

ON AN OPEN PROBLEM OF SIMSEK CONCERNING THE COMPUTATION OF A FAMILY OF SPECIAL NUMBERS

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Simsek [Y. Simsek, New families of special numbers for computing negative order Euler numbers and related numbers and polynomials, *Appl. Anal. Discrete Math.* 12(2018), 1-35.] conjectured that

$$B(d, k) = (k^d + x_1 k^{d-1} + x_2 k^{d-2} + \cdots + x_{d-1} k) 2^{k-d},$$

where $x_1, x_2, \dots, x_{d-1}, d$ are positive integers, and proposed the following open problem: (I) How can we compute the coefficients x_1, x_2, \dots, x_{d-1} ? (II) Is it possible to find the function $f_d(x) = \sum_{k=1}^{\infty} B(d, k) x^k$? By using the familiar Stirling numbers of the first and second kind, we solve this problem. We further obtain a general result on the generalized numbers of $B(d, k)$.

1. INTRODUCTION

Simsek [12] constructed a new family of special numbers denote by $y_1(n, k; \lambda)$:

$$(1) \quad F_{y_1}(t, k; \lambda) = \frac{1}{k!} (\lambda e^t + 1)^k = \sum_{n=0}^{\infty} y_1(n, k; \lambda) \frac{t^n}{n!}.$$

The numbers $y_1(n, k; \lambda)$ are related to the many well-known numbers including the Bernoulli numbers, the Fibonacci numbers, the Lucas numbers, the Stirling

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numbers of the second kind and the central numbers. By using (1), an explicit expression of $y_1(n, k; \lambda)$ is given by

$$(2) \quad y_1(n, k; \lambda) = \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} j^n \lambda^j.$$

In particular, the first few values of the numbers $y_1(n, k; \lambda)$ are listed by the table in [12]. Substituting $\lambda = 1$ into (2) yields

$$(3) \quad B(n, k) = k! y_1(n, k; 1),$$

where $B(n, k)$ is defined by the Golombek's identity [6]:

$$B(n, k) = \frac{d^n}{dt^n} (e^t + 1)^k \Big|_{t=0}.$$

The numbers $B(n, k)$ are related to the following numbers $a_k 2^k$. The first few cases are

$$\begin{aligned} B(0, k) &= 2^k, \\ B(1, k) &= k \cdot 2^{k-1}, \\ B(2, k) &= k(k+1)2^{k-2}, \end{aligned}$$

see for example [12]. Simsek [12] obtained the following recurrence

$$B(d, k) = \frac{2^{k-d}}{m_0} \binom{k}{d} - \sum_{\nu=1}^d \frac{m_\nu}{m_0} B(d-\nu, k),$$

where d is a positive integer and $m_0, m_1, \dots, m_d \in \mathbb{Q}$. Simsek [12] further conjectured that

$$(4) \quad B(d, k) = (k^d + x_1 k^{d-1} + x_2 k^{d-2} + \dots + x_{d-1} k) 2^{k-d},$$

where $x_1, x_2, \dots, x_{d-1}, d$ are positive integers, and proposed the following open problem:

- (I) How can we compute the coefficients x_1, x_2, \dots, x_{d-1} ?
- (II) We assume that for $|x| < r$

$$(5) \quad \sum_{k=1}^{\infty} B(d, k) x^k = f_d(x).$$

Is it possible to find $f_d(x)$ function? For the open problem see also [13].

The aim of this paper is to give an answer to this open problem, see Corollaries 1, 3, 4 and 5 in Section 3, respectively. We further generalize the open problem to the numbers $Y(d, k; \lambda) = k! y_1(d, k; \lambda)$.

2. PRELIMINARIES

2.1. Faà di Bruno's formula and the Stirling numbers

Let $h = f \circ g$ be a composite function. There are several ways to represent the n -th derivative of the composite function h . One of the best known ways is given by Faà di Bruno's formula [5, 8, 15]

$$(6) \quad h^{(n)}(t) = \sum_{\pi(n)} \frac{n!}{k_1!k_2!\cdots k_n!} f^{(k)}(g(t)) \left(\frac{g'(t)}{1!}\right)^{k_1} \left(\frac{g''(t)}{2!}\right)^{k_2} \cdots \left(\frac{g^{(n)}(t)}{n!}\right)^{k_n},$$

where the sum runs over all partitions $\pi(n)$ of the integer n , k_i denotes the number of parts of size i , and $k = k_1 + k_2 + \cdots + k_n$ denotes the number of parts of the considered partition. This well-known formula may be also described in terms of the Bell polynomials:

$$(7) \quad h^{(n)}(t) = \sum_{k=1}^n f^{(k)}(g(t)) B_{n,k} \left(g'(t), g''(t), \dots, g^{(n-k+1)}(t) \right),$$

where the Bell polynomial $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ is given by an explicit expression

$$(8) \quad B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{c_1+2c_2+\dots+n \\ c_1+c_2+\dots=k}} \frac{n!}{c_1!c_2!\cdots c_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{c_1} \left(\frac{x_2}{2!}\right)^{c_2} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{c_{n-k+1}}.$$

The first few cases of the Bell polynomial are

$$B_{1,1}(x_1) = x_1, B_{2,1}(x_1, x_2) = x_2, B_{2,2}(x_1) = x_1^2, \\ B_{3,1}(x_1, x_2, x_3) = x_3, B_{3,2}(x_1, x_2) = 3x_1x_2, B_{3,3}(x_1) = x_1^3.$$

There are many interesting and important identities for the Bell polynomials. One of the useful identities is

$$(9) \quad B_{n,k}(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}) = a^k b^n B_{n,k}(x_1, x_2, \dots, x_{n-k+1}),$$

where a and b are any complex numbers. See Chapter 3 in [5].

The Stirling number of the second kind denoted by $S(n, k)$ counts the number of partitionks of a set of n elements into k indistinguishable boxes in which no box is empty, and the Stirling number of the first kind denoted by $s(n, k)$ counts the number of arrangements of n objects into k nonempty circular permutations. It is worth noting that

$$(10) \quad S(n, k) = B_{n,k}(\underbrace{1, 1, \dots, 1}_{n-k+1}),$$

$$(11) \quad s(n, k) = B_{n,k}(0!, -1!, 2!, \dots, (-1)^{n-k}(n-k)!).$$

In particular, $S(0,0) = s(0,0) = 1$, $S(n,0) = s(n,0) = 0$ for $n \geq 1$ and $S(0,k) = s(0,k) = 0$ for $k \geq 1$. The Stirling numbers of the first and second kind have the "horizontal" generating function:

$$(12) \quad (x)_n = \sum_{k=0}^n s(n,k)x^k,$$

$$(13) \quad x^n = \sum_{k=0}^n S(n,k)(x)_k,$$

where the falling factorial $(x)_i = x(x-1)\cdots(x-i+1)$. They also have the "vertical" generating function:

$$(14) \quad \frac{(e^t - 1)^k}{k!} = \sum_{n \geq k} S(n,k) \frac{t^n}{n!},$$

$$(15) \quad \frac{(\log(1+t))^k}{k!} = \sum_{n \geq k} s(n,k) \frac{t^n}{n!}.$$

For more details see Chapter 5 in [5].

2.2. An explicit expression of $y_1(n, k; \lambda)$

By using Faà di Bruno's formula we establish a new explicit expression for $y_1(n, k; \lambda)$ which is different from (2).

Proposition 1. *If n is a nonnegative integer, then*

$$(16) \quad y_1(n, k; \lambda) = \frac{1}{k!} \sum_{j=0}^n \binom{k}{j} j! S(n, j) \lambda^j (\lambda + 1)^{k-j}.$$

Proof. By (1) we have

$$y_1(n, k; \lambda) = \frac{1}{k!} \frac{d^n}{dt^n} (\lambda e^t + 1)^k |_{t=0}.$$

Thus, (16) is obviously valid for $n = 0$. According to (7), we obtain for $n \geq 1$,

$$y_1(n, k; \lambda) = \frac{1}{k!} \sum_{j=1}^n \binom{k}{j} (\lambda + 1)^{k-j} B_{n,j}(\lambda, \lambda, \dots, \lambda).$$

Combining with (9) and (10) yields

$$y_1(n, k; \lambda) = \frac{1}{k!} \sum_{j=1}^n \binom{k}{j} j! \lambda^j (\lambda + 1)^{k-j} S(n, j),$$

which is equivalent to (16) because $S(n, 0) = 0$ for $n \geq 1$. □

Remark 1. Here we apply the well-known Faà di Bruno's formula to obtain the explicit expression of $y_1(n, k; \lambda)$ which is helpful in solving the open problem in next section. In fact, by the work of Boyadzhiev [3], we also arrive at (16). But our method is very different from that of Boyadzhiev.

Remark 2. If we let $\lambda = 1$, then (16) reduces to

$$(17) \quad B(n, k) = k!y_1(n, k; 1) = \sum_{j=0}^n \binom{k}{j} j! S(n, j) 2^{k-j},$$

see [14, Identity 12].

3. MAIN RESULTS

In this section we will answer the open problem proposed by Simsek in the first section.

3.1. Answer to the open problem

Let $Y(d, k; \lambda) = k!y_1(d, k; \lambda)$. Obviously, by (16) we arrive at

$$(18) \quad Y(d, k; \lambda) = \sum_{j=0}^d \binom{k}{j} j! S(d, j) \lambda^j (\lambda + 1)^{k-j}.$$

Now, let us consider the following general problem concerning $Y(d, k; \lambda)$.

Theorem 1. Let d be a nonnegative integer. Then

$$(19) \quad Y(d, k; \lambda) = \{x_0(\lambda)k^d + x_1(\lambda)k^{d-1} + x_2(\lambda)k^{d-2} + \cdots + x_{d-1}(\lambda)k\}(\lambda + 1)^{k-d},$$

where the coefficients $x_{d-i}(\lambda)$ ($i = 1, 2, \dots, d$) can be explicitly given by

$$(20) \quad x_{d-i}(\lambda) = \sum_{j=i}^d s(j, i) S(d, j) \lambda^j (\lambda + 1)^{d-j}.$$

Proof. The case for $d = 0$ is trivial. So let us consider the case for $d \geq 1$. By (18) we have

$$\begin{aligned} Y(d, k; \lambda) &= \sum_{j=1}^d \binom{k}{j} S(d, j) j! \lambda^j (\lambda + 1)^{k-j} \\ &= \sum_{j=1}^d (k)_j S(d, j) \lambda^j (\lambda + 1)^{k-j}. \end{aligned}$$

Using (12) we have

$$Y(d, k; \lambda) = \sum_{j=1}^d S(d, j) \lambda^j (\lambda + 1)^{k-j} \sum_{i=1}^j k^i s(j, i).$$

Interchanging the order of summations yields

$$\begin{aligned} Y(d, k; \lambda) &= \sum_{i=1}^d k^i \sum_{j=i}^d s(j, i) S(d, j) \lambda^j (\lambda + 1)^{k-j} \\ &= (\lambda + 1)^{k-d} \sum_{i=1}^d k^i \sum_{j=i}^d s(j, i) S(d, j) \lambda^j (\lambda + 1)^{d-j}, \end{aligned}$$

which leads to (19). □

Taking $\lambda = 1$ in Theorem 1 we immediately obtain the following corollary.

Corollary 1. *Let d be a nonnegative integer. Then*

$$(21) \quad B(d, k) = (k^d + x_1 k^{d-1} + x_2 k^{d-2} + \cdots + x_{d-1} k) 2^{k-d},$$

where the coefficients x_{d-i} ($i = 1, 2, \dots, d-1$) are given by

$$(22) \quad x_{d-i} = \sum_{j=i}^d s(j, i) S(d, j) 2^{d-j}.$$

Remark 3. *Eq.(22) shows that the coefficients x_{d-i} ($i = 1, 2, \dots, d-1$) can be explicitly expressed in terms of two kinds of Stirling numbers, which gives an answer to the open problem (I) in Section 1.*

Remark 4. *By (22), we can easily calculate that*

$$\begin{aligned} B(0, k) &= 2^k, \\ B(1, k) &= k \cdot 2^{k-1}, \\ B(2, k) &= (k^2 + k) 2^{k-2}, \\ B(3, k) &= (k^3 + 3k^2) 2^{k-3}, \\ B(4, k) &= (k^4 + 6k^3 + 3k^2 - 2k) 2^{k-4}. \end{aligned}$$

In fact, both of the formulas for $B(3, k)$ and $B(4, k)$ were given in [12]. In [12], Simsek was aware of the fact that the numbers x_j , $j = 1, 2, \dots, d-1$, are integers and d is a positive integer since $x_3 = -2$. Therefore there is a typing error in the expression of open problems, so it would be more appropriate to specify in Remark 4 that x_j , $j = 1, 2, \dots, d-1$, are integers and d is a positive integer.

Theorem 1 gives an explicit expression for $Y(d, k; \lambda)$. We further obtain a recursive formula for $Y(d, k; \lambda)$ below. First we need the following identity.

Lemma 1. *If d is a positive integer, then*

$$(23) \quad \sum_{\nu=0}^{d-1} s(d, d-\nu)Y(d-\nu, k; \lambda) = \binom{k}{d} d! \lambda^d (\lambda+1)^{k-d}.$$

Proof. We have

$$\begin{aligned} \frac{1}{d!} \sum_{\nu=0}^{d-1} s(d, d-\nu)Y(d-\nu, k; \lambda) &= \frac{1}{d!} \sum_{\nu=0}^{d-1} s(d, d-\nu) \sum_{j=0}^k \binom{k}{j} j^{d-\nu} \lambda^j \\ &= \frac{1}{d!} \sum_{j=0}^k \binom{k}{j} \lambda^j \sum_{\nu=1}^d s(d, \nu) j^\nu \\ &= \frac{1}{d!} \sum_{j=0}^k \binom{k}{j} \lambda^j (j)_d \\ &= \sum_{j=0}^k \binom{k}{j} \binom{j}{d} \lambda^j = \sum_{j=0}^k \binom{k}{d} \binom{k-d}{k-j} \lambda^j \\ &= \binom{k}{d} \lambda^d (\lambda+1)^{k-d}, \end{aligned}$$

which implies that (23) is true. □

Since $s(d, 0) = 0$ for $d \geq 1$, (23) can be rewritten as

$$(24) \quad \sum_{\nu=0}^d s(d, d-\nu)Y(d-\nu, k; \lambda) = \binom{k}{d} d! \lambda^d (\lambda+1)^{k-d}.$$

By using Identity (24), we immediately have the following theorem.

Theorem 2. *We have the recurrence relation:*

$$(25) \quad \begin{aligned} Y(0, k; \lambda) &= (\lambda+1)^k, \\ Y(d, k; \lambda) &= \binom{k}{d} d! \lambda^d (\lambda+1)^{k-d} - \sum_{\nu=1}^d s(d, d-\nu)Y(d-\nu, k; \lambda), \quad d \geq 1. \end{aligned}$$

Taking $\lambda = 1$ we obtain a recursive formula for $B(d, k)$.

Corollary 2. *If d is a nonnegative integer, then the numbers $B(d, k)$ can be computed recursively by*

$$(26) \quad \begin{aligned} B(0, k) &= 2^k, \\ B(d, k) &= \frac{2^{k-d}}{m_0} \binom{k}{d} - \sum_{\nu=1}^d \frac{m_\nu}{m_0} B(d-\nu, k), \quad d \geq 1, \end{aligned}$$

where the coefficients

$$(27) \quad m_\nu = \frac{1}{d!} s(d, d - \nu), \quad \nu = 0, 1, \dots, d.$$

Remark 5. In fact, Simsek [12] obtained the recursive formula (26). He pointed out that all $m_\nu \in \mathbb{Q}$, $\nu = 0, 1, \dots, d$. However, he did not give the explicit formula for the coefficients m_ν ($\nu = 0, 1, \dots, d$).

Let $f_d(x; \lambda) = \sum_{k=1}^{\infty} Y(d, k; \lambda) x^k$ be a generating function of $Y(d, k; \lambda)$. Then we have the following theorem.

Theorem 3. If d is a nonnegative integer, then

$$(28) \quad f_d(x; \lambda) = \begin{cases} \frac{(\lambda+1)x}{1-(\lambda+1)x}, & d = 0, \\ \sum_{j=1}^d j! S(d, j) \frac{(\lambda x)^j}{(1-(\lambda+1)x)^{j+1}}, & d \geq 1. \end{cases}$$

Proof. If $d = 0$, we have

$$(29) \quad f_0(x; \lambda) = \sum_{k=1}^{\infty} Y(0, k; \lambda) x^k = \sum_{k=1}^{\infty} (\lambda + 1)^k x^k = \frac{(\lambda + 1)x}{1 - (\lambda + 1)x}.$$

If $d > 0$, then by (19) we have

$$\begin{aligned} f_d(x; \lambda) &= \sum_{k=1}^{\infty} Y(d, k; \lambda) x^k \\ &= \sum_{k=1}^{\infty} ((\lambda + 1)x)^k \sum_{i=1}^d k^i \sum_{j=i}^d s(j, i) S(d, j) \left(\frac{\lambda}{\lambda + 1} \right)^j \\ &= \sum_{k=1}^{\infty} ((\lambda + 1)x)^k \sum_{j=1}^d S(d, j) \left(\frac{\lambda}{\lambda + 1} \right)^j \sum_{i=1}^j k^i s(j, i) \\ &= \sum_{k=1}^{\infty} ((\lambda + 1)x)^k \sum_{j=1}^d S(d, j) \left(\frac{\lambda}{\lambda + 1} \right)^j (k)_j \end{aligned}$$

Interchanging the order of summations we obtain

$$\begin{aligned} f_d(x; \lambda) &= \sum_{j=1}^d S(d, j) \left(\frac{\lambda}{\lambda + 1} \right)^j \sum_{k=1}^{\infty} ((\lambda + 1)x)^k (k)_j \\ &= \sum_{j=1}^d S(d, j) \left(\frac{\lambda}{\lambda + 1} \right)^j j! \frac{((\lambda + 1)x)^j}{(1 - (\lambda + 1)x)^{j+1}} \\ &= \sum_{j=1}^d j! S(d, j) \frac{(\lambda x)^j}{(1 - (\lambda + 1)x)^{j+1}}. \end{aligned}$$

The desired result is derived. \square

Let us consider the generating function of $f_d(x; \lambda)$.

Theorem 4. *We have*

$$(30) \quad \sum_{k=0}^{\infty} f_k(x; \lambda) \frac{t^k}{k!} = \frac{x(1 + \lambda e^t)}{1 - x - \lambda x e^t}.$$

Proof. By (28) we have

$$\begin{aligned} \sum_{k=0}^{\infty} f_k(x; \lambda) \frac{t^k}{k!} &= \frac{(\lambda + 1)x}{1 - (\lambda + 1)x} + \sum_{k=1}^{\infty} \frac{t^k}{k!} \sum_{j=1}^k j! S(k, j) \frac{(\lambda x)^j}{(1 - (\lambda + 1)x)^{j+1}} \\ &= \frac{(\lambda + 1)x}{1 - (\lambda + 1)x} + \sum_{j=1}^{\infty} j! \frac{(\lambda x)^j}{(1 - (\lambda + 1)x)^{j+1}} \sum_{k=j}^{\infty} S(k, j) \frac{t^k}{k!}. \end{aligned}$$

Using (14) we have

$$\begin{aligned} \sum_{k=0}^{\infty} f_k(x; \lambda) \frac{t^k}{k!} &= \frac{(\lambda + 1)x}{1 - (\lambda + 1)x} + \sum_{j=1}^{\infty} \frac{(\lambda x)^j}{(1 - (\lambda + 1)x)^{j+1}} (e^t - 1)^j \\ &= \frac{(\lambda + 1)x}{1 - (\lambda + 1)x} + \frac{1}{1 - (\lambda + 1)x} \left\{ \frac{1}{1 - \frac{\lambda x}{1 - (\lambda + 1)x} (e^t - 1)} - 1 \right\}. \end{aligned}$$

After simplifying the right-hand side of the last identity, we arrive at (30). □

By the generating function (30) we find the recurrence relation of $f_d(x; \lambda)$.

Theorem 5. *The sequence of the functions $f_d(x; \lambda)$ ($d = 0, 1, \dots$) satisfies the following recurrence relation:*

$$(31) \quad \begin{aligned} f_0(x; \lambda) &= \frac{(\lambda + 1)x}{1 - (\lambda + 1)x}, \\ f_d(x; \lambda) &= \frac{\lambda x}{1 - (\lambda + 1)x} \left\{ 1 + \sum_{j=0}^{d-1} \binom{d}{j} f_j(x; \lambda) \right\}, \quad d \geq 1. \end{aligned}$$

Proof. Multiplying both sides of (30) by $1 - x - \lambda x e^t$ and equating the coefficient of t^d , we obtain (31). □

Taking $\lambda = 1$ we naturally find the function $f_d(x) = f_d(x; 1)$ which was proposed in the open problem (II).

Corollary 3. *The $f_d(x)$ can be explicitly computed by*

$$(32) \quad f_d(x) = \sum_{k=1}^{\infty} B(d, k) x^k = \begin{cases} \frac{2x}{1-2x}, & d = 0, \\ \sum_{j=1}^d j! S(d, j) \frac{x^j}{(1-2x)^{j+1}}, & d \geq 1. \end{cases}$$

Corollary 4. *The sequence of the functions $f_d(x)$ ($d = 0, 1, \dots$) can be computed by the following recurrence relation:*

$$(33) \quad \begin{aligned} f_0(x) &= \frac{2x}{1-2x}, \\ f_d(x) &= \frac{x}{1-2x} \left\{ 1 + \sum_{j=0}^{d-1} \binom{d}{j} f_j(x) \right\}, \quad d \geq 1. \end{aligned}$$

Remark 6. *Corollaries 3 and 4 give the explicit expression and the recursive formula of $f_d(x)$, respectively, which answer the open problem (II) in Section 1.*

3.2. The Relation between $f_d(x; \lambda)$ and the Eulerian polynomials

Recall the following classical result [4, 5, 7]

$$\sum_{k=0}^{\infty} k^n x^k = \frac{A_n(x)}{(1-x)^{n+1}}, \quad |x| < 1,$$

where $A_n(x)$ is the Eulerian polynomial of degree n , and it may be explicitly written by

$$A_n(x) = \sum_{k=1}^n A(n, k) x^k, \quad A_0(x) = 1.$$

In the above, $A(n, k)$ are known as Eulerian numbers and they can be expressed by

$$A(n, k) = \sum_{i=1}^k (-1)^i \binom{n+1}{i} (k-i)^n.$$

The Eulerian polynomials $A_n(x)$ can be defined by the exponential generating function

$$(34) \quad \sum_{n=0}^{\infty} A_n(x) \frac{t^n}{n!} = \frac{1-x}{1-xe^{t(1-x)}}.$$

From (34) one may easily obtain

$$A_0(x) = 1, \quad A_1(x) = x, \quad A_2(x) = x^2 + x, \quad A_3(x) = x^3 + 4x^2 + x, \quad \dots$$

The Eulerian polynomials are closely related to the Apostol-Bernoulli numbers [2]:

$$(35) \quad A_n(x) = -\frac{(1-x)^{n+1}}{n+1} \mathcal{B}_{n+1}(x),$$

where $\mathcal{B}_n(x)$ are the Apostol-Bernoulli numbers [1, 9, 10, 11, 16] defined by

$$(36) \quad \frac{t}{xe^t - 1} = \sum_{n=0}^{\infty} \mathcal{B}_n(x) \frac{t^n}{n!}.$$

Theorem 6. *If d is a nonnegative integer, then*

$$(37) \quad f_d(x; \lambda) = \begin{cases} \frac{(\lambda+1)x}{1-(\lambda+1)x}, & d = 0, \\ \frac{(1-x)^d}{(1-(\lambda+1)x)^{d+1}} A_d\left(\frac{\lambda x}{1-x}\right), & d \geq 1, \end{cases}$$

where $A_d(x)$ is the Eulerian polynomial of degree d .

Proof. By (30) we have

$$\sum_{k=0}^{\infty} f_k(x; \lambda) \frac{t^k}{k!} = \frac{1}{x-1} \frac{1}{\frac{\lambda x}{1-x} e^t - 1} - 1.$$

According to (36), if $d = 0$, then

$$(38) \quad f_0(x; \lambda) = \frac{1}{x-1} \mathcal{B}_1\left(\frac{\lambda x}{1-x}\right) - 1 = \frac{(\lambda+1)x}{1-(\lambda+1)x},$$

and if $d \geq 1$, then

$$(39) \quad f_d(x; \lambda) = \frac{1}{(d+1)(x-1)} \mathcal{B}_{d+1}\left(\frac{\lambda x}{1-x}\right).$$

Replacing x by $\frac{\lambda x}{1-x}$ in (35) yields

$$(40) \quad A_d\left(\frac{\lambda x}{1-x}\right) = -\frac{(1-(\lambda+1)x)^{d+1}}{(d+1)(1-x)^{d+1}} \mathcal{B}_{d+1}\left(\frac{\lambda x}{1-x}\right).$$

Combining (38), (39) with (40) we immediately arrive at (37). □

Taking $\lambda = 1$ we establish the relation between $f_d(x)$ and the Eulerian polynomials.

Corollary 5. *If d is a nonnegative integer, then*

$$(41) \quad f_d(x) = \begin{cases} \frac{2x}{1-2x}, & d = 0, \\ \frac{(1-x)^d}{(1-2x)^{d+1}} A_d\left(\frac{x}{1-x}\right), & d \geq 1. \end{cases}$$

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