X_1 \cup X_2 \cup \dots \cup X_k$. If $\delta = 1$, then the minimum degree vertices are in I . Hence, G is super-edge connected.

In what follows, we assume that $\delta \geq 2$. Let F be a minimum edge cut of G

with $|F| = \lambda \leq \delta$. Note that a complete k -partite graph is super-edge connected for any integer k not less than three.

For simplicity, we distinguish the following three cases:

Case 1. $F \subseteq G[X]$.

If $G[X] - F$ is connected, then $G - F$ is connected, a contradiction. Hence $G[X] - F$ is not connected. It follows that there exists a vertex $u \in X$ such that $E(u : G[X]) = F$. Note that the initial condition $N(I) = X$, it yields that there must exist a vertex $v \in I$ which is adjacent to u . Hence, v must be connected to other vertices of $X \setminus \{u\}$ since $\delta \geq 2$. Therefore $G - F$ is connected, again a contradiction. Hence G is super-edge connected.

Case 2. $F \subseteq [X, I]$.

It follows that $G[X]$ is connected. Since $G - F$ is not connected, there must exist a vertex $u \in I$ such that $E(u : G[X]) = F$. Hence G is super-edge connected.

Case 3. $F \cap [X, I] \neq \emptyset$ and $F \cap G[X] \neq \emptyset$.

If $G[X] - F$ is connected, then we know each vertex of I is connected to $G[X] - F$ in $G - F$. So $G - F$ is connected, a contradiction.

Hence assume that $G[X] - F$ is not connected. Because F is the minimum edge cut of G , $F \cap E(G[X])$ is the minimum edge cut of $G[X]$ by Lemma 2.6. Let $F_1 = F \cap [X, I]$ and $F_2 = F \cap E(G[X])$. Then $G[X] - F_2$ has only two components G_1 and an isolated vertex u .

If u has some neighbors u_1, u_2, \dots, u_m ($m \geq 1$) of I in $G - F$, then $G[\{u, u_1, u_2, \dots, u_m\}] \cong K_{1,m}$ and $|\{u, u_1, u_2, \dots, u_m\}, X| \geq (m + 1)\delta - (m + 1) > \delta$, a contradiction. Then u has no neighbors in I . That is G is super-edge connected.

This completes the proof of Theorem 2.7. \square

From Theorem 2.7, we directly obtain the following result.

Corollary 2.8. *Let $G = (X_1 \cup X_2 \cup \dots \cup X_k, I, E)$ be a non-complete multipartite connected k -multisplit graph of order n . Then $\lambda(G) = \delta(G)$.*

3. MAXIMALLY EDGE-CONNECTED AND SUPER-EDGE-CONNECTED DIGRAPHS

In this section, we will explore sufficient conditions for digraphs to be maximally edge-connected and super-edge connected in terms of a new topological index.

Let D be a digraph with minimum degree $\delta \geq 1$. If α is a real number, the **zeroth-order general Randić index** of D was proposed by Volkmann in [13]:

$$R_\alpha^0(D) = \sum_{x \in V(D)} [d(x)]^\alpha.$$

We begin with several basic lemmas, which we use in the proofs of our main results.

Lemma 3.9. ([9]) *If $x - 2 \geq y \geq 1$ and $\alpha \in (-\infty, 0) \cup (1, +\infty)$ is a real number, then*

$$(x - 1)^\alpha + (y + 1)^\alpha < x^\alpha + y^\alpha.$$

Lemma 3.10. ([12]) *Let $\alpha > 1$ be a real number, and let a_1, a_2, \dots, a_p and A be positive reals such that $\sum_{i=1}^p a_i \leq A$. If in addition, a_1, a_2, \dots, a_p, A are positive integers, and a, b are integers with $A = ap + b$ and $0 \leq b < p$, then*

$$\sum_{i=1}^p a_i^\alpha \geq (p - b)a^\alpha + b(a + 1)^\alpha.$$

The **associated digraph** $D(G)$ of a graph G is obtained by replacing each edge of G by a pair of mutually opposite oriented arcs.

The following observation is simple but useful.

Observation 3.11. *Let G be a graph and $D(G)$ its associated digraph, then $\lambda(G) = \lambda(D(G))$.*

Theorem 3.12. *Let D be a strongly connected digraph of order $n \geq 3$, minimum degree δ and edge-connectivity λ , and let $\alpha > 1$ be a real number. If*

$$R_\alpha^0(D) < n \left(\frac{n-2}{2} \right)^\alpha + (\delta - 1) \left[2 \left(\frac{n}{2} \right)^\alpha - \delta^\alpha - (n - \delta - 2)^\alpha \right],$$

then D is maximally-edge connected.

Proof. If $\delta = 1$, then $\lambda = \delta$ in every case. In what follows, we always assume that $\delta \geq 2$. Suppose to the contrary that $\lambda \leq \delta - 1$, then there exist two disjoint sets $X, Y \subset V(D)$ such that $X \cup Y = V(D)$ and $|(X, Y)| = \lambda$. We first show that $\delta + 1 \leq |X|, |Y| \leq n - \delta - 1$. Suppose that X contains at most δ vertices. Since every vertex in X has at most $|X| - 1$ out-neighbors in X and there are at most λ arcs from X to Y , we obtain the contradiction

$$\delta|X| \leq \sum_{x \in X} d^+(x) \leq |X|(|X| - 1) + \lambda \leq \delta(|X| - 1) + \delta - 1 = \delta|X| - 1.$$

Therefore $|X| \geq \delta + 1$. By the similar approach, we can show that $|Y| \geq \delta + 1$.

It follows from the above

$$\sum_{x \in X} d(x) \leq \sum_{x \in X} d^-(x) \leq |X|(|X| - 1) + \delta - 1.$$

Applying Lemma 3.10, we deduce that

$$\begin{aligned} \sum_{x \in X} [d(x)]^\alpha &\geq (|X| - (\delta - 1))(|X| - 1)^\alpha + (\delta - 1)|X|^\alpha \\ &= (|X| - 1)^\alpha + (|X| - 1)^{\alpha+1} + (\delta - 1)[|X|^\alpha - (|X| - 1)^\alpha]. \end{aligned}$$

Similarly, we have

$$\sum_{y \in Y} [d(y)]^\alpha \geq (|Y| - 1)^\alpha + (|Y| - 1)^{\alpha+1} + (\delta - 1)[|Y|^\alpha - (|Y| - 1)^\alpha].$$

Combing the previous two inequalities, together with the initial condition $|X| + |Y| = n$, we get

$$\begin{aligned} R_\alpha^0(D) &= \sum_{x \in X} [d(x)]^\alpha + \sum_{y \in Y} [d(y)]^\alpha \\ &\geq (|X| - 1)^{\alpha+1} + (|X| - 1)^\alpha + (|Y| - 1)^{\alpha+1} + (|Y| - 1)^\alpha \\ &\quad + (\delta - 1)[|X|^\alpha - (|X| - 1)^\alpha + |Y|^\alpha - (|Y| - 1)^\alpha]. \\ &= (|X| - 1)^{\alpha+1} + (n - |X| - 1)^{\alpha+1} + (|X| - 1)^\alpha + (n - |X| - 1)^\alpha \\ &\quad + (\delta - 1)[|X|^\alpha + (n - |X|)^\alpha - ((|X| - 1)^\alpha + (n - |X| - 1)^\alpha)]. \end{aligned}$$

To accomplish the proof, it is sufficient to find the sharp lower bound of the following function

$$\begin{aligned} Q_1(X, Y) &= (|X| - 1)^{\alpha+1} + (n - |X| - 1)^{\alpha+1} + (|X| - 1)^\alpha + (n - |X| - 1)^\alpha \\ &\quad + (\delta - 1)[|X|^\alpha + (n - |X|)^\alpha - ((|X| - 1)^\alpha + (n - |X| - 1)^\alpha)]. \end{aligned}$$

Note that the functions $H_1(t) = (t - 1)^{\alpha+1} + (n - t - 1)^{\alpha+1}$, $H_2(t) = (t - 1)^\alpha + (n - t - 1)^\alpha$ and $H_3(t) = t^\alpha + (n - t)^\alpha$ have their minimum for $t = n/2$, and the function $H_2(t)$ has its maximum for $H_2(t) = \delta^\alpha + (n - \delta - 2)^\alpha$ when $\delta + 1 \leq t \leq n - \delta - 1$ and $\alpha > 1$, which immediately yields that

$$\begin{aligned} R_\alpha^0(D) &\geq \left(\frac{n}{2} - 1\right)^{\alpha+1} + \left(\frac{n}{2} - 1\right)^\alpha + \left(\frac{n}{2} - 1\right)^{\alpha+1} + \left(\frac{n}{2} - 1\right)^\alpha \\ &\quad + (\delta - 1) \left[\left(\frac{n}{2}\right)^\alpha + \left(\frac{n}{2}\right)^\alpha - \delta^\alpha - (n - \delta - 2)^\alpha \right] \\ &= n \left(\frac{n-2}{2}\right)^\alpha + (\delta - 1) \left[2 \left(\frac{n}{2}\right)^\alpha - \delta^\alpha - (n - \delta - 2)^\alpha \right], \end{aligned}$$

which contradicts the hypothesis. Therefore $\lambda = \delta$. \square

We write $D(K_n)$ for the complete digraph of order n . The following example will show that the condition in Theorem 3.12 is best possible.

Example 3.13. Let n and δ be integers such that $n = 2\delta + 2 \geq 6$. Furthermore, let $H_1 = D(K_{\delta+1})$ with vertex set $V(H_1) = \{x_1^1, x_2^1, \dots, x_{\delta+1}^1\}$, and let $H_2 = D(K_{\delta+1})$ with vertex set $V(H_2) = \{x_1^2, x_2^2, \dots, x_{\delta+1}^2\}$. Define the digraph H by the union of H_1 and H_2 together with the $2\delta - 2$ arcs $x_1^1x_1^2, x_2^1x_2^2, \dots, x_{\delta-1}^1x_{\delta-1}^2$ as well as $x_1^2x_1^1, x_2^2x_2^1, \dots, x_{\delta-1}^2x_{\delta-1}^1$. Then $n(H) = n$, $\delta(H) = \delta$ and

$$R_\alpha^0(H) = 4\delta^\alpha + (2\delta - 2)(\delta + 1)^\alpha.$$

Therefore

$$R_\alpha^0(H) = n \left(\frac{n-2}{2} \right)^\alpha + (\delta - 1) \left[2 \left(\frac{n}{2} \right)^\alpha - \delta^\alpha - (n - \delta - 2)^\alpha \right]$$

and so equality in the inequality of Theorem 3.12. However, $\lambda(H) = \delta(H) - 1$.

Corollary 3.14. Let G be a connected graph of order $n \geq 3$, minimum degree δ and edge-connectivity λ , and let $\alpha > 1$ be a real number. If

$$R_\alpha^0(G) < n \left(\frac{n-2}{2} \right)^\alpha + (\delta - 1) \left[2 \left(\frac{n}{2} \right)^\alpha - \delta^\alpha - (n - \delta - 2)^\alpha \right],$$

then D is maximally-edge connected.

Proof. Since $N_G(v) = N_{D(G)}^+(v) = N_{D(G)}^-(v)$, for each vertex $v \in V(G) = V(D(G))$, we observe that $n(G) = n(D(G))$, $\delta(G) = \delta(D(G))$ and $R_\alpha^0(G) = R_\alpha^0(D(G))$. Thus Theorem 3.12 and Observation 3.11 imply the desired result. \square

Theorem 3.15. Let D be a strongly connected digraph of order n , minimum degree $\delta \geq 2$ and edge-connectivity λ , and let $\alpha > 1$ be a real number. If

$$R_\alpha^0(D) < n \left(\frac{n-2}{2} \right)^\alpha + \delta \left[2 \left(\frac{n}{2} \right)^\alpha - (\delta - 1)^\alpha - (n - \delta - 1)^\alpha \right],$$

then D is super-edge connected.

Proof. Suppose to the contrary that D is not super- λ , then there exist two disjoint sets $X, Y \subset V(D)$ such that $X \cup Y = V(D)$, $|X|, |Y| \geq 2$ and $|(X, Y)| = \lambda$. We first show that $\delta \leq |X|, |Y| \leq n - \delta$. Suppose that X contains at most $\delta - 1$ vertices when $\delta \geq 3$. Since every vertex in X has at most $|X| - 1$ out-neighbors in X and there are at most λ arcs from X to Y , we obtain the contradiction

$$\delta|X| \leq \sum_{x \in X} d^+(x) \leq |X|(|X| - 1) + \lambda \leq (\delta - 1)(|X| - 1) + \delta = \delta|X| - |X| + 1.$$

Therefore $|X| \geq \delta$. Similar analysis shows that $|Y| \geq \delta$.

It immediately follows that

$$\sum_{x \in X} d(x) \leq \sum_{x \in X} d^-(x) \leq |X|(|X| - 1) + \delta.$$

Applying Lemma 3.10 in the case $|X| \geq \delta + 1$, we deduce that

$$\begin{aligned} \sum_{x \in X} [d(x)]^\alpha &\geq (|X| - \delta)(|X| - 1)^\alpha + \delta|X|^\alpha \\ &= (|X| - 1)^\alpha + (|X| - 1)^{\alpha+1} + \delta[|X|^\alpha - (|X| - 1)^\alpha]. \end{aligned}$$

If $|X| = \delta$, then

$$\sum_{x \in X} d(x) \leq \sum_{x \in X} d^-(x) \leq |X|(|X| - 1) + \delta = |X|^2.$$

Applying again Lemma 3.10, we see that

$$\sum_{x \in X} [d(x)]^\alpha \geq \delta^{\alpha+1} = (|X| - 1)^\alpha + (|X| - 1)^{\alpha+1} + \delta[|X|^\alpha - (|X| - 1)^\alpha].$$

Similar calculations shows that

$$\sum_{y \in Y} [d(y)]^\alpha \geq (|Y| - 1)^\alpha + (|Y| - 1)^{\alpha+1} + \delta[|Y|^\alpha - (|Y| - 1)^\alpha].$$

It follows from the previous two inequalities, in conjunction with the initial condition $|X| + |Y| = n$, that

$$\begin{aligned} R_\alpha^0(D) &= \sum_{x \in X} [d(x)]^\alpha + \sum_{y \in Y} [d(y)]^\alpha \\ &\geq (|X| - 1)^{\alpha+1} + (|X| - 1)^\alpha + (|Y| - 1)^{\alpha+1} + (|Y| - 1)^\alpha \\ &\quad + \delta[|X|^\alpha - (|X| - 1)^\alpha + |Y|^\alpha - (|Y| - 1)^\alpha]. \\ &= (|X| - 1)^{\alpha+1} + (n - |X| - 1)^{\alpha+1} + (|X| - 1)^\alpha + (n - |X| - 1)^\alpha \\ &\quad + \delta[|X|^\alpha + (n - |X|)^\alpha - ((|X| - 1)^\alpha + (n - X - 1)^\alpha)]. \end{aligned}$$

To accomplish the proof, it is sufficient to find the minimum of the function

$$\begin{aligned} Q_2(X, Y) &= (|X| - 1)^{\alpha+1} + (n - |X| - 1)^{\alpha+1} + (|X| - 1)^\alpha + (n - |X| - 1)^\alpha \\ &\quad + \delta[|X|^\alpha + (n - |X|)^\alpha - ((|X| - 1)^\alpha + (n - X - 1)^\alpha)]. \end{aligned}$$

Note that the functions $H_1(t) = (t - 1)^{\alpha+1} + (n - t - 1)^{\alpha+1}$, $H_2(t) = (t - 1)^\alpha + (n - t - 1)^\alpha$ and $H_3(t) = t^\alpha + (n - t)^\alpha$ have their minimum for $t = n/2$, and the function $H_2(t)$ has its maximum $H_2(t) = (\delta - 1)^\alpha + (n - \delta - 1)^\alpha$ when $\delta \leq t \leq n - \delta$ and $\alpha > 1$, it yields that

$$\begin{aligned} R_\alpha^0(D) &\geq \left(\frac{n}{2} - 1\right)^{\alpha+1} + \left(\frac{n}{2} - 1\right)^\alpha + \left(\frac{n}{2} - 1\right)^{\alpha+1} + \left(\frac{n}{2} - 1\right)^\alpha \\ &\quad + \delta \left[\left(\frac{n}{2}\right)^\alpha + \left(\frac{n}{2}\right)^\alpha - (\delta - 1)^\alpha - (n - \delta - 1)^\alpha \right] \\ &= n \left(\frac{n-2}{2}\right)^\alpha + \delta \left[2 \left(\frac{n}{2}\right)^\alpha - (\delta - 1)^\alpha - (n - \delta - 1)^\alpha \right] \end{aligned}$$

again a contradiction. Therefore D is super-edge connected. \square

The next example shows that the condition in Theorem 3.15 is best possible.

Example 3.16. Let n and δ be integers such that $n = 2\delta \geq 6$. Furthermore, let $H_1 = D(K_\delta)$ with vertex set $V(H_1) = \{x_1^1, x_2^1, \dots, x_\delta^1\}$, and let $H_2 = D(K_\delta)$ with vertex set $V(H_2) = \{x_1^2, x_2^2, \dots, x_\delta^2\}$. Define the digraph H by the union of H_1 and H_2 together with the 2δ arcs $x_1^1x_1^2, x_2^1x_2^2, \dots, x_\delta^1x_\delta^2$ as well as $x_1^2x_1^1, x_2^2x_2^1, \dots, x_\delta^2x_\delta^1$. Then $n(H) = n$, $\delta(H) = \delta$ and

$$R_\alpha^0(H) = 2\delta^{\alpha+1}.$$

Therefore

$$R_\alpha^0(H) = n \left(\frac{n-2}{2} \right)^\alpha + \delta \left[2 \left(\frac{n}{2} \right)^\alpha - (\delta-1)^\alpha - (n-\delta-1)^\alpha \right]$$

and so equality in the inequality of Theorem 3.15. However, H is not super-edge connected.

The following is an immediate consequence of Theorem 3.15.

Corollary 3.17. Let G be a connected graph of order n , minimum degree $\delta \geq 2$ and edge-connectivity λ , and let $\alpha > 1$ be a real number. If

$$R_\alpha^0(G) < n \left(\frac{n-2}{2} \right)^\alpha + \delta \left[2 \left(\frac{n}{2} \right)^\alpha - (\delta-1)^\alpha - (n-\delta-1)^\alpha \right],$$

then G is super-edge connected.

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Litao Guo

School of Applied Mathematics,
Xiamen University of Technology,
Fujian, 361024, PR China
E-mail: *ltguo2012@126.com*

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Guifu Su

School of Science,
Beijing University of Chemical Technology,
Beijing, 100029, PR China
E-mail: *gfs@mail.buct.edu.cn*

Lutz Volkmann

Lehrstuhl II für Mathematik,
RWTH Aachen University,
Aachen, 52056, Germany
E-mail: *volkm@math2.rwth-aachen.de*

Xingke Zhao

Editorial Department of Journal
of Xinjiang University,
Xinjiang, 830046, PR China