

ON A NEW (p, q) -MATHIEU-TYPE POWER SERIES
AND ITS APPLICATIONS

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Dedicated to Prof. Gradimir Milovanovic on the occasion on his 70th birthday.

Our aim in this paper, is to establish certain new integral representations for the (p, q) -Mathieu-type power series. In particular, we investigate the Mellin-Barnes type integral representations for a particular case of these special function. Moreover, we introduce the notion of the (p, q) -Mittag-Leffler functions and we present a relationships between these two functions. Some other applications are proved, in particular two Turán type inequalities for the (p, q) -Mathieu-type series are derived.

1. INTRODUCTION

The following familiar infinite series

$$(1) \quad S(r) = \sum_{n=1}^{\infty} \frac{2n}{(n^2 + r^2)^2},$$

is called a Mathieu series. It was introduced and studied by Émile Leonard Mathieu in his book [7] devoted to the elasticity of solid bodies. Bounds for this series are needed for the solution of boundary value problems for the biharmonic equations in a two-dimensional rectangular domain, see [11, Eq. (54), p. 258].

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Several interesting problems and solutions dealing with integral representations and bounds for the following generalization of the Mathieu series, which is so-called generalized Mathieu series with a fractional power can be found in [3, 8, 14, 16]:

$$S_\mu(r) = \sum_{n=1}^\infty \frac{2n}{(n^2 + r^2)^{\mu+1}} \quad (\mu > 0, r > 0).$$

In [14], the authors derived the following new Laplace type integral representation via Schlömilch series:

$$S_\mu(r) = \frac{\sqrt{\pi}}{2^{\mu-\frac{1}{2}}\Gamma(\mu+1)} \int_0^\infty e^{-rt} \mathcal{K}_\mu(t) dt \quad (\mu > \frac{3}{2}),$$

where

$$\mathcal{K}_\mu(t) = t^{\mu+\frac{1}{2}} \sum_{k=1}^\infty \frac{J_{\mu+\frac{1}{2}}(kt)}{k^{\mu-\frac{1}{2}}},$$

and $J_\mu(z)$ is the Bessel function. Motivated essentially by the works of Cerone and Lenard [1], Srivastava and Tomovski in [13] defined a family of generalized Mathieu series

$$(2) \quad S_\mu^{(\alpha,\beta)}(r; \mathbf{a}) = S_\mu^{(\alpha,\beta)}(r; \{a_k\}_{k=0}^\infty) = \sum_{k=1}^\infty \frac{2a_k^\beta}{(a_k^\alpha + r^2)^\mu} \quad (\alpha, \beta, \mu, r > 0),$$

where it is tacitly assumed that the positive sequence

$$\mathbf{a} = \{a_k\} = \{a_1, a_2, \dots\}, \text{ such that } \lim_{k \rightarrow \infty} a_k = \infty,$$

is so chosen that the infinite series in definition (2) converges, that is, that the following auxiliary series

$$\sum_{k=1}^\infty \frac{1}{a_k^{\mu\alpha-\beta}},$$

is convergent.

Definition 1. (see [12, Eq. (6.1), p. 256]) The extended Beta function $B_{p,q}(x, y)$ is defined by

$$(3) \quad B_{p,q}(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} E_{p,q}(t) dt$$

$$(x, y, p, q \in \mathbb{C}, \min(\Re(x), \Re(y)) > 0, \min(\Re(p), \Re(q)) \geq 0),$$

where $E_{p,q}(t)$ is defined by

$$E_{p,q}(t) = \exp\left(-\frac{p}{t} - \frac{q}{1-t}\right)$$
$$(p, q \in \mathbb{C}, \min(\Re(p), \Re(q)) \geq 0).$$

In particular, Chaudhry et al. [2, p. 591, Eq. (1.7)], introduced the p-extension of Euler’s Beta function B(x, y) :

$$B_p(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}e^{-\frac{p}{t(1-t)}} dt \quad (\Re(p) > 0),$$

whose special case, when p = 0 (or p = q = 0 in (3)), is the familiar Beta integral

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt \quad (\Re(x), \Re(y) > 0).$$

Definition 2. [5] Assume that p, q ∈ C_> = {z ∈ C : Re(z) > 0}, λ, μ, s ∈ C and ν, a ∈ C \ Z_0^-. The extended Hurwitz-Lerch zeta function is defined by

$$(4) \quad \Phi_{\lambda, \mu, \nu}(z, s, a; p, q) = \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \frac{B_{p,q}(\mu+n, \nu-\mu)}{B(\mu, \nu-\mu)} \frac{z^n}{(a+n)^s} \quad (|z| < 1),$$

where (λ)_n denotes the Pochhammer symbol (or the shifted factorial) defined, in terms of Euler’s Gamma function, by

$$(\lambda)_\mu = \frac{\Gamma(\lambda + \mu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\mu = 0; \lambda \in \mathbb{C} \setminus \{0\}), \\ \lambda(\lambda + 1)\dots(\lambda + n - 1) & (\mu = n \in \mathbb{N}; \lambda \in \mathbb{C}). \end{cases}$$

On the unit circle |z| = 1, the series (4) converges absolutely if, in addition, we set one of the following conditions [5, Theorem 3]:

$$\Re(s - \lambda) > 0, \text{ or } \Re(\nu) > \Re(\mu) > 0 \text{ and } \Re(\nu - \mu) > \Re(\lambda - s).$$

Upon setting λ = 1, (4) reduces to

$$\Phi_{\mu, \nu}(z, s, a; p, q) = \sum_{n=0}^{\infty} \frac{B_{p,q}(\mu+n, \nu-\mu)}{B(\mu, \nu-\mu)} \frac{z^n}{(a+n)^s} \quad (|z| < 1).$$

It is easy to observe that

$$(5) \quad \Phi_{\lambda, \mu, \nu}(z, s, a; p, q) = \frac{1}{\Gamma(\lambda)} D_z^{\lambda-1} \{z^{\lambda-1} \Phi_{\mu, \nu}(z, s, a; p, q)\} \quad (\Re(\lambda) > 0),$$

where D_z^λ denotes the well-known Riemann-Liouville fractional derivative operator defined by

$$D_z^\lambda f(z) = \begin{cases} \frac{1}{\Gamma(-\lambda)} \int_0^z (z-t)^{-\lambda-1} f(t) dt & (\Re(\lambda) < 0), \\ \frac{d^m}{dz^m} D_z^{\lambda-m} f(z) & (m-1 \leq \Re(\lambda) < m, (m \in \mathbb{N})). \end{cases}$$

In [5, Theorem 3.8] Luo et al. proved the following integral representation for the extended Hurwitz-Lerch zeta function Φ_{λ, μ, ν}(z, s, a; p, q) :

$$(6) \quad \Phi_{\lambda, \mu, \nu}(z, s, a; p, q) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-at} {}_2F_1 \left[\begin{matrix} \lambda, \mu \\ \nu \end{matrix}; ze^{-t}; p, q \right] dt$$

$$(|z| < 1, p, q, a, s \in \mathbb{C}_>, \lambda, \mu \in \mathbb{C}, \nu \in \mathbb{C} \setminus \mathbb{Z}_0^-)$$

where ${}_2F_1 \left[\begin{smallmatrix} a, b \\ c \end{smallmatrix}; z; p, q \right]$ is the extended Gauss hypergeometric function defined by [12, Eq. (6.2)]

$${}_2F_1 \left[\begin{smallmatrix} a, b \\ c \end{smallmatrix}; z; p, q \right] = \sum_{n=0}^{\infty} (a)_n \frac{B_{p,q}(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!}$$

$$(|z| < 1, \Re(c) > \Re(b) > 0, \min(\Re(p), \Re(q)) \geq 0).$$

When $p = q$, we obtain the extended Gaussian hypergeometric function ${}_2F_1$ defined by [2]:

$${}_2F_1 \left[\begin{smallmatrix} a, b \\ c \end{smallmatrix}; z; p \right] = \sum_{n=0}^{\infty} (a)_n \frac{B_p(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!}$$

$$(|z| < 1, \Re(p) \geq 0, \Re(c) > \Re(b) > 0).$$

The integral representation of the extended Hurwitz-Lerch zeta function (6) has an important role to find the integral representation for the extended Mathieu-type power series defined in (8).

The Fox-Wright function ${}_p\Psi_q[\cdot]$ with p numerator parameters $\alpha_1, \dots, \alpha_p$ and q denominator parameters β_1, \dots, β_q is defined by

$$(7) \quad {}_p\Psi_q \left[\begin{smallmatrix} (\alpha_1, A_1), \dots, (\alpha_p, A_p) \\ (\beta_1, B_1), \dots, (\beta_q, B_q) \end{smallmatrix} \middle| z \right] = {}_p\Psi_q \left[\begin{smallmatrix} (\alpha_p, A_p) \\ (\beta_q, B_q) \end{smallmatrix} \middle| z \right] = \sum_{k=0}^{\infty} \frac{\prod_{l=1}^p \Gamma(\alpha_l + kA_l)}{\prod_{j=1}^q \Gamma(\beta_j + kB_j)} \frac{z^k}{k!}.$$

The defining series in (7) converges in the whole complex z -plane when

$$\Delta = \sum_{j=1}^q B_j - \sum_{j=1}^p A_j > -1;$$

when $\Delta = 0$, then the series in (7) converges for $|z| < \nabla$, where

$$\nabla = \left(\prod_{j=1}^p A_j^{-A_j} \right) \left(\prod_{j=1}^q B_j^{B_j} \right).$$

If, in the definition (7) we set

$$A_1 = \dots = A_p = 1 \quad \text{and} \quad B_1 = \dots = B_q = 1,$$

we get the relatively more familiar generalized hypergeometric function ${}_pF_q[\cdot]$ given by

$${}_pF_q \left[\begin{smallmatrix} \alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q \end{smallmatrix} \middle| z \right] = \frac{\prod_{j=1}^q \Gamma(\beta_j)}{\prod_{i=1}^p \Gamma(\alpha_i)} {}_p\Psi_q \left[\begin{smallmatrix} (\alpha_1, 1), \dots, (\alpha_p, 1) \\ (\beta_1, 1), \dots, (\beta_q, 1) \end{smallmatrix} \middle| z \right].$$

In terms of the extended Beta function (3) we now define the (p, q)-Mathieu-type power series by:

$$(8) \quad S_{\mu, \nu, \tau, \omega}^{(\alpha, \beta)}(r; \mathbf{a}; p, q; z) = \sum_{n=1}^{\infty} \frac{2a_n^\beta(\nu)_n B_{p, q}(\tau + n, \omega - \tau) z^n}{n! B(\tau, \omega - \tau) (a_n^\alpha + r^2)^\mu}$$

$$(r, \alpha, \beta, \nu > 0, \omega > \tau > 0, p, q \in \mathbb{C}, \min(\Re(p), \Re(q)) \geq 0, |z| \leq 1).$$

In particular case when $p = q$, we define the p -Mathieu-type power series defined by:

$$(9) \quad S_{\mu, \nu, \tau, \omega}^{(\alpha, \beta)}(r; \mathbf{a}; p; z) = \sum_{n=1}^{\infty} \frac{2a_n^\beta(\nu)_n B_p(\tau + n, \omega - \tau) z^n}{n! B(\tau, \omega - \tau) (a_n^\alpha + r^2)^\mu}$$

$$(r, \alpha, \beta, \nu, \omega, \tau > 0, p \in \mathbb{C}_>, |z| \leq 1).$$

The function $S_{\mu, \nu, \tau, \omega}^{(\alpha, \beta)}(r; \mathbf{a}; p, q; z)$ has many other special cases. If we set $p = q = 0$, we get

$$(10) \quad S_{\mu, \nu, \tau, \omega}^{(\alpha, \beta)}(r; \mathbf{a}; z) = S_{\mu, \nu, \tau, \omega}^{(\alpha, \beta)}(r; \mathbf{a}; 0, 0; z) = \sum_{n=1}^{\infty} \frac{2a_n^\beta(\nu)_n (\tau)_n z^n}{n! (\omega)_n (a_n^\alpha + r^2)^\mu}$$

$$(r, \alpha, \beta, \nu, \tau, \omega > 0, |z| \leq 1).$$

On the other hand, by letting $\tau = \omega$ in (10) we obtain [15, Eq. 5, p. 974]:

$$(11) \quad S_{\mu, \nu}^{(\alpha, \beta)}(r; \mathbf{a}; z) = S_{\mu, \nu, \tau, \tau}^{(\alpha, \beta)}(r; \mathbf{a}; z) = \sum_{n=1}^{\infty} \frac{2a_n^\beta(\nu)_n z^n}{n! (a_n^\alpha + r^2)^\mu}$$

$$(r, \alpha, \beta, \nu > 0, |z| \leq 1).$$

Furthermore, in the special cases when $\nu = z = 1$, we get the generalized Mathieu series (2).

The contents of our paper is organized as follows. In section 2, we present new integral representation for the (p, q)-Mathieu-type series. In particular, we derive the Mellin-Barnes type integral representations for (p, q)-Mathieu-type series. As applications, In Section 3, we introduce the (p, q)-Mittag-Leffler functions and we derive some relationships between these two special functions, in particular we derive new series representations for the (p, q)-Mathieu-type series. Relationships between the (p, q)- and generalized Mathieu-type series are proved and two Turán type inequalities are established.

2. INTEGRAL REPRESENTATION FOR THE (p, q)-MATHIEU TYPE POWER SERIES

In the course of our investigation, one of the main tools is the following result providing the integral representation for the (p, q)-Mathieu-type series

$$S_{\mu, \nu, \tau, \omega}^{(\alpha, \beta)}(r; \{k^\gamma\}_{k=0}^\infty; p, q; z).$$

Theorem 1. Let $r, \alpha, \beta, \nu, \mu, \omega, \tau > 0$, $p, q \in \mathbb{C}_>$, such that $\tau < \omega$ and $\gamma(\mu\alpha - \beta) > 0$. Then (p, q) -Mathieu-type series $S_{\mu, \nu, \tau, \omega}^{(\alpha, \beta)}(r; \{k^\gamma\}_{k=0}^\infty; p, q; z)$ possesses the integral representation given by:

$$(12) \quad S_{\mu, \nu, \tau, \omega}^{(\alpha, \beta)}(r; \{k^\gamma\}_{k=0}^\infty; p, q; z) = \frac{2\nu\tau z}{\omega\Gamma(\mu)} \int_0^\infty t^{\gamma(\mu\alpha - \beta)} e^{-t} K_{p, q}^{(\alpha, \beta)}(\mu, \gamma, \tau, \omega; t) dt,$$

where $K_{p, q}^{(\alpha, \beta)}(\mu, \gamma, \tau, \omega; t)$ is defined by

$$K_{p, q}^{(\alpha, \beta)}(\mu, \gamma, \tau, \omega; t) = {}_2F_1 \left[\begin{matrix} \nu+1, \tau+1 \\ \omega+1 \end{matrix}; ze^{-t}; p, q \right] {}_1\Psi_1 \left[\begin{matrix} (\mu, 1) \\ (\gamma(\mu\alpha - \beta) + 1, \gamma\alpha) \end{matrix} \middle| -r^2 t^{\gamma\alpha} \right].$$

Proof. By using the definition (8), we can write the extended Mathieu-type series $S_{\mu, \nu, \tau, \omega}^{(\alpha, \beta)}(r; \mathbf{a}; p, q; z)$ in the following form:

$$(13) \quad S_{\mu, \nu, \tau, \omega}^{(\alpha, \beta)}(r; \mathbf{a}; p, q; z) = 2 \sum_{m=0}^\infty \binom{\mu + m - 1}{m} (-r^2)^m \sum_{n=1}^\infty \frac{(\nu)_n}{a_n^{(\mu+m)\alpha - \beta}} \frac{B_{p, q}(\tau + n, \omega - \tau)}{B(\tau, \omega - \tau)} \frac{z^n}{n!}.$$

Putting,

$$\chi_{n, m}^{(\alpha, \beta)}(\mu, \gamma) = (n + 1)^{\gamma((\mu+m)\alpha - \beta) + 1},$$

$$\Phi_{m, \mu, \nu}^{(\alpha, \beta)}(\tau, \omega; p, q; z) = \Phi_{\nu+1, \tau+1, \omega+1}(z, \gamma[(\mu + m)\alpha - \beta] + 1, 1; p, q),$$

by (6), we get

$$\begin{aligned} & S_{\mu, \nu, \tau, \omega}^{(\alpha, \beta)}(r; \{k^\gamma\}_{k=0}^\infty; p, q; z) = \\ &= 2z \sum_{m=0}^\infty \binom{\mu + m - 1}{m} (-r^2)^m \sum_{n=0}^\infty \frac{(\nu)_{n+1}}{(n + 1)!} \frac{B_{p, q}(\tau + 1 + n, \omega - \tau)}{B(\tau, \omega - \tau)} \frac{z^n}{\chi_{n, m}^{(\alpha, \beta)}(\mu, \gamma)} \\ &= 2\nu z \sum_{m=0}^\infty \binom{\mu + m - 1}{m} (-r^2)^m \sum_{n=0}^\infty \frac{(\nu + 1)_n}{n!} \frac{B_{p, q}(\tau + 1 + n, \omega - \tau)}{B(\tau, \omega - \tau)} \frac{z^n}{\chi_{n, m}^{(\alpha, \beta)}(\mu, \gamma)} \\ &= \frac{2\nu z B(\tau + 1, \omega - \tau)}{B(\tau, \omega - \tau)} \sum_{m=0}^\infty \binom{\mu + m - 1}{m} (-r^2)^m \sum_{n=0}^\infty \frac{(\nu + 1)_n B_{p, q}(\tau + 1 + n, \omega - \tau) z^n}{n! B(\tau + 1, \omega - \tau) \chi_{n, m}^{(\alpha, \beta)}(\mu, \gamma)} \\ &= \frac{2\nu z B(\tau + 1, \omega - \tau)}{B(\tau, \omega - \tau)} \sum_{m=0}^\infty \binom{\mu + m - 1}{m} \Phi_{m, \mu, \nu}^{(\alpha, \beta)}(\tau, \omega; p, q; z) (-r^2)^m \\ &= \frac{2\nu\tau z}{\omega} \int_0^\infty t^{\gamma(\mu\alpha - \beta)} e^{-t} {}_2F_1 \left[\begin{matrix} \nu+1, \tau+1 \\ \omega+1 \end{matrix}; ze^{-t}; p, q \right] \left(\sum_{m=0}^\infty \frac{\binom{\mu+m-1}{m} (-r^2 t^{\gamma\alpha})^m}{\Gamma(\gamma[(\mu + m)\alpha - \beta] + 1)} \right) dt \\ &= \frac{2\nu\tau z}{\omega\Gamma(\mu)} \int_0^\infty t^{\gamma(\mu\alpha - \beta)} e^{-t} {}_2F_1 \left[\begin{matrix} \nu+1, \tau+1 \\ \omega+1 \end{matrix}; ze^{-t}; p, q \right] {}_1\Psi_1 \left[\begin{matrix} (\mu, 1) \\ (\gamma(\mu\alpha - \beta) + 1, \gamma\alpha) \end{matrix} \middle| -r^2 t^{\gamma\alpha} \right] dt. \end{aligned}$$

This completes the proof of Theorem 1. □

Now, in the case $p = q$, theorem 1 reduces to the following corollary.

Corollary 1. *Let $r, \alpha, \beta, \nu, \mu, \tau, \omega > 0$, $p \in \mathbb{C}_>$ such that $\tau < \omega$ and $\gamma(\mu\alpha - \beta) > 0$. Then p -Mathieu-type series $S_{\mu, \nu, \tau, \omega}^{(\alpha, \beta)}(r; \{k^\gamma\}_{k=0}^\infty; p; z)$ possesses the integral representation given by:*

$$(14) \quad S_{\mu, \nu, \tau, \omega}^{(\alpha, \beta)}(r; \{k^\gamma\}_{k=0}^\infty; p; z) = \frac{2\nu\tau z}{\omega\Gamma(\mu)} \int_0^\infty t^{\gamma(\mu\alpha - \beta)} e^{-t} K_p^{(\alpha, \beta)}(\mu, \gamma, \tau, \omega; t) dt,$$

where $K_p^{(\alpha, \beta)}(\mu, \gamma, \tau, \omega; t)$ is defined by

$$K_p^{(\alpha, \beta)}(\mu, \gamma, \tau, \omega; t) = {}_2F_1 \left[\begin{matrix} \nu+1, \tau+1 \\ \omega+1 \end{matrix}; ze^{-t}; p \right] {}_1\Psi_1 \left[\begin{matrix} (\mu, 1) \\ (\gamma(\mu\alpha - \beta) + 1, \gamma\alpha) \end{matrix} \middle| -r^2 t^{\gamma\alpha} \right].$$

Remark 1. 1. *By letting $p = q = 0$ in (14) we deduce that the function $S_{\mu, \nu, \tau, \omega}^{(\alpha, \beta)}(r; \{k^\gamma\}_{k=0}^\infty; z)$ possesses the following integral representation:*

$$(15) \quad S_{\mu, \nu, \tau, \omega}^{(\alpha, \beta)}(r; \{k^\gamma\}_{k=0}^\infty; z) = \frac{2\nu\tau z}{\omega\Gamma(\mu)} \int_0^\infty t^{\gamma(\mu\alpha - \beta)} e^{-t} K^{(\alpha, \beta)}(\mu, \gamma, \tau, \omega; t) dt,$$

where $K^{(\alpha, \beta)}(\mu, \gamma, \tau, \omega; t)$ is defined by

$$K^{(\alpha, \beta)}(\mu, \gamma, \tau, \omega; t) = {}_2F_1 \left[\begin{matrix} \nu+1, \tau+1 \\ \omega+1 \end{matrix}; ze^{-t} \right] {}_1\Psi_1 \left[\begin{matrix} (\mu, 1) \\ (\gamma(\mu\alpha - \beta) + 1, \gamma\alpha) \end{matrix} \middle| -r^2 t^{\gamma\alpha} \right].$$

2. *Setting $\tau = \omega$ in (15) and using the fact that*

$${}_2F_1 \left[\begin{matrix} a, b \\ b \end{matrix}; z \right] = (1 - z)^{-a},$$

we obtain the following integral representation for the function $S_{\mu, \nu}^{(\alpha, \beta)}(r; \{k^\gamma\}_{k=0}^\infty; z)$ [15, Theorem 1, (Eq. 8), p. 975]

$$(16) \quad S_{\mu, \nu}^{(\alpha, \beta)}(r; \{k^\gamma\}_{k=0}^\infty; z) = \frac{2\nu z}{\Gamma(\mu)} \int_0^\infty \frac{t^{\gamma(\mu\alpha - \beta)} e^{-t}}{(1 - ze^{-t})^{\nu+1}} {}_1\Psi_1 \left[\begin{matrix} (\mu, 1) \\ (\gamma(\mu\alpha - \beta) + 1, \gamma\alpha) \end{matrix} \middle| -r^2 t^{\gamma\alpha} \right] dt.$$

In the next Theorem we present the Mellin-Barnes integral representation for the alternating Mathieu-type series $S_{\mu, \nu, \tau, \omega}^{(2, 1)}(r; \{k\}_{k=0}^\infty; p, q; -z)$.

Theorem 2. *Let $r, \nu, \mu, \tau, \omega > 0$, $\Re(p), \Re(q) \geq 0$. Then the following integral representation*

$$(17) \quad S_{\mu, \nu, \tau, \omega}^{(2, 1)}(r; \{k\}_{k=0}^\infty; p, q; -z) = \frac{-z}{i\pi\Gamma(\nu)} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)\Gamma(\nu - s + 1)\mathbb{B}_{p, q}(\tau, \omega; s) [\Gamma(-s + ir + 1)\Gamma(-s - ir + 1)]^\mu}{z^s [\Gamma(-s + ir + 2)\Gamma(-s - ir + 2)]^\mu} ds,$$

holds true for all $|\arg(-z)| < \pi$, where $\mathbb{B}_{p, q}$ is defined by

$$\mathbb{B}_{p, q}(\tau, \omega; s) = \frac{B_{p, q}(\tau - s + 1, \omega - \tau)}{B(\tau, \omega - \tau)}.$$

Proof. The contour of integration extends from $c - i\infty$ to $c + i\infty$, such that all the poles of the Gamma function $\Gamma(\nu - s + 1)$ at the points $s = k + \nu + 1$, $k \in \mathbb{N}$ are separated from the poles of the Gamma function $\Gamma(s)$ at the points $s = -k$, $k \in \mathbb{N}$. Suppose that the poles of the integrand are simple and using the fact that

$$\text{res}[\Gamma, -k] = \lim_{s \rightarrow -k} (s + k)\Gamma(s) = \frac{(-1)^k}{k!},$$

we find that

$$\begin{aligned} & \frac{z}{i\pi\Gamma(\nu)} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)\Gamma(\nu - s + 1)\mathbb{B}_{p,q}(\tau, \omega; s) [\Gamma(-s + ir + 1)\Gamma(-s - ir + 1)]^\mu}{[\Gamma(-s + ir + 2)\Gamma(-s - ir + 2)]^\mu} z^{-s} ds \\ &= \frac{2z}{\Gamma(\nu)} \sum_{k=0}^{\infty} \lim_{s \rightarrow -k} \frac{(s + k)\Gamma(s)\mathbb{B}_{p,q}(\tau, \omega; s)\Gamma(\nu - s + 1) [\Gamma(-s + ir + 1)]^\mu}{[\Gamma(-s - ir + 1)]^{-\mu} [\Gamma(-s + ir + 2)\Gamma(-s - ir + 2)]^\mu} \\ &= \frac{2z}{\Gamma(\nu)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{B_{p,q}(\tau + k + 1, \omega - \tau)\Gamma(\nu + k + 1)}{B(\tau, \omega - \tau)((k + 1)^2 + r^2)^\mu} z^k \\ &= -2 \sum_{k=1}^{\infty} \frac{(\nu)_k k B_{p,q}(\tau + k, \omega - \tau)}{B(\tau, \omega - \tau)(k^2 + r^2)^\mu} \frac{(-z)^k}{k!} \\ &= -S_{\mu, \nu, \tau, \omega}^{(2,1)}\left(r; \{k\}_{k=0}^\infty; p, q; -z\right). \end{aligned}$$

This completes the proof of Theorem 2 □

Corollary 2. Let $r, \nu, \mu, \tau, \omega > 0$, $\Re(p) \geq 0$. Then the following integral representation

$$\begin{aligned} & S_{\mu, \nu, \tau, \omega}^{(2,1)}\left(r; \{k\}_{k=0}^\infty; p; -z\right) = \\ (18) \quad &= -\frac{z}{i\pi\Gamma(\nu)} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)\Gamma(\nu - s + 1)\mathbb{B}_{p,p}(\tau, \omega; s) [\Gamma(-s + ir + 1)\Gamma(-s - ir + 1)]^\mu}{z^s [\Gamma(-s + ir + 2)\Gamma(-s - ir + 2)]^\mu} ds, \end{aligned}$$

holds true for all $|\arg(-z)| < \pi$, where $\mathbb{B}_{p,p}(\tau, \omega; s)$ is defined by

$$\mathbb{B}_{p,p}(\tau, \omega; s) = \frac{B_p(\tau - s + 1, \omega - \tau)}{B(\tau, \omega - \tau)}.$$

Remark 2. If we set $p = 0$ in Corollary 2, then we get the Mellin-Barnes representation of the function $S_{\mu, \nu, \tau, \omega}^{(2,1)}\left(r; \{k\}_{k=0}^\infty; -z\right)$:

$$\begin{aligned} & S_{\mu, \nu, \tau, \omega}^{(2,1)}\left(r; \{k\}_{k=0}^\infty; -z\right) = \\ (19) \quad &= -\frac{z\Gamma(\omega)}{i\pi\Gamma(\nu)\Gamma(\tau)} \int_{c-i\infty}^{c+i\infty} \frac{\kappa(\nu, \tau, \omega; s) [\Gamma(-s + ir + 1)\Gamma(-s - ir + 1)]^\mu}{[\Gamma(-s + ir + 2)\Gamma(-s - ir + 2)]^\mu} z^{-s} ds, \end{aligned}$$

where,

$$\kappa(\nu, \tau, \omega; s) = \frac{\Gamma(s)\Gamma(\nu - s + 1)\Gamma(\tau - s + 1)}{\Gamma(\omega - s + 1)}.$$

In particular, for $\tau = \omega$ we get

$$\begin{aligned} S_{\mu, \nu}^{(2,1)}\left(r; \{k\}_{k=0}^{\infty}; -z\right) &= \\ &= -\frac{z}{i\pi\Gamma(\nu)} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)\Gamma(\nu - s + 1) [\Gamma(-s + ir + 1)\Gamma(-s - ir + 1)]^{\mu}}{[\Gamma(-s + ir + 2)\Gamma(-s - ir + 2)]^{\mu}} z^{-s} ds. \end{aligned}$$

Moreover, if we set $\mu = 2$ and $\nu = 1$ in the above equation we get the Mellin-Barnes for the alternating Mathieu-type series proved by Saxena et al. [10, Theorem 3.1].

3. APPLICATIONS

In our first application in this section we present the relationships between the (p, q)-Mathieu-type series $S_{2, \nu, \tau, \omega}^{(2,1)}\left(r; \{k\}_{k=0}^{\infty}; p, q; z\right)$ and the Rieman-Liouville operator.

3.1 Relationships with (p, q)-Mathieu-type series and the Rieman-Liouville operator

Our first main application is asserted by the following theorem.

Theorem 3. Let $r, \mu, \tau, \omega > 0$, $\Re(p), \Re(q) \geq 0$ and $0 \leq \nu < 1$. Then

$$\begin{aligned} S_{2, \nu, \tau, \omega}^{(2,1)}\left(r; \{k\}_{k=0}^{\infty}; p, q; z\right) &= \frac{1}{2ir\Gamma(\nu)} \left[D_z^{\nu-1} (z^{\nu-1} \Phi_{\tau, \omega}(z, 2, -ir; p, q)) \right. \\ &\quad \left. - D_z^{\nu-1} (z^{\nu-1} \Phi_{\tau, \omega}(z, 2, ir; p, q)) \right]. \end{aligned}$$

Proof. By using the definition of the (p, q)-Mathieu-type series, we can write the Mathieu-type series

$S_{2, \nu, \tau, \omega}^{(2,1)}\left(r; \{k\}_{k=0}^{\infty}; p, q; z\right)$ in the following form:

$$S_{2, \nu, \tau, \omega}^{(2,1)}\left(r; \{k\}_{k=0}^{\infty}; p, q; z\right) = \frac{1}{2ir} [\Phi_{\nu, \tau, \omega}(z, 2, -ir; p, q) - \Phi_{\nu, \tau, \omega}(z, 2, ir; p, q)].$$

Combining the above equation with (5) we get the desired result. □

3.2 Relationships with (p, q)-Mittag-Leffler function and (p, q)-Mathieu-type series. In this section, we introduce the definition of the (p, q)-Mittag-Leffler function, we establish an integral representation for this function and we present some relationships with the (p, q)-Mathieu-type series. For $\lambda, \tau, \omega, \theta, \sigma, \delta > 0$, $\omega > \tau > 0$, and $\min(\Re(p), \Re(q)) \geq 0$, we define the (p, q)-Mittag-Leffler function by

$$(20) \quad E_{\delta, \theta, \sigma; p, q}^{(\lambda, \tau, \omega)}(z) = \sum_{k=0}^{\infty} \frac{(\lambda)_k}{[\Gamma(\theta k + \sigma)]^{\delta}} \frac{B_{p, q}(\tau + k, \omega - \tau)}{B(\tau, \omega - \tau)} \frac{z^k}{k!} \quad (z \in \mathbb{C}).$$

In the case $p = q$ we define the p -Mittag-Leffler function by

$$(21) \quad E_{\delta, \theta, \sigma; p}^{(\lambda, \tau, \omega)}(z) = \sum_{k=0}^{\infty} \frac{(\lambda)_k}{[\Gamma(\theta k + \sigma)]^\delta} \frac{B_p(\tau + k, \omega - \tau)}{B(\tau, \omega - \tau)} \frac{z^k}{k!} \quad (z \in \mathbb{C}).$$

Namely, when $p = 0$, from (21) we observe

$$E_{\delta, \theta, \sigma}^{(\lambda, \tau, \omega)}(z) = \sum_{k=0}^{\infty} \frac{(\lambda)_k (\tau)_k}{[\Gamma(\theta k + \sigma)]^\delta (\omega)_k} \frac{z^k}{k!} \quad (z \in \mathbb{C}),$$

and then, also with the condition $\omega = \tau$ we can immediately get [15]

$$(22) \quad E_{\delta, \theta, \sigma}^{(\lambda)}(z) = \sum_{k=0}^{\infty} \frac{(\lambda)_k}{[\Gamma(\theta k + \sigma)]^\delta} \frac{z^k}{k!} \quad (z \in \mathbb{C}).$$

For $\lambda = 1$ the above series was introduced by S. Gerhold [4].

Lemma 1. *Let $\tau, \omega, \theta, \sigma, \delta > 0$, such that $\tau < \omega$. Assume that $\min(\Re(p), \Re(q)) \geq 0$. Then the following representation*

$$(23) \quad E_{\delta, \theta, \sigma + \theta; p, q}^{(1, \tau + 1, \omega + 1)}(z) = \frac{\omega}{z\tau} \left[E_{\delta, \theta, \sigma; p, q}^{(1, \tau, \omega)}(z) - \frac{B_{p, q}(\tau, \omega - \tau)}{[\Gamma(\sigma)]^\delta B(\tau, \omega - \tau)} \right],$$

holds true.

Proof. By computation, we get

$$\begin{aligned} E_{\delta, \theta, \sigma + \theta; p, q}^{(1, \tau + 1, \omega + 1)}(z) &= \sum_{k=0}^{\infty} \frac{B_{p, q}(\tau + k + 1, \omega - \tau) z^k}{B(\tau + 1, \omega - \tau) [\Gamma(\theta k + \sigma + \theta)]^\delta} \\ &= \frac{B(\tau, \omega - \tau)}{z B(\tau + 1, \omega - \tau)} \sum_{k=1}^{\infty} \frac{B_{p, q}(\tau + k, \omega - \tau) z^k}{B(\tau, \omega - \tau) [\Gamma(\theta k + \sigma)]^\delta} \\ &= \frac{\omega}{z\tau} \left[E_{\delta, \theta, \sigma; p, q}^{(1, \tau, \omega)}(z) - \frac{B_{p, q}(\tau, \omega - \tau)}{[\Gamma(\sigma)]^\delta B(\tau, \omega - \tau)} \right]. \end{aligned}$$

The proof of Lemma 1 is completed. □

Theorem 4. *Let $\lambda, \tau, \omega, \theta, \sigma > 0, \delta \in \mathbb{N}, \omega > \tau > 0$, and $\min(\Re(p), \Re(q)) \geq 0$. Then the (p, q) -Mathieu-type series admits the following series representation:*

$$(24) \quad \begin{aligned} &S_{\mu, \nu, \tau, \omega}^{(\alpha, \beta)} \left(r; \{[\Gamma(\theta k + \sigma)]^\gamma\}_{k=0}^\infty; p, q; z \right) \\ &= 2 \sum_{m=0}^{\infty} \binom{\mu + m - 1}{m} (-r^2)^m \left[E_{\gamma[(\mu+m)\alpha-\beta], \theta, \sigma; p, q}^{(\nu, \tau, \omega)}(z) - \frac{\mathbb{B}_{p, q}(\tau, \omega, 1)}{[\Gamma(\sigma)]^\gamma [(\mu+m)\alpha-\beta]} \right]. \end{aligned}$$

Moreover, the following series representation

$$\begin{aligned}
S_{\mu,1,\tau,\omega}^{(\alpha,\beta)}\left(r; \{\Gamma(\theta k + \sigma)\}^\gamma_{k=0}; p, q; z\right) &= \\
(25) \qquad &= \frac{2z\tau}{\omega} \sum_{m=0}^{\infty} \binom{\mu + m - 1}{m} (-r^2)^m E_{\gamma[(\mu+m)\alpha-\beta],\theta,\sigma+\theta;p,q}^{(1,\tau+1,\omega+1)}(z),
\end{aligned}$$

holds true.

Proof. In view of the definition of the (p, q)-Mittag-Leffler function (20) and the equation (13) we obtain (24). Finally, combining the equation (24) with the relation (23) derived in Lemma 1, we obtain the formula (25). \square

Taking in (24) the values $\theta = \sigma = 1$ we obtain the following representation:

Corollary 3. *Let $\lambda, \tau, \omega, \theta, \sigma > 0, \delta \in \mathbb{N}$ such that $\tau < \omega, .$ In addition, assume that $\min(\Re(p), \Re(q)) \geq 0$. Then the (p, q)-Mathieu-type series admits the following series representations:*

$$\begin{aligned}
S_{\mu,\nu,\tau,\omega}^{(\alpha,\beta)}\left(r; \{(k!)^\gamma\}_{k=0}; p, q; z\right) &= \\
(26) \qquad &= 2 \sum_{m=0}^{\infty} \binom{\mu + m - 1}{m} (-r^2)^m \left[E_{\gamma[(\mu+m)\alpha-\beta],1,1;p,q}^{(\nu,\tau,\omega)}(z) - \mathbb{B}_{p,q}(\tau, \omega, 1) \right],
\end{aligned}$$

and

$$(27) \qquad S_{\mu,1,\tau,\omega}^{(\alpha,\beta)}\left(r; \{(k!)^\gamma\}_{k=0}; p, q; z\right) = \frac{2z\tau}{\omega} \sum_{m=0}^{\infty} \binom{\mu + m - 1}{m} (-r^2)^m E_{\gamma[(\mu+m)\alpha-\beta],1,2;p,q}^{(1,\tau+1,\omega+1)}(z).$$

Lemma 2. *Let $p, q > 0$ and $0 < x, y < 1/2$. Then*

$$(28) \qquad B_{p,q}(x, y) \leq (2p)^{\frac{2x-1}{2}} (2q)^{\frac{2y-1}{2}} \sqrt{\Gamma(-2x + 1, 2p)\Gamma(-2y + 1, 2q)},$$

where $\Gamma(a, x)$ is the incomplete Gamma function defined by

$$\Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t} dt, \quad a > 0, x \geq 0.$$

Proof. By using the Cauchy-Schwartz inequality we have

$$\begin{aligned}
B_{p,q}(x, y) &\leq \left[\int_0^1 (t^{x-1} e^{-\frac{t}{p}})^2 dt \right]^{\frac{1}{2}} \left[\int_0^1 ((1-t)^{y-1} e^{-\frac{t}{q}})^2 dt \right]^{\frac{1}{2}} \\
&= \left[\int_1^\infty t^{-2x} e^{-2pt} dt \right]^{\frac{1}{2}} \left[\int_1^\infty t^{-2y} e^{-2qt} dt \right]^{\frac{1}{2}} \\
&= (2p)^{\frac{2x-1}{2}} (2q)^{\frac{2y-1}{2}} \left[\int_{2p}^\infty t^{-2x} e^{-t} dt \right]^{\frac{1}{2}} \left[\int_{2q}^\infty t^{-2y} e^{-t} dt \right]^{\frac{1}{2}} \\
&= (2p)^{\frac{2x-1}{2}} (2q)^{\frac{2y-1}{2}} \sqrt{\Gamma(-2x + 1, 2p)\Gamma(-2y + 1, 2q)},
\end{aligned}$$

which completes the proof of Lemma 2. □

Lemma 3. For $\lambda, \tau, \omega, \theta, \sigma > 0, \omega > \tau > 0, \delta \in \mathbb{N}$ and $\min(\Re(p), \Re(q)) \geq 0$. Then the (p, q)-Mittag-Leffler function $E_{\delta, \theta, \sigma; p, q}^{(\lambda, \tau, \omega)}(z)$ possesses the following integral representation:

$$(29) \quad E_{\delta, \theta, \sigma; p, q}^{(\lambda, \tau, \omega)}(z) = \frac{1}{B(\tau, \omega - \tau)} \int_0^1 t^{\tau-1} (1-t)^{\omega-\tau-1} E_{p, q}(t) E_{\delta, \theta, \sigma}^{(\lambda)}(zt) dt,$$

holds true.

Proof. By using the definition of the (p, q)-Beta function we get

$$\begin{aligned} & \int_0^1 t^{\tau-1} (1-t)^{\omega-\tau-1} E_{p, q}(t) E_{\delta, \theta, \sigma}^{(\lambda)}(zt) dt = \\ &= \int_0^1 t^{\tau-1} (1-t)^{\omega-\tau-1} E_{p, q}(t) \left(\sum_{k=0}^{\infty} \frac{(\lambda)_k (zt)^k}{[\Gamma(\theta k + \sigma)]^\delta k!} \right) dt \\ &= \sum_{k=0}^{\infty} \frac{(\lambda)_k z^k}{[\Gamma(\theta k + \sigma)]^\delta k!} \int_0^1 t^{\tau+k-1} (1-t)^{\omega-\tau-1} E_{p, q}(t) dt \\ &= B(\tau, \omega - \tau) \sum_{k=0}^{\infty} \frac{(\lambda)_k}{[\Gamma(\theta k + \sigma)]^\delta} \frac{B_{p, q}(\tau + k, \omega - \tau)}{B(\tau, \omega - \tau)} \frac{z^k}{k!} \\ &= B(\tau, \omega - \tau) E_{\delta, \theta, \sigma; p, q}^{(\lambda, \tau, \omega)}(z). \end{aligned}$$

The proof of Lemma 3 is completed.

Theorem 5. For $\lambda, \tau, \omega, \theta, \sigma > 0, \omega > \tau > 0$, and $\min(\Re(p), \Re(q)) \geq 0$. Then the following integral representation

$$(30) \quad \begin{aligned} & S_{\mu, \nu, \tau, \omega}^{(\alpha, \beta)} \left(r; \{[\Gamma(\theta k + \sigma)]^\gamma\}_{k=0}^\infty; p, q; z \right) = \\ &= \frac{1}{B(\tau, \omega - \tau)} \int_0^1 t^{\tau-1} (1-t)^{\omega-\tau-1} E_{p, q}(t) S_{\mu, \nu}^{(\alpha, \beta)} \left(r; \{[\Gamma(\theta k + \sigma)]^\gamma\}_{k=0}^\infty; zt \right) dt \\ &- \frac{2B_{p, q}(\tau, \omega - \tau)}{B(\tau, \omega - \tau)} \cdot \frac{[\Gamma(\sigma)]^{\gamma\beta}}{(r^2 + [\Gamma(\sigma)]^{\gamma\alpha})^\mu} \end{aligned}$$

holds true for all $|z| < 1$. Moreover, the following integral representation

$$(31) \quad \begin{aligned} & S_{\mu, 1, \tau, \omega}^{(\alpha, \beta)} \left(r; \{[\Gamma(\theta k + \sigma)]^\gamma\}_{k=0}^\infty; p, q; z \right) = \\ &= \frac{z\tau}{\omega B(\tau + 1, \omega - \tau)} \int_0^1 t^{\tau-1} (1-t)^{\omega-\tau-1} E_{p, q}(t) S_{\mu, 1}^{(\alpha, \beta)} \left(r; \{[\Gamma(\theta k + \sigma)]^\gamma\}_{k=0}^\infty; zt \right) dt, \end{aligned}$$

holds true for all $|z| < 1$.

□

Proof. By means of Lemma 3 and the integral representation (24) we get

$$\begin{aligned}
& S_{\mu, \nu, \tau, \omega}^{(\alpha, \beta)}\left(r; \{[\Gamma(\theta k + \sigma)]^\gamma\}_{k=0}^\infty; p, q; z\right) = 2 \sum_{m=0}^\infty \binom{\mu + m - 1}{m} (-r^2)^m \\
& \times \left[\frac{1}{B(\tau, \omega - \tau)} \int_0^1 t^{\tau-1} (1-t)^{\omega-\tau-1} E_{p,q}(t) E_{\gamma[(\mu+m)\alpha-\beta], \theta, \sigma}^{(\nu)}(zt) dt \right] \\
& - \frac{2B_{p,q}(\tau, \omega - \tau)}{[\Gamma(\sigma)]^{\gamma(\mu\alpha-\beta)} B(\tau, \omega - \tau)} \sum_{m=0}^\infty \binom{\mu + m - 1}{m} \left(\frac{-r^2}{[\Gamma(\sigma)]^{\gamma\alpha}} \right)^m \\
& = \frac{2}{B(\tau, \omega - \tau)} \\
& \times \int_0^1 t^{\tau-1} (1-t)^{\omega-\tau-1} E_{p,q}(t) \left(\sum_{m=0}^\infty \binom{\mu + m - 1}{m} (-r^2)^m E_{\gamma[(\mu+m)\alpha-\beta], \theta, \sigma}^{(\nu)}(zt) \right) dt \\
& - \frac{2B_{p,q}(\tau, \omega - \tau)}{[\Gamma(\sigma)]^{\gamma(\mu\alpha-\beta)} B(\tau, \omega - \tau)} \cdot \frac{1}{\left(1 + \frac{r^2}{[\Gamma(\sigma)]^{\gamma\alpha}}\right)^\mu} \\
& = \frac{2}{B(\tau, \omega - \tau)} \int_0^1 t^{\tau-1} (1-t)^{\omega-\tau-1} E_{p,q}(t) \\
& \times \left[\sum_{k=0}^\infty \frac{(\nu)_k}{[\Gamma(\theta k + \sigma)]^{\gamma(\mu\alpha-\beta)}} \left(\sum_{m=0}^\infty \frac{\binom{\mu+m-1}{m} (-r^2)^m}{[\Gamma(\theta k + \sigma)]^{\gamma\alpha m}} \right) \frac{(zt)^k}{k!} \right] dt \\
& - \frac{2B_{p,q}(\tau, \omega - \tau)}{B(\tau, \omega - \tau)} \cdot \frac{[\Gamma(\sigma)]^{\gamma\beta}}{(r^2 + [\Gamma(\sigma)]^{\gamma\alpha})^\mu} \\
& = \frac{2}{B(\tau, \omega - \tau)} \int_0^1 t^{\tau-1} (1-t)^{\omega-\tau-1} E_{p,q}(t) \sum_{k=0}^\infty \frac{(\nu)_k}{[\Gamma(\theta k + \sigma)]^{\gamma(\mu\alpha-\beta)}} \\
& \times \left(1 + \frac{r^2}{[\Gamma(\theta k + \sigma)]^{\gamma\alpha}} \right)^{-\mu} \frac{(zt)^k}{k!} dt - \frac{2B_{p,q}(\tau, \omega - \tau)}{B(\tau, \omega - \tau)} \cdot \frac{[\Gamma(\sigma)]^{\gamma\beta}}{(r^2 + [\Gamma(\sigma)]^{\gamma\alpha})^\mu} \\
& = \frac{2}{B(\tau, \omega - \tau)} \int_0^1 t^{\tau-1} (1-t)^{\omega-\tau-1} E_{p,q}(t) \sum_{k=0}^\infty \frac{(\nu)_k [\Gamma(\theta k + \sigma)]^{\gamma\beta}}{([\Gamma(\theta k + \sigma)]^{\gamma\alpha} + r^2)^\mu} \frac{(zt)^k}{k!} dt \\
& - \frac{2B_{p,q}(\tau, \omega - \tau)}{B(\tau, \omega - \tau)} \cdot \frac{[\Gamma(\sigma)]^{\gamma\beta}}{(r^2 + [\Gamma(\sigma)]^{\gamma\alpha})^\mu} \\
& = \frac{1}{B(\tau, \omega - \tau)} \int_0^1 t^{\tau-1} (1-t)^{\omega-\tau-1} E_{p,q}(t) S_{\mu, \nu}^{(\alpha, \beta)}\left(r; \{[\Gamma(\theta k + \sigma)]^\gamma\}_{k=0}^\infty; zt\right) dt \\
& - \frac{2B_{p,q}(\tau, \omega - \tau)}{B(\tau, \omega - \tau)} \cdot \frac{[\Gamma(\sigma)]^{\gamma\beta}}{(r^2 + [\Gamma(\sigma)]^{\gamma\alpha})^\mu},
\end{aligned}$$

which evidently completes the proof of the representation (30). Finally, combining (25) and (29) and repeating the same calculations as above we get (31). The proof of Theorem 5 is completed. □

Corollary 4. Let $\mu, \omega, \theta, \sigma, \alpha > 0$, $\min(\Re(p), \Re(q)) \geq 0$. If $0 < \tau < \frac{1}{2}$ and $\tau < \omega < \tau + \frac{1}{2}$. Then, the following inequality

$$\left| S_{\mu,1,\tau,\omega}^{(\alpha,\beta)}(r; \{\Gamma(\theta k + \sigma)\}^\gamma_{k=0}^\infty; p, q; z) \right| \leq \frac{\tau}{\omega} (2p)^{\frac{2\tau-1}{2}} (2q)^{\frac{2(\omega-\tau)-1}{2}}$$

$$\times \sqrt{\Gamma(1-2\tau, 2p)\Gamma(-2(\omega-\tau)+1, 2q)} \frac{S_{\mu,1}^{(\alpha,\beta)}(r; \{\Gamma(\theta k + \sigma)\}^\gamma_{k=0}^\infty)}{\Gamma(\tau+1, \omega-\tau)}, \quad |z| \leq 1,$$

holds true, where $S_{\mu,1}^{(\alpha,\beta)}(r; \{\Gamma(\theta k + \sigma)\}^\gamma_{k=0}^\infty; 1) = S_{\mu,1}^{(\alpha,\beta)}(r; \{\Gamma(\theta k + \sigma)\}^\gamma_{k=0}^\infty)$.

Proof. By using the integral representation (31) we get

$$\left| S_{\mu,1,\tau,\omega}^{(\alpha,\beta)}(r; \{\Gamma(\theta k + \sigma)\}^\gamma_{k=0}^\infty; p, q; z) \right| \leq \frac{2\tau S_{\mu,1}^{(\alpha,\beta)}(r; \{\Gamma(\theta k + \sigma)\}^\gamma_{k=0}^\infty)}{\omega \Gamma(\tau+1, \omega-\tau)}$$

$$\times \int_0^1 t^{\tau-1} (1-t)^{\omega-\tau-1} E_{p,q}(t) dt$$

$$= \frac{2\tau B_{p,q}(\tau, \omega-\tau)}{\omega \Gamma(\tau+1, \omega-\tau)} S_{\mu,1}^{(\alpha,\beta)}(r; \{\Gamma(\theta k + \sigma)\}^\gamma_{k=0}^\infty).$$

Applying Lemma 2 to the above inequality, we obtain the desired result. □

3.3 Turán type inequalities for the (p, q)-Mathieu-type series

Theorem 6. Let $r, \alpha, \beta, \nu, \mu > 0, \omega > \tau > 0$, $\min(\Re(p), \Re(q)) \geq 0$. Then the following assertions are true:

1. The (p, q)-Mathieu-type series considered as a function in p (or q) is completely monotonic and log-convex on (0, ∞). Furthermore, the following Turán type inequality

$$(32) \quad \left[S_{\mu,\nu,\tau,\omega}^{(\alpha,\beta)}(r; \mathbf{a}; p+1, q; z) \right]^2 \leq S_{\mu,\nu,\tau,\omega}^{(\alpha,\beta)}(r; \mathbf{a}; p, q; z) S_{\mu,\nu,\tau,\omega}^{(\alpha,\beta)}(r; \mathbf{a}; p+2, q; z),$$

holds true for all $z \in (0, 1)$.

2. Assume that $r^2 + \mathbf{a} \geq 1$. Then the (p, q)-Mathieu-type series considered as a function in μ is completely monotonic and log-convex on (0, ∞). Furthermore, the following Turán type inequality

$$(33) \quad \left[S_{\mu+1,\nu,\tau,\omega}^{(\alpha,\beta)}(r; \mathbf{a}; p, q; z) \right]^2 \leq S_{\mu,\nu,\tau,\omega}^{(\alpha,\beta)}(r; \mathbf{a}; p, q; z) S_{\mu+2,\nu,\tau,\omega}^{(\alpha,\beta)}(r; \mathbf{a}; p, q; z),$$

holds true for all $z \in (0, 1)$ such that $r^2 + \mathbf{a} \geq 1$.

Proof. 1. In [5, Corollary 2.7], the authors proved that the extended Beta function p (or q) $\mapsto B_{p,q}(x, y)$ is completely monotonic function on (0, ∞) and using the fact that sums of completely monotonic functions are completely monotonic too, we deduce that the p (or q) $\mapsto S_{\mu,\nu,\tau,\omega}^{(\alpha,\beta)}(r; \mathbf{a}; p, q; z)$ is completely monotonic and

log-convex on $(0, \infty)$, since every completely monotonic function is log-convex (see [17, p.167]). Thus, for all $p_1, p_2 > 0$, and $t \in [0, 1]$, we obtain

$$S_{\mu, \nu, \tau, \omega}^{(\alpha, \beta)}(r; \mathbf{a}; tp_1 + (1-t)p_2, q; z) \leq \left[S_{\mu, \nu, \tau, \omega}^{(\alpha, \beta)}(r; \mathbf{a}; p_1, q; z) \right]^t \left[S_{\mu, \nu, \tau, \omega}^{(\alpha, \beta)}(r; \mathbf{a}; p_2, q; z) \right]^{1-t}.$$

Letting $t = \frac{1}{2}$, $p_1 = p$ and $p_2 = p + 2$ in the above inequality we get the Turán type inequality (32).

2. We note that the function $\mu \mapsto (r^2 + \mathbf{a})^{-\mu}$ is completely monotonic on $(0, \infty)$ such that $r^2 + \mathbf{a} \geq 1$, and consequently the function $\mu \mapsto S_{\mu, \nu, \tau, \omega}^{(\alpha, \beta)}(r; \mathbf{a}; p, q; z)$ is completely monotonic and log-convex on $(0, \infty)$. \square

Remark 3. *The condition $r^2 + \mathbf{a} \geq 1$ is not necessary to prove the Turán type inequality (33), a similar proof of the Theorem 2 in [14], we obtain (33).*

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