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ACYCLIC TOTAL DOMINATING SETS IN CUBIC GRAPHS

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We show that every cubic graph has a total dominating set D such that the subgraph induced by D is acyclic. As a consequence, an old result attributed to Berge follows.

1. Introduction

A dominating set D of a graph is a set of vertices such that every vertex not in D has a neighbor in D. Restricting a dominating set D such that the subgraph induced by D has some property is well studied. For example, requiring the graph induced by D to have no isolated vertices is equivalent to D being a total dominating set. (Recall that a set D of vertices is a total dominating set if every vertex in the graph has a neighbor in D; for more on total domination, see [**4**].) Similarly, requiring D to have no edges means that D is an independent dominating set. And there are more general ideas such as acyclic domination, introduced by Hedetniemi, Hedetniemi and Rall [**3**] and explored further for example in [**8**].

In this paper we consider the same idea for total domination. Specifically we look for a total dominating set D such that the subgraph induced by D has no cycles. Equivalently, this is a dominating set such that the subgraph induced by D is an isolate-free forest. Some complexity issues for generalized total domination were investigated by Schaudt and Schrader [**7**]. Here we prove that every cubic graph has an acyclic total dominating set. As a consequence, we obtain a (new)

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proof of Berge's result (see [**2**]) which asserts that every cubic graph contains two disjoint independent dominating sets.

2. Main Result

We prove that:

Theorem 1. *If a (simple) graph G has minimum degree at least 2 and maximum degree at most 3, then there exists a total dominating set D of G such that the subgraph induced by D is acyclic.*

The proof is by induction. The main part starts with a cycle C of shortest length and is broken into two cases based on how the cycle interacts with the rest of the graph. But first we provide some preliminaries.

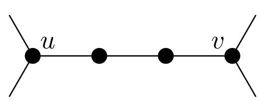
2.1. Preparation

We use ***ATD-set*** to stand for "acyclic total dominating set". If we construct a total dominating set, then a ***bad cycle*** is one all of whose vertices are in the total dominating set.

Lemma 1. *We may assume G is connected. Further, we may assume that every bridge has at least one end of degree 2.*

Proof. If the graph is disconnected, then one can simply induct on the components. If some bridge joins two vertices of degree 3, then one can similarly delete the bridge and induct. QED

Define a ***linkage*** as a path with ends u and v of degree 3 and internal vertices (if any) of degree 2; we say that such u and v are ***linked***. Define a ***handle*** as a cycle with exactly one vertex of degree 3. Here is a linkage.



Lemma 2. *We may assume that*
(a) there is no linkage with at least two internal vertices.
(b) there is no handle.

Proof. (a) Suppose vertices u and v are linked by a path with at least two internal vertices. Let G' be the graph obtained from G by deleting the internal vertices of the linkage. This graph has minimum degree 2. So by induction it has an ATD-set, say D'. If the linkage has exactly two internal vertices and both ends u and v are in D', then the set D' works for G as well. Otherwise, add to D' a minimal total dominating set of the internal vertices of the linkage to yield D''. By construction, the set D'' dominates both the internal vertices of the linkage and the rest of the graph; by minimality, D'' does not contain every vertex in the linkage, so that we have not created a cycle. That is, D'' is an ATD-set D for G.

(b) Suppose there is a handle L with u as its (unique) vertex of degree 3. Let v be the neighbor of u not on L. We note that uv is a bridge. By Lemma 1, the vertex v has degree 2. Let w be the other neighbor of v. By part (a), the vertex w has degree 3. Let G' be the graph obtained from G by deleting both L and u. By induction, the graph G' has an ATD-set. Every such set in G' can be extended to an ATD-set of G by adding to it a minimum TD-set of the handle L that contains the vertex u. QED

Thus:

Lemma 3. *We may assume that no two degree-2 vertices are adjacent.*

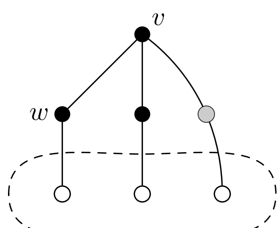
Proof. By Lemma 2, if there are two adjacent vertices of degree 2, then G is a cycle. But, Theorem 1 is trivial for a cycle. QED

For a set W of vertices,

- Let $\mathcal{S}(W)$ denote the degree-2 neighbors of W not in W.
- Let $\mathcal{T}(W)$ be the boundary $N[W \cup \mathcal{S}(W)] - (W \cup \mathcal{S}(W))$. Note that, by the definition of $\mathcal{S}(W)$ and the above lemma, every vertex of $\mathcal{T}(W)$ has degree 3.

Lemma 4. *We may assume that there is no vertex v with $|\mathcal{S}(\{v\})| \geq 2$ and $|\mathcal{T}(\{v\})| = 3$.*

Proof. Suppose there is such a vertex v. Here is a picture, where the gray vertex might or might not exist.



Let G' be the graph obtained from G by deleting $\{v\} \cup \mathcal{S}(\{v\})$. By the assumption, every vertex in $\mathcal{T}(\{v\})$ is adjacent to a different vertex of $\{v\} \cup \mathcal{S}(\{v\})$. This means that the graph G' has minimum degree 2. So by induction, the graph G' has an ATD-set, say D'.

The first case is that all of $\mathcal{S}(\{v\})$ is already dominated by D'. Then add all of $\mathcal{S}(\{v\})$ to D', and one has an ATD-set of G. The second case is that some vertex $w \in \mathcal{S}(\{v\})$ is not dominated by D'. Then add v and w to D', and similarly one is done. QED

Now

- Let C be a shortest cycle and let $S = \mathcal{S}(V(C))$ and $T = \mathcal{T}(V(C))$.

By Lemma 3, S is independent. There are two cases depending on whether some vertex outside C has more than one neighbor in C and/or some vertex in S has its non-C neighbor adjacent to C.

We will also need the following result, which surely is known:

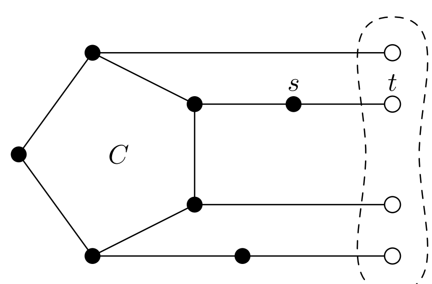
Lemma 5. *Every multi-graph with minimum degree at least* 2 *has an orientation without a source or sink.*

Proof. Let B be the set of bridges. These induce a forest. Orient the forest so that no non-leaf is a source or sink (e.g. by rooting at a leaf and orienting every arc towards the root). Robbins' theorem [**5**] says that every bridgeless graph has a strong orientation. So we can find a strong orientation of each component of the graph $G - B$. The resultant orientation of the overall graph has no sink or source. QED

2.2. Case 1

For Case 1 we assume: *No vertex outside C has more than one neighbor in C, and no vertex in S has its non-C neighbor adjacent to C.*

This means that every vertex in S has one neighbor in T. Consider a vertex t of T and suppose it is adjacent to exactly two vertices of S. Then by the assumption of Case 1, the set $\mathcal{T}(\{t\})$ consists of two distinct elements of C and one vertex outside $C \cup S$. This contradicts Lemma 4. A similar contradiction is obtained if T is adjacent to three vertices of S. Thus vertex t is adjacent to at most one vertex of S. By the assumption of Case 1, if t is adjacent to a vertex of S then it has no neighbor in C. That is, every vertex in T is adjacent to exactly one vertex in $C \cup S$. Here is a picture:



There are two subcases.

Case 1a. Assume $|S| < |C|$.

Let G' be the graph with C and S deleted. By the induction hypothesis, the graph G' has an ATD-set, say D'. Let S' be the vertices of S not dominated by D'. Start by adding to D', all of S' and each vertex of C with a neighbor in S'.

If none of C is yet dominated, then add to D' all but one vertex from C, and we have an ATD-set of G. So assume otherwise. Pick some orientation of C. Let C' be the vertices on C not yet dominated. If a vertex u of C' has a neighbor in C', then add u to D' (and note that by symmetry its neighbor is added as well). If a vertex u of C' has no neighbor in C', then add to D' its successor v on C. Call the resultant set D.

Clearly we have added a neighbor of every vertex in S' and then a neighbor of every vertex in C'. Since every vertex not in S' or C' is dominated by D', it follows that D totally dominates G. It remains to verify that this process did not create a bad cycle (consisting of vertices of D).

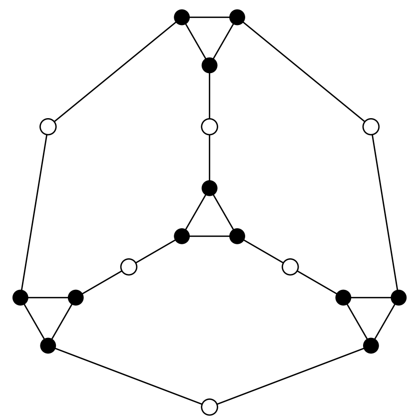
We show first that C is not the bad cycle. If C' is empty, then the set D is obtained from D' by adding to it all of S' and each vertex of C with a neighbor in S'. In this case, since $|S| < |C|$, at least one vertex on C does not belong to D. So assume C' is nonempty. By the above assumption, C' is not all of C. This means that there must exist a vertex u of C' whose successor v is in $C - C'$. Then vertex v is in D if and only if vertex u is not. This implies once again that C is not a bad cycle.

Second, note that a bad cycle cannot contain a vertex $s \in S$, since we never take both s and its neighbor in T. Nor can the bad cycle be contained completely within G', since we chose an ATD-set there. So a bad cycle must contain two edges t_1v_1 and t_2v_2 where $v_1, v_2 \in C$ and $t_1, t_2 \in T$, as well as a portion of C joining v_1 and v_2. Since v_1 and v_2 are dominated by D', they are not in C'. Thus they were added to D since their predecessors on the cycle needed dominating. In particular, this means that their predecessors are not in D. That is, they are not joined in $D \cap C$, a contradiction. This shows there is no bad cycle, as required.

Case 1b. $|S| = |C|$. *That is, every vertex of* C *has a neighbor in* S.

Consider a vertex $s \in S$ and let t be its neighbor in T. If t is not in a triangle, then let G'' be the graph obtained from $G - (C \cup S)$ with vertex t contracted out (that is, with t removed and an edge added between its two remaining neighbors). This is a valid graph (it is simple and has no leaves). Let D'' be an ATD-set of G''. To extend this to an ATD-set for G, start by adding both s and t. Note that this does not create a bad cycle, since such a cycle would contain t and its neighbors other than s, and every such cycle corresponds to a cycle in G''. Then proceed as in case 1a, noting that we cannot end up adding all of C to D.

It remains to resolve the case that every possible t is in a triangle. So first note that this implies that C is a triangle, since we chose the shortest cycle. Second, we can consider all triangles for our choice of C. So either there is a choice that works or that moves us to case 2, or: for every triangle C, all three vertices have degree 2-neighbors and the other neighbors of these vertices are in triangles. From this, it follows that the graph G has the following form. Start with a cubic graph H that has a spanning subgraph X consisting of triangles. Then subdivide every edge not in X. Here is an example:



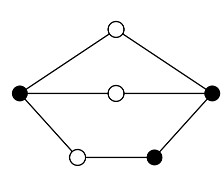
Such a graph has an ATD-set D as follows. We will choose exactly two vertices from each triangle of X and none of the subdivision vertices. It is immediate that any such set totally dominates all of X and does not have a bad cycle. So we need to choose the vertices such that they dominate the subdivision vertices. Create an auxiliary graph H' from H by contracting each triangle to a single vertex. Note that the graph H' might have multi-edges. By Lemma 5, the graph H' has an orientation without sinks or sources. This lifts to an orientation of some of the edges of H. For the set D, take every vertex in X that is the head of an arc in the partially oriented H; and then if a triangle of X still has only one vertex, add any other vertex of that triangle. And we are done with Case 1.

2.3. Case 2

For Case 2 we assume: *Some vertex has more than one neighbor in C and/or some vertex in S has a neighbor in T that is adjacent to C.*

We will need the following concept. Define a ***ported subgraph*** (H, P) as an induced subgraph H with some subset P of the vertices designated ***ports*** such that (a) every non-port has no neighbor outside H, and (b) every port has at most one neighbor outside H.

If P is nonempty, we say that a ported subgraph (H, P) is ***insular*** if for every nonempty subset $Q \subseteq P$, there is an ATD-set $A(Q)$ of H that contains Q but no other port. If P is empty, then we require only that H have an ATD-set. Here is an example of an insular ported subgraph, where the ports are the white vertices. (The requisite ATD-set $A(Q)$ is obtained in each case by adding the two adjacent non-ports to Q.)

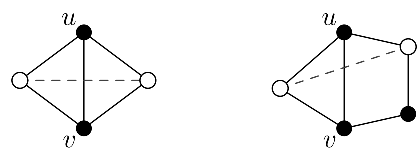


We now resume the proof of the main theorem.

Lemma 6. *We may assume that there is an insular ported subgraph.*

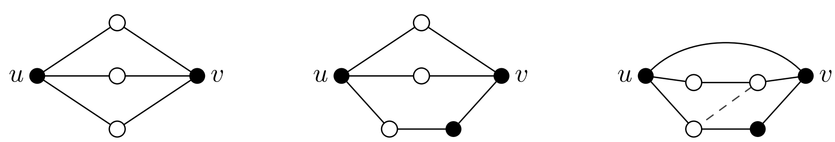
Proof. The conditions of Case 2 imply that some pair of vertices u and v on C are joined by a path R of length 2 or 3 with internal vertex(s) outside C, and if length 3, (exactly) one of these has degree 2. In particular, this implies the girth of the graph is at most 6.

Assume the girth is 3. Up to symmetry, there are two possibilities, drawn here: in the picture the black vertices can have no further edges and the dashes indicate possible edges.



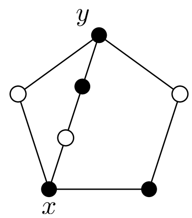
We claim the above pictures each represent an insular ported subgraph. In each case, the ports are those white vertices that do have degree 2. If the two white vertices are adjacent, then there are no ports and $\{u, v\}$ is an ATD-set. If the two white vertices are not adjacent, then one can form $A(Q)$ by adding to Q all non-ports except u.

Assume the girth is 4. Up to symmetry, there are three possibilities, drawn here:



We claim the above pictures each represent an insular ported subgraph. In each case the ports are the white vertices of degree 2. In the first two cases, one can form $A(Q)$ by adding to Q all non-ports except u. In the third case, when the dashed edge is not present, the same construction works in some instances but fails to be totally dominating if the port incident with v is not in Q; if that occurs, then let $A(Q) = Q \cup \{u, v\}$ instead. If the dashed edge is present, then it is easily checked that the result with one port is an insular ported subgraph.

Assume the girth is 5. Then u and v are distance 2 apart on C and the exterior path R has length 3 and contains a degree 2 vertex. By using R instead of one of the pieces of the cycle, this means there is a 5-cycle C' that contains a vertex of degree 2. If the cycle C' satisfies the conditions for Case 1, then we are done. So assume it doesn't. Again two vertices on it, say x and y, are joined by a path R' of length 3 with internal vertex(s) outside C', one of which has degree 2. The situation is drawn here.



We claim that this is an insular ported subgraph. By the girth condition it is induced. For a nonempty set Q of ports, the desired ATD-set can be obtained by adding to Q both x and y, and then one other non-port if x or y is not yet totally dominated.

Finally, assume the girth is 6. Then u and v are antipodal on C and the exterior path R has length 3 and contains a degree 2 vertex. By using R instead of one of the pieces of the cycle, this means there is a 6-cycle C' that contains a vertex of degree 2. If the cycle C' satisfies the conditions for Case 1, then we are done. So assume it doesn't. Again two vertices on it are joined by a path R' of length 3 with internal vertex(s) outside C', one of which has degree 2. By using R' instead, this means there is a 6-cycle C'' that contains two vertices of degree 2. By Lemma 4, these two vertices are antipodal. It similarly follows that C'' must satisfy the conditions for Case 1, because by Lemma 4 a vertex on C'' cannot be adjacent to more than one degree-2 vertex. QED

We show next that if the graph contains a suitable insular ported subgraph. then we are done.

Lemma 7. *Let (H, P) be an insular ported subgraph such that no two ports have a common neighbor in $\mathcal{S}(V(H))$. If $V(H) \cup \mathcal{S}(V(H))$ is the whole graph, or if $V(H) \cup \mathcal{S}(V(H))$ is not the whole graph and $G' = G - (V(H) \cup \mathcal{S}(V(H)))$ has minimum degree at least 2, then G has an ATD-set.*

Proof. If $V(H) \cup \mathcal{S}(V(H))$ is the whole graph, then the hypothesis means that $\mathcal{S}(V(H))$ is empty and $H = G$ has an ATD-set. So assume otherwise.

By the induction hypothesis, the subgraph G' has an ATD-set, say D'. Let S' be the subset of $\mathcal{S}(V(H))$ that is not dominated by D'. If S' is nonempty, then let $R = A(Q)$ where Q is the subset of the ports incident with S'; if S' is empty but P is not, then let $R = A(\{q\})$ for some port q; and if $S' = P = \emptyset$, then let R be any ATD-set of H. Let $D = D' \cup R$.

The set D is clearly a total dominating set of G, since it contains a total dominating set of both H and G' as well as a neighbor of every vertex in $\mathcal{S}(V(H))$. It remains to check that D is acyclic.

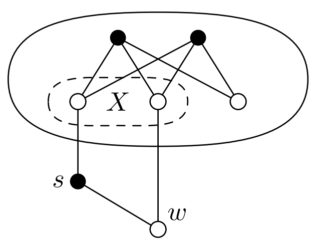
Note that we take no vertex of $\mathcal{S}(V(H))$, and we take acyclic dominating sets of both H and G'. So any bad cycle must use vertices of both H and G'. Consider an edge uv where $u \in V(H)$, $v \in V(G')$ and both u and v are taken in D. This means that u is a port but not a neighbor of S'; therefore S' is empty and $u = q$. By the definition of ported subgraph, u has at most one neighbor outside $V(H)$.

Hence, there is only one such edge, and so there is no bad cycle using vertices of both H and G', as required. QED

Now, since by Lemma 6 there exists an insular ported subgraph, we are done unless the conditions of the above lemma fail. This means at least one of the following conditions holds.

1. There is some vertex w in $\mathcal{S}(V(H))$ with two neighbors in $V(H)$. In this case let $X = N(w)$.
2. The subgraph G' has nonempty vertex set but does not have minimum degree 2. So some vertex w in $\mathcal{T}(V(H))$ has at least two linkages to H. Let X be the port(s) of H linked to w by linkages, and let s be the neighbor of w in $\mathcal{S}(V(H))$ if there is such a vertex. (By Lemma 4, w has at most one degree-2 neighbor.)

Here is a picture of an example of Condition 2.



Under either condition we proceed as follows. Let H' be the subgraph induced by $V(H)$ together with w and the vertex s, if it exists. Define P' by taking $P - (N(w) \cap V(H))$ and adding w if and only if w has a neighbor outside H'. We call this an ***extension***.

Lemma 8. *If (H, P) is an insular ported subgraph, then the extension (H', P') is an insular ported subgraph.*

Proof. It is immediate that H' is a ported subgraph. It remains to check that it is insular. If P' is empty, then $A(X)$ is an ATD-set of H', as required. So assume P' is nonempty. We will write A' for the required mapping from subsets of P' to ATD-set's of H'.

Let Q' be a nonempty subset of P'. There are two cases. If Q' does not contain w, then define $A'(Q')$ as $A(Q' \cup X)$. This clearly totally dominates all of $V(H)$ and, since it contains X, it totally dominates the one or two vertices in $V(H') - V(H)$. There is no bad cycle since we assumed $A(Q' \cup X)$ is acyclic.

If Q' does contain w, then pick some vertex $x \in N(w) \cap V(H)$ and define $A(Q')$ as $\{w\} \cup A((Q' - \{w\}) \cup \{x\})$. This totally dominates all of $V(H)$ and since it contains $\{x, w\}$, it totally dominates the one or two vertices of $V(H') - V(H)$. There is no bad cycle completely within $V(H)$ since $A((Q' - \{w\}) \cup \{x\})$ is acyclic. And there is no bad cycle containing w, since $A(Q')$ contains at most one neighbor of w (namely x). That is, $A(Q')$ is an ATD-set of H'. QED

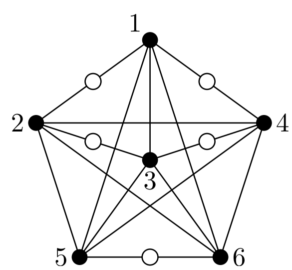
So this is how we proceed once we have an insular ported subgraph of a graph G. If two ports have a common neighbor in $\mathcal{S}(V(H))$ then do an extension. If there is a vertex of $\mathcal{T}(H)$ with two linkages to H, then do an extension. Either we get to an insular ported subgraph that we can apply Lemma 7 to, or we do an extension in which P' is empty, and we're done.

This concludes the proof of Theorem 1.

3. Discussion

We note that the above result does not generalize to allowing vertices of degree 1. Since a total dominating set must contain the neighbors of all such vertices, allowing end-vertices prevents any restriction on the subgraph induced by the total dominating set.

There is also a limit to how high the maximum degree can be. For example, consider the graph H formed from K_6 with vertices $\{1,2,3,4,5,6\}$ by subdividing the five edges 12, 23, 34, 14 and 56. Then every total dominating set includes one of 5 or 6, and contains either $\{1,3\}$ or $\{2,4\}$, and thus contains a triangle. This graph H has maximum degree 5. But perhaps it is true that the above theorem extends to maximum degree 4. Here is a picture of the example.



Computer search on small cases suggests that one might be able to improve Theorem 1. For example:

Question 1. *1. Does every graph with minimum degree at least* 2 *and maximum degree at most* 3 *have a* ***minimum*** *total dominating set that is acyclic?*
2. Is it true that if the total domination number is less than $n/2$*, where* n *is the order, then* ***every*** *minimum total dominating set is acyclic?*

Recall that the total domination number of a cubic graph is at most half the order; see [**1**]. If the total domination number is $n/2$ or more, then there can exist minimum total dominating sets that are not acyclic; consider for example the cube on 8 vertices.

Finally, we note that the main theorem implies a result about the idomatic number. Recall that the idomatic number is the maximum number of disjoint independent dominating sets in the graph. By Theorem 1, the graph G has a total dominating set D such that the subgraph induced by D is bipartite. Schaudt [**6**] observed that this condition is equivalent to the graph G containing two disjoint independent dominating sets. Thus:

Theorem 2. *If graph G has minimum degree at least 2 and maximum degree at most 3, then G has two disjoint independent-dominating sets.*

As a corollary it follows that every cubic graph has two disjoint independent-dominating sets. This result is attributed to Berge in [**2**].

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