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THE PROOF OF THE PEREPECHKO's CONJECTURE CONCERNING NEAR-PERFECT MATCHINGS ON $C_m \times P_n$ CYLINDERS OF ODD ORDER

Rade Doroslovački, Jelena Đokić, Bojana Pantić, Olga Bodroža-Pantić*

For all odd values of m, we prove that the sequence of the numbers of near-perfect matchings on $C_m \times P_{2n+1}$ cylinder with a vacancy on the boundary obeys the same recurrence relation as the sequence of the numbers of perfect matchings on $C_m \times P_{2n}$. Further more, we prove that for all odd values of m denominator of the generating function for the total number of the near-perfect matchings on $C_m \times P_{2n+1}$ graph is always the square of denominator of generating function for the sequence of the numbers of perfect matchings on $C_m \times P_{2n}$ graph, as recently conjectured by Perepechko.

1. INTRODUCTION

The monomer-dimer problem, that of counting the exact number of coverings of a rectangular lattice by a previously specified number of monomers and dimers, arises in several models in statistical physics. When the number of monomers is zero this problem basically comes down to the enumeration of perfect matchings, and has been widely studied ([**1**], [**8**]). Whereas in case the number of monomers is one (in the odd sized graphs) the term we use for these configurations is *near-perfect matchings*.

This paper deals with a monomer-dimer problem on $C_m \times P_n$ graphs which shall further on be referred to as *cylinders* (Figure 1). *The vacancy* is a single monomer on a non-bipartite lattice such as these cylinders and rectangular lattices of the form $P_m \times P_n$ where both parameters m and n are odd. A closed-form expression for the number of near-perfect matchings when the vacancy is on the boundary of a cylinder is given in [**10**].

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The similarity of asymptotic expansions for the total number of near-perfect matching on $C_m \times P_n$ cylinders and $P_m \times P_n$ rectangular lattices is reported by Kong in [**5**]. More to the point, Perepechko recently extended these results [**6**] revealing that if the location of the vacancy remains unchanged both the number of near-perfect matchings and the number of perfect matchings [**4**] on a cylinder with odd fixed m are solutions to the same recurrence relation. In order to enumerate matchings he applied a universal method proposed by Wilf [**9**], with the help of an algebraic computer system for whose implementation he required a 64-bit version of Maple 17. The end result was, obtaining the generating functions for $m \leq 13$ which enabled him to put forth the following theorem as a conjecture:

Theorem 1. *[**6**] For the all odd values of m the denominator of the generating function $G_m^N(z)$ for the total number of near-perfect matchings on $C_m \times P_{2n+1}$ is always the square of the denominator of generating function $G_m^P(z)$ for the number of perfect matchings on $C_m \times P_{2n}$.*

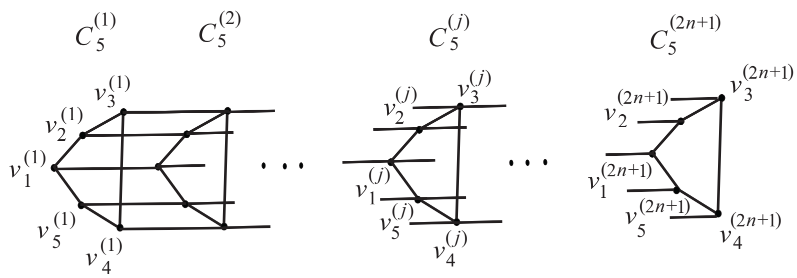

Figure 1: Cylinder $C_5 \times P_{2n+1}$.

2. PRELIMINARIES

From this point forward, we shall assume that m is a fixed odd integer greater than 1. Now, let $C_m^{(j)}$ be the j-th copy of the cycle C_m with vertices $v_1^{(j)}, v_2^{(j)}, \ldots, v_m^{(j)}$ $(1 \leq j \leq n)$ in the $C_m \times P_n$ graph labeled by $G_{m,n}$ (Figure 1). For practical reasons we shall be using the following vertex labels for $v_0^{(i)}$ and $v_{m+1}^{(i)}$ interchangeably with $v_m^{(i)}$ and $v_1^{(i)}$, respectively.

Figure 2 a) depicts a $C_3 \times P_6$ graph together with one of his perfect matchings (represented by the bold edges). In Figure 2 b) and 2 c) an odd sized graph $C_3 \times P_7$ is shown together with a couple of his near-perfect matchings: the first one having a vacancy in the form of vertex $v_1^{(1)}$ on the cylinder's boundary ($C_3^{(1)} \cup C_3^{(7)}$) (Figure 2b) and the second one's vacancy being vertex $v_1^{(4)}$ that belongs to $C_3^{(4)}$ cycle (Figure 2c). Note that the edges of the second near-perfect matching (in bold) which belong to the subgraph of $C_3 \times P_7$ induced by the set of vertices from cycles $C_3^{(1)}$, $C_3^{(2)}$ and $C_3^{(3)}$ determine a near-perfect matching of this subgraph (which is itself a cylinder, too) with a vacancy in $v_2^{(3)}$ on its boundary. We would obtain a perfectly similar situation if we were to consider the last three cycles of $C_3 \times P_7$ instead of the first three. In that case the vacancy would be represented with vertex $v_3^{(5)}$ (Figure 2c).

Let $K_m(n)$ be the number of perfect matchings on the cylinder $G_{m,n}$, when $n \geq 1$ setting $K_m(0) \stackrel{\text{def}}{=} 1$. Let $K_m^v(n)$ be the number of near-perfect matchings on the cylinder $G_{m,n}$ when the vacancy is fixed at vertex v ($v \in V(G_{m,n})$) or, in other words, the number of perfect matchings of the graph $G_{m,n} - v$. Note that the number $K_m(n)$ is non-zero iff n is even and the number $K_m^v(n)$ is non-zero iff n is odd. Further, if the vertices v and w both belong to the same cycle C_m^j ($1 \leq j \leq n$) then $K_m^v(n) = K_m^w(n)$ so let us denote it by $\widehat{K}_m^{(j)}(n)$. This comes as a consequence of the rotational symmetry of the cylinder. However, there is yet another symmetry of it which implies that

$$\widehat{K}_m^{(j)}(n) = \widehat{K}_m^{(n+1-j)}(n), \text{ for all } j \ (1 \leq j \leq n). \tag{1}$$

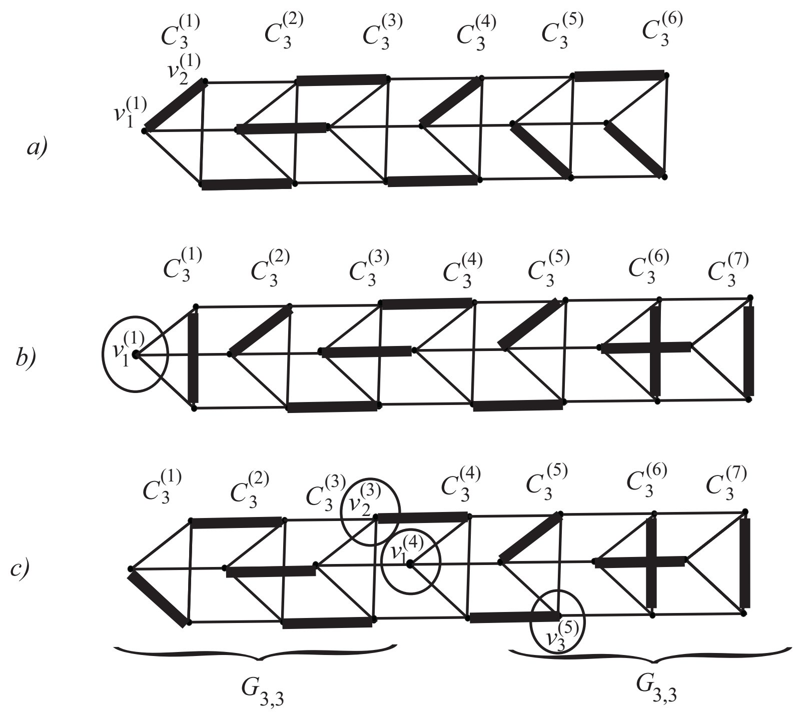

Figure 2: a) Graph $G_{3,6}$ with one of his perfect matchings b) Graph $G_{3,7}$ with one of his near-perfect matchings and vacancy $v_1^{(1)}$ on the cylinder's boundary; c) Graph $G_{3,7}$ with one of his near-perfect matchings and vacancy $v_1^{(4)}$.

We shall adopt the following labels used in [**6**]:

$\mathcal{G}_m^B(z) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \widehat{K}_m^B(n)z^n$ - generating function for the numbers of near-perfect matchings on $G_{m,2n+1}$ graph with one fixed vacant vertex on the boundary, i.e. for the sequence $\widehat{K}_m^B(n) \stackrel{\text{def}}{=} \widehat{K}_m^{(1)}(2n+1)$, $n \geq 0$.

$\mathcal{G}_m^P(z) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \widehat{K}_m^P(n)z^n$ - generating function for the numbers of perfect matchings on the graph $G_{m,2n}$, i.e. for the sequence $\widehat{K}_m^P(n) \stackrel{\text{def}}{=} K_m(2n)$, $n \geq 0$ ($K_m(0) \stackrel{\text{def}}{=} 1$).

$\mathcal{G}_m^N(z) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \widehat{K}_m^N(n)z^n$ - generating function for the total number of near-perfect matchings on $G_{m,2n+1}$ graph, i.e. for the sequence $\widehat{K}_m^N(n) \stackrel{\text{def}}{=} m \cdot \sum_{j=1}^{2n+1} \widehat{K}_m^{(j)}(2n+1)$, $n \geq 0$.

.1. THE MAIN RESULT

With the aim of proving Theorem 1 we shall be in need of the definition bellow.

Definition 1. *For some perfect matching (near-perfect matching) of the graph $G_{m,n}$, where n is even (n is odd), the* **state** *of (vertices belonging to) a cycle $C_m^{(j)}$, where $1 \leq j \leq n$ (where $j' \leq j \leq n$ and $C_m^{(j')}$ comprises vacancy v) is by the definition a cyclic word $p \equiv p_1p_2\ldots p_m$ ($p_0 \stackrel{\text{def}}{=} p_m$ and $p_{m+1} \stackrel{\text{def}}{=} p_1$) over alphabet $\{L,R,M\}$ formed in the following manner:*

$$p_i = \begin{cases} L, & \text{either if vertex } v_i^{(j)} \text{ is a vacancy or the edge } v_i^{(j)}v_i^{(j-1)} \ (j \geq 2) \text{ belongs to} \\ & \text{that perfect matching (near-perfect matching),} \\ R, & \text{if the edge } v_i^{(j)}v_i^{(j+1)} \ (j \leq n-1) \text{ belongs to that perfect matching} \\ & \text{(near-perfect matching),} \\ M, & \text{either if the edge } v_i^{(j)}v_{i+1}^{(j)} \text{ or the edge } v_i^{(j)}v_{i-1}^{(j)} \text{ belongs to that perfect} \\ & \text{matching (near-perfect matching).} \end{cases}$$

For instance, the state of the vertices belonging to $C_3^{(1)}$ cycle in Figure 2a) i b) are MMR and LMM, respectively, whereas the state of vertices belonging to $C_3^{(1)}$ cycle in Figure 2c) is not defined as the vacancy belongs to the forth cycle. For the last near-perfect matching, the state of vertices from the last four cycles are as follows: LLR, MML, RMM and LMM. It is worthy a noticing that all the maximal subwords of the cyclic word $p_1p_2\ldots p_m$ consisting only of the letter M are of even size.

Let us denote by $\mathcal{D}_m \equiv (V(\mathcal{D}_m), E(\mathcal{D}_m))$ ($m \geq 3$) the digraph whose set of vertices $V(\mathcal{D}_m)$ consists of all the possible states of cycles $C_m^{(j)}$ of graph $G_{m,n}$ for some perfect matching (n an arbitrary even integer) or near-perfect matching (n an arbitrary odd integer), whilst the set of edges $E(\mathcal{D}_m)$ is defined in the following way: there exists an edge from vertex $p_1p_2\ldots p_m$ to $q_1q_2\ldots q_m$ iff the following is fulfilled $p_i = R \Leftrightarrow q_i = L$, for all i ($1 \leq i \leq m$). Now, let $\mathcal{P}_m$ be a subset of $V(\mathcal{D}_m)$ which contains all the words which have an even number of letter L (including zero). We shall denote the set $V(\mathcal{D}_m)\backslash\mathcal{P}_m$, which consists of all the words with an odd number of letter L, with $\mathcal{B}_m$. Let us notice the digraph $\mathcal{D}_m$ is in fact the bipartite graph $(\mathcal{P}_m, \mathcal{B}_m)$ which comes as a result of m being an odd number and out of the definition of $E(\mathcal{D}_m)$.

Lemma 1. *$\mathcal{P}_m$ and $\mathcal{B}_m$ are sets of the same cardinality, to be more precise*

$$| \mathcal{P}_m |=| \mathcal{B}_m |= \frac{(1-\sqrt{2})^m + (1+\sqrt{2})^m}{2}.$$

Proof. We begin by defining a mapping $\varphi : \mathcal{P}_m \to \mathcal{B}_m$ between these two sets in the following way. At first, each vertex $p \in \mathcal{P}_m$ is being associated with the element $\varphi(p) \in \mathcal{B}_m$ which as a cyclic word is obtained from the word p substituting each appearance of L with R and reverse, each appearance of R with L. Since m is an odd number, the number of letters R in the word p is consequently odd, thus the number of letters L in the word q is odd. Clearly, this particular mapping is therefore well defined. Additionally, for each element $b \in \mathcal{B}_m$ there is a uniquely determined element from $\mathcal{P}_m$ which maps into it by the same principal - substituting each letter L with R and conversely, so the mapping φ is a bijection.

With the aim of determining the exact cardinality of the set $V(\mathcal{D}_m)$ let l_n be defined as the number of all the (non-cyclic) words of length n over the alphabet $\{L, M, R\}$ such that all of its maximal subwords which consist solely of consecutive Ms are of even length. Having analised all the potential candidates for the choice of the letter corresponding to vertex $v_1^{(j)}$ what we obtain is the following recurrence formula: $l_n = 2l_{n-1} + l_{n-2}$, with the initial conditions $l_1 = 2, l_2 = 5$ ($l_0 \stackrel{\text{def}}{=} 1$) alongside with $| \mathcal{P}_m \cup \mathcal{B}_m | = 2l_{m-1} + 2l_{m-2}$. Solving the above recurrence equation for l_n in the standard fashion we have that $| \mathcal{P}_m \cup \mathcal{B}_m | = (1-\sqrt{2})^m + (1+\sqrt{2})^m$, which implies the very statement of the lemma. □

Let $\mathcal{P}_m^*$, $\mathcal{B}_m^*$ and $\mathcal{E}_m$ be subsets of $V(\mathcal{D}_m)$ which consist of the words without the letter L, words with exactly one letter L and those words which do not contain the letter R at all, respectively. Evidently, $\mathcal{P}_m^* \subseteq \mathcal{P}_m$ i $\mathcal{B}_m^* \subseteq \mathcal{B}_m$. Note that if the cyclic word $p \equiv p_1 p_2 \ldots p_m$ over the alphabet $\{L, R, M\}$ belongs to one of these sets $V(\mathcal{D}_m)$, $\mathcal{P}_m$, $\mathcal{B}_m$, $\mathcal{P}_m^*$, $\mathcal{B}_m^*$ or $\mathcal{E}_m$, then the word $\sigma(p) \stackrel{\text{def}}{\equiv} p_m p_{m-1} \ldots p_2 p_1$ (obtained under reversal) belongs to the very same set, as well as the word $\rho_k(p) \stackrel{\text{def}}{\equiv} p_k p_{k+1} \ldots p_m p_1 \ldots p_{k-1}$ for an arbitrary k ($1 \le k \le m$) (this is a consequence of the cylinder's symmetry with respect to the plane which contains the cylinder's axe and of its rotational symmetry). We say that the words $\sigma(p)$ and $\rho_k(p)$ ($1 \le k \le m$) are *of the same type* as p. Representatives of any possible type of states for $m = 3$ (of some cycle $C_3^{(j)}$) are shown in Figure 3, whereas the same for $m = 5$ (and cycle $C_5^{(j)}$) are depicted in Figure 4 and Figure 5). Additionally, note that in case there exists an edge in the digraph $\mathcal{D}_m$ which points from v to some vertex w, than there exists an edge pointing from vertex $\rho_k(v)$ to vertex $\rho_k(w)$ ($1 \le k \le m$), but also an edge pointing from vertex $\sigma(v)$ to vertex $\sigma(w)$.

Lemma 2. *The number of different types of words belonging to the set $\mathcal{P}_m$ is equal to the number of different types of words from the set $\mathcal{B}_m$.*

Proof. Let us form a bijection between these two sets in the following manner. With each of the representatives $p \in \mathcal{P}_m$ from the class of words that are of the same type we associate a representative of the class of words $\varphi(p) \in \mathcal{B}_m$ (substituting each letter L with R and conversely, each letter R with L). It follows trivially that the mapping defined as such is a bijection from the facts that the mapping $\varphi : \mathcal{P}_m \to \mathcal{B}_m$ is a bijection and that the words of the same type are being mapped into adequate words of the same type. □

Let us denote with $f^p(k)$ the number of all the walks of length k ($k \geq 0$) in the digraph $\mathcal{D}_m$ which begin at vertex p ($p \in V(\mathcal{D}_m)$) and end at a vertex from the set $\mathcal{E}_m$.

It is then fairly simple to prove the following two lemmas:

Lemma 3. *Each of the perfect matchings of the graph $G_{m,2n}$ ($n \geq 1$), corresponds to a unique walk of length $2n-1$ in the digraph $\mathcal{D}_m$ which begins at a vertex from the set $\mathcal{P}_m^*$ and ends at a vertex from the set $\mathcal{E}_m$, whilst the j-th vertex ($1 \leq j \leq 2n$) in that walk represents the state of the j-th cycle $C_m^{(j)}$ from the observed perfect matching, and vice versa. Each walk of length $2n-1$ in the digraph $\mathcal{D}_m$ which starts at a vertex from the set $\mathcal{P}_m^*$ yet ends at a vertex from the set $\mathcal{E}_m$ defines a unique perfect matching of the graph $G_{m,2n}$ whose each cycle $C_m^{(j)}$ ($1 \leq j \leq 2n$) is in the state corresponding to the j-th vertex in that walk.*

Lemma 4. *Each near-perfect matching of the graph $G_{m,2n+1}$ ($n \geq 0$) with a vacancy on the cycle $C_m^{(1)}$ of this graph corresponds to a unique walk of length $2n$ in the digraph $\mathcal{D}_m$ which begins at a vertex from the set $\mathcal{B}_m^*$ and ends at a vertex from the set $\mathcal{E}_m$, whilst the j-th vertex ($1 \leq j \leq 2n+1$) in that walk represents the state of the j-th cycle $C_m^{(j)}$ for the given near-perfect matching, and vice versa. Each walk of length $2n$ in the digraph $\mathcal{D}_m$ which starts at a vertex from the set $\mathcal{B}_m^*$ and ends at a vertex from the set $\mathcal{E}_m$ defines a unique near-perfect matching of the graph $G_{m,2n+1}$ with a vacancy on the cycle $C_m^{(1)}$ whose state of the cycle $C_m^{(j)}$ for each j, where $1 \leq j \leq n$, is exactly the word corresponding to the j-th vertex of that walk.*

Forming the above mentioned bijections we have that

$$\widehat{K}_m^P(n) = K_m(2n) = \sum_{p \in \mathcal{P}_m^*} f^p(2n-1) \quad (n \geq 1) \tag{2}$$

$$\widehat{K}_m^B(n) = \widehat{K}_m^{(1)}(2n+1) = \frac{1}{m} \sum_{b \in \mathcal{B}_m^*} f^b(2n) \quad (n \geq 0) \tag{3}$$

Note that $f^b(0) = 0$ for $b \in \mathcal{B}_m^* \backslash \mathcal{E}_m$, whereas $f^b(0) = 1$ for $b \in \mathcal{B}_m^* \cap \mathcal{E}_m$, and that $\mid \mathcal{B}_m^* \cap \mathcal{E}_m \mid = m$ and $\widehat{K}_m^B(0) = \widehat{K}_m^{(1)}(1) = 1$.

Definition 2. *Two cyclic words $p \equiv p_1 p_2 \dots p_m$ and $q \equiv q_1 q_2 \dots q_m$ over the alphabet $\{L, R, M\}$ are called* **equivalent** *iff there exists a word q^* of the same type as q in which all the letters R (in case there are any at all) appear on exactly the same positions as in the word p*

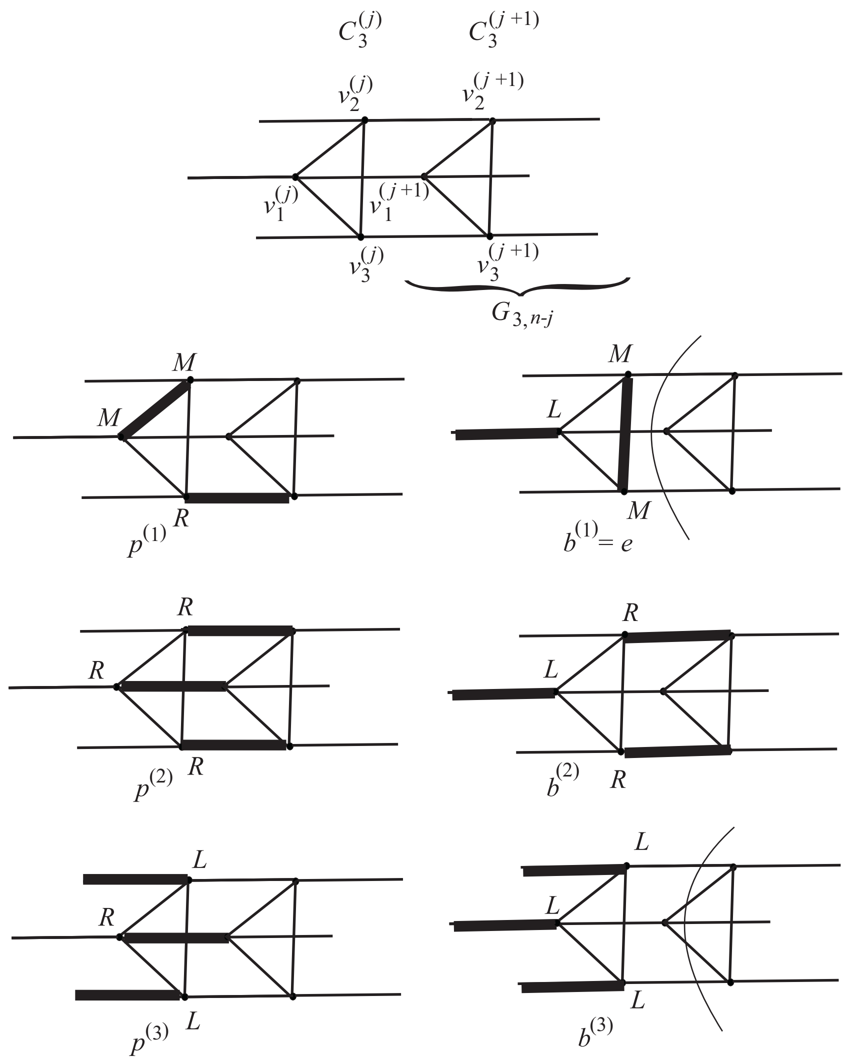

Figure 3: Possible types of states in cycle $C_3^{(j)}$.

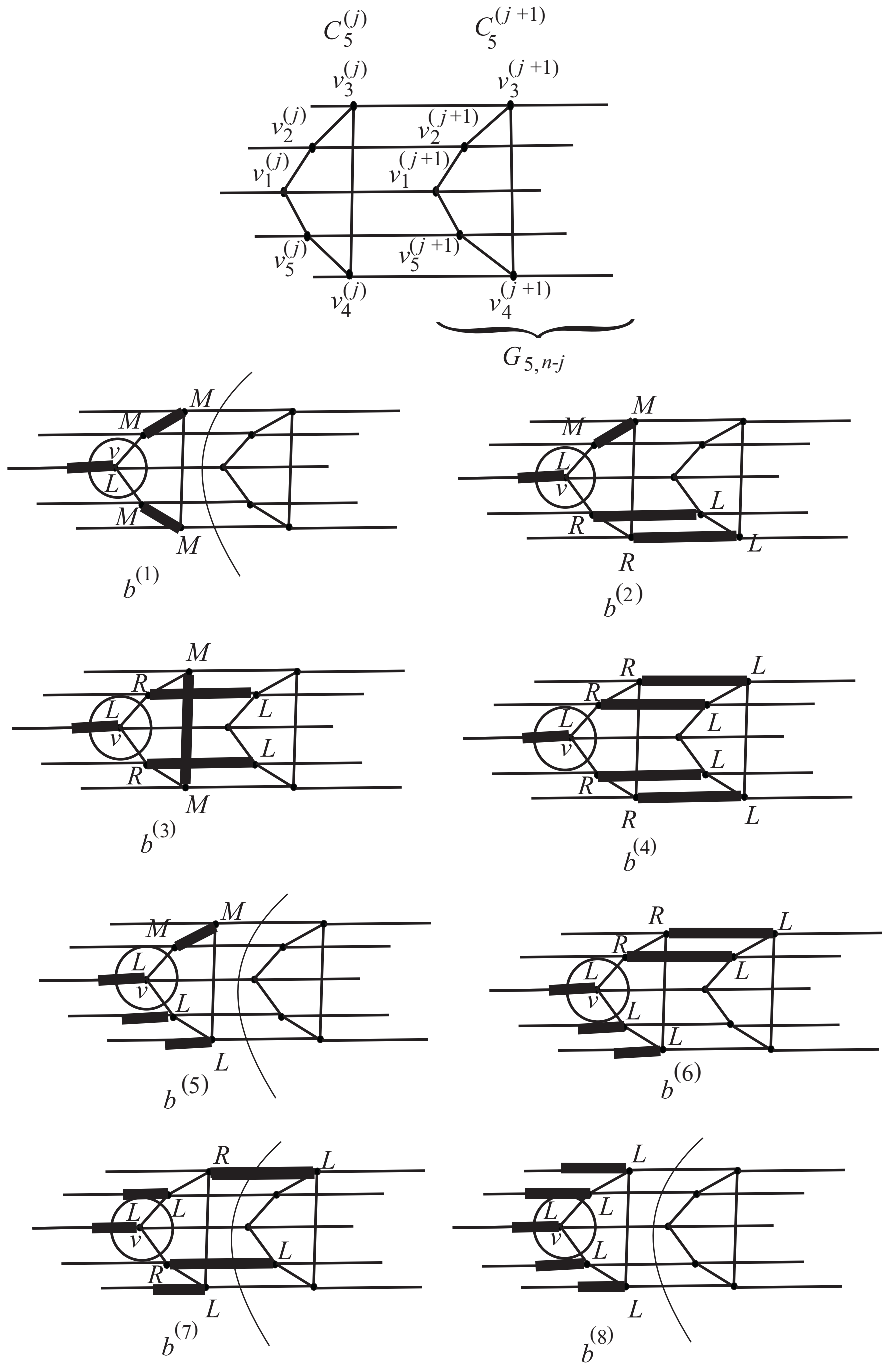

Figure 4: Possible types of states in cycle $C_5^{(j)}$: $b_1 - b_8$ (with an odd number of letters L).

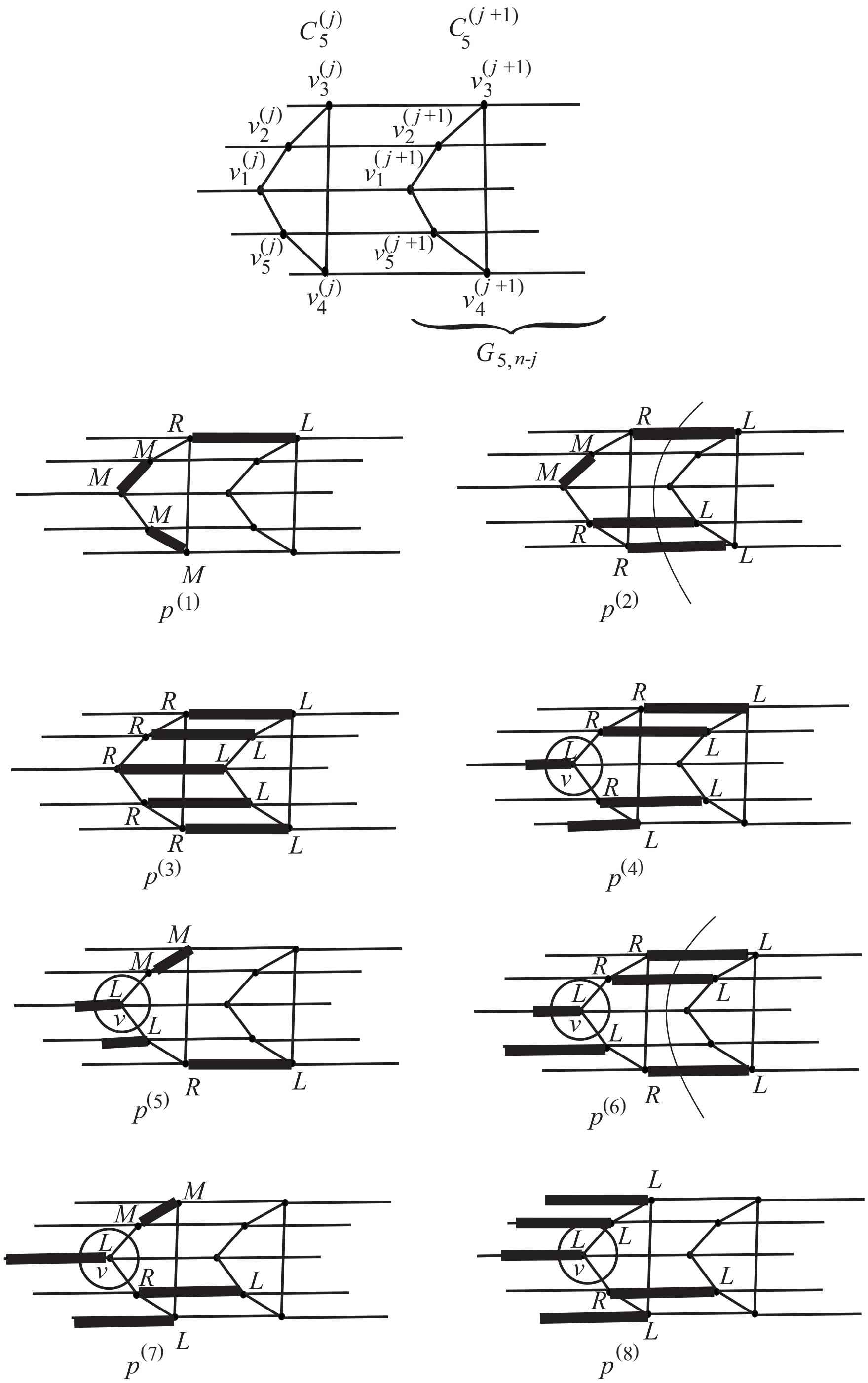

Figure 5: Possible types of states in cycle $C_5^{(j)}$: $p_1 - p_8$ (with an even number of letters L).

For example, vertices $b^{(1)} \equiv LMMMM$ i $b^{(5)} \equiv LMMLL$ are two vertices of the digraph $\mathcal{D}_m$ of a different type, in which the letter R does not appear. Consequently, these two words are equivalent. Vertices $p^{(5)} \equiv LMMRL$, $p^{(7)} \equiv LMMLR$, $p^{(8)} \equiv LLLLR$ i $p^{(1)} \equiv MMRMM$ are also equivalent as e.g. the words $\rho_4(p^{(5)}) \equiv RLLMM$, $\sigma(p^{(7)}) \equiv RLMML$, $\rho_5(p^{(8)}) \equiv RLLLL$ and $\rho_3(p^{(1)}) \equiv RMMMM$ are all the words in which the unique R appears in the initial position. Additionally, when observing the vertices $b^{(3)} \equiv LRMMR$ and $\rho_4(b^{(7)}) \equiv LRLLR$ we come to notice that the letters R appear in the second and the fifth position, thus proving the vertices $b^{(3)}$ i $b^{(7)}$ to be equivalent. Note that all the vertices of the same type are also mutually equivalent.

Lemma 5. *In case the vertices v and w of the digraph $\mathcal{D}_m$ are equivalent, then $f^v(k) = f^w(k)$, for all $k \geq 0$.*

Proof. Let us consider, at first, the case of different vertices of the same type. Let $w = \rho_r(v)$, for some integer r ($2 \leq r \leq m$). Every walk $v_0 \equiv v, v_1, v_2, \ldots, v_k$ of length k in the digraph $\mathcal{D}_m$ which starts with v and ends at a vertex from the set $\mathcal{E}_m$ ($v_k \in \mathcal{E}_m$) corresponds to a unique walk $\rho_r(v_0) \equiv \rho_r(v) \equiv w, \rho_r(v_1), \rho_r(v_2), \ldots, \rho_r(v_k)$ of length k in the digraph $\mathcal{D}_m$ which begins at w and ends at a vertex from the set $\mathcal{E}_m$ ($\rho_r(v_k) \in \mathcal{E}_m$), and vice versa. Every walk $w_0 \equiv w, w_1, w_2, \ldots, w_k$ of length k in the digraph $\mathcal{D}_m$ which begins with w and ends at a vertex belonging to the set $\mathcal{E}_m$ ($w_k \in \mathcal{E}_m$) corresponds to a unique walk $\rho_{m-r}(w_0) \equiv \rho_{m-r}(w) \equiv v, \rho_{m-r}(w_1), \rho_{m-r}(w_2), \ldots, \rho_{m-r}(w_k)$ of length k in the digraph $\mathcal{D}_m$ which begins at v and ends at a vertex from the set $\mathcal{E}_m$ ($\rho_{m-r}(w_k) \in \mathcal{E}_m$). Consequently we have $f^v(k) = f^w(k)$. The case of $w = \sigma(v)$ is pretty similar.

If the vertices v and w are not of the same type, yet equivalent, then there exists the vertex w^* of the same type as w in which the letter R appears on the exact same positions as in v. Then each successor of vertex v is additionally the successor of vertex w^* in the digraph $\mathcal{D}_m$ and the other way around (having letters L in exactly those positions where Rs are in v or w^*). This implies a possibility of forming a bijection between the set of all the walks of length k which begin at v and end at a vertex from the set $\mathcal{E}_m$ with the set of all such walks which instead of beginning at vertex v start at vertex w^* simply by substituting the initial vertex v with w^* in each of those walks. This is why $f^v(k) = f^{w^*}(k) = f^w(k)$. □

Let M_m be the adjacency matrix of the digraph $\mathcal{D}_m$. Based on the Cayley-Hamilton theorem, all the sequences $f^v(k), k \geq 0, v \in V(\mathcal{D}_m)$ fulfill the same recurrence relation which is determined by the characteristic equation of the matrix M_m. Since all of these sequences are the same for all the equivalent vertices, then contracting all of such into one single vertex, with edges being multiplied in case one of the vertices had more different successors which are mutually equivalent, we obtain a multidigraph $\overline{\mathcal{D}}_m$ with the adjacency matrix $\overline{M}_m$ whose characteristic equation also determines the recurrence formulae of the observed sequences. Sets of vertices $\mathcal{P}_m$, $\mathcal{P}_m^*$, $\mathcal{B}_m$ and $\mathcal{B}_m^*$ are in this way being reduced to the sets $\overline{\mathcal{P}}_m$, $\overline{\mathcal{P}}_m^*$, $\overline{\mathcal{B}}_m$ and $\overline{\mathcal{B}}_m^*$, respectively. Clearly, $\overline{\mathcal{P}}_m^* \subseteq \overline{\mathcal{P}}_m$ and $\overline{\mathcal{B}}_m^* \subseteq \overline{\mathcal{B}}_m$.
Multidigraph $\overline{\mathcal{D}}_3$ is shown in Figure 6 whereas $\overline{\mathcal{D}}_5$ is in Figure 7. For instance, from the vertex which came into existence by the subtraction of vertex $b^{(1)} = LMMMM$ and the vertices which were equivalent to it five edges point into the vertex obtained by the

subtraction of vertex $p^{(2)} = MMRRR$ and the vertices equivalent to it as amongst them there exist five vertices in total ($MMRRR, RMMRR, RRMMR, RRRMM$ and $MRRRM$) that are all of the same type as $p^{(2)}$ (including this vertex itself) all of which are successors of $b^{(1)}$ in the digraph $\mathcal{D}_m$. From the vertex which was created by the contraction of vertex $p^{(1)} = MMRMM$ the vertices equivalent to it point two edges into the vertex obtained by the contraction of $b^{(2)} = LMMRR$ and the vertices equivalent to it, for there exist exactly two vertices of the digraph $\mathcal{D}_5$ from the ten vertices that are of the same type as vertex $b^{(2)}$ which are the successors of vertex $p^{(1)}$. Those are $\rho_4(b^{(2)}) \equiv RRLMM$ and $\rho_4(\sigma(b^{(2)})) \equiv MMLRR$ (with the letter L on the third position).

Note that all the vertices from the set $\mathcal{E}_m$ are being contracted into one single vertex $e \in \overline{\mathcal{B}}_m$. In case $m = 5$, vertex e has been obtained by contracting the following few vertices $b^{(1)}$, $b^{(5)}$, $b^{(8)}$ and the vertices of the same type as theirs (Figure 7). However, this is not the case for the set $\mathcal{P}_m^*$. Namely, already for the case of $m = 3$ we have that $|\overline{\mathcal{P}}_m^*| > 1$.

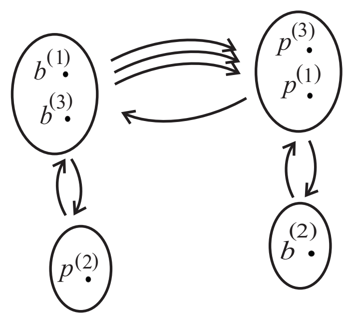

Figure 6: Multidigraph $\overline{\mathcal{D}}_3$.

Lemma 6. *The sets $\overline{\mathcal{P}}_m$ and $\overline{\mathcal{B}}_m$ are of the same cardinality.*

Proof. Note that the number of all the elements of the set $\overline{\mathcal{P}}_m$ (considered as words) in which the letter R appears k times (k is odd) is equal to the number of all the elements from the set $\overline{\mathcal{B}}_m$ in which the letter R appears $m-k$ ($m-k$ is even) times. In other words, the number of different (binary) bracelets with k red beads and $m-k$ green beads is equal to the number of different bracelets with $m-k$ red beads and k green beads. (What we consider under the term of bracelet is a circle of m colored beads with up to two different colors that can be turned over.) Therefore, $|\overline{\mathcal{P}}_m| = |\overline{\mathcal{B}}_m| = |V(\overline{\mathcal{D}}_m)|/2$. □

To illustrate this, for instance, the number of vertices in $\overline{\mathcal{D}}_5$ with exactly two Rs ($b^{(2)}$ and $b^{(4)}$) is equal to the number of vertices with exactly 3 Rs ($p^{(2)}$ and $p^{(4)}$). Additionally, the number of vertices in $\overline{\mathcal{D}}_7$ containing $0, 2, 4$ and 6 letters R is $1, 3, 4$ and 1, respectively, which is exactly the number of vertices we have for $\overline{\mathcal{D}}_7$ with $7, 5, 3$ and 1 letter R, respectively. The exact number of the bracelets with m beads is

$$|V(\overline{\mathcal{D}}_m)| = \frac{1}{2}\left(\frac{1}{m}\sum_{d|m}\phi(d)2^{m/d} + 2^{(m+1)/2}\right)$$

with $\phi(d)$ the Euler's totient function [**2**]. For an efficient way to generate them see [**7**].

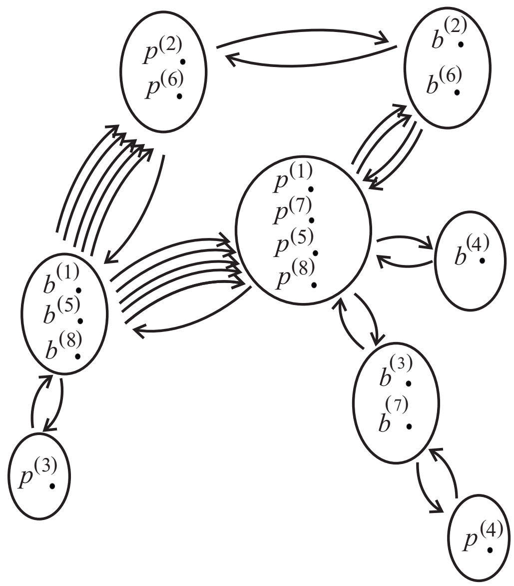

Figure 7: Multidigraph $\overline{\mathcal{D}}_5$.

Lemma 7. *All the sequences* $g^b(n) \stackrel{\text{def}}{=} f^b(2n)$ $(n \geq 0)$*, for an arbitrary* $b \in \mathcal{B}_m$*, as well as the sequences* $g^p(n) \stackrel{\text{def}}{=} f^p(2n+1)$ $(n \geq 0)$*, for an arbitrary* $p \in \mathcal{P}_m$ *obey the very same difference equation of order* $\mid V(\overline{\mathcal{D}}_m) \mid /2$.

Proof. Owing to the fact that the multidigraph $\overline{\mathcal{D}}_m = (\overline{\mathcal{P}}_m, \overline{\mathcal{B}}_m)$ is bipartite (as well as the digraph $\mathcal{D}_m = (\mathcal{P}_m, \mathcal{B}_m)$ itself), his adjacency matrix may be represented as a block matrix $\begin{bmatrix} 0 & A_m \\ B_m & 0 \end{bmatrix}$, where the matrices A_m and B_m are square matrices of the same order $\mid \overline{\mathcal{P}}_m \mid = \mid \overline{\mathcal{B}}_m \mid = \mid V(\overline{\mathcal{D}}_m) \mid /2$.

Let $\overline{\mathcal{P}}_m = \{p_1, p_2, \ldots, p_s\}$ and $\overline{\mathcal{B}}_m = \{b_1, b_2, \ldots, b_s\}$, where $s = \mid V(\overline{\mathcal{D}}_m) \mid /2$. For the sequences $f^v(k), k \geq 0, v \in V(\overline{\mathcal{D}}_m)$ the following holds:

$$(4) \qquad \begin{bmatrix} f^{p_1}(k) \\ f^{p_2}(k) \\ \vdots \\ f^{p_s}(k) \\ f^{b_1}(k) \\ f^{b_2}(k) \\ \vdots \\ f^{b_s}(k) \end{bmatrix} = \begin{bmatrix} 0 & A_m \\ B_m & 0 \end{bmatrix}^k \begin{bmatrix} f^{p_1}(0) \\ f^{p_2}(0) \\ \vdots \\ f^{p_s}(0) \\ f^{b_1}(0) \\ f^{b_2}(0) \\ \vdots \\ f^{b_s}(0) \end{bmatrix} \quad \text{and } f^x(0) = \begin{cases} 0, & \text{for } x \neq e \\ 1, & \text{for } x = e. \end{cases}$$

From here we reach a conclusion that the sequence $g^b(n) \stackrel{\text{def}}{=} f^b(2n)$, for each $b \in \mathcal{B}_m$ obeys the difference equation which is determined by a characteristic equation of the matrix $B_m A_m$, whereas the sequence $g^p(n) \stackrel{\text{def}}{=} f^p(2n+1)$, for each $p \in \mathcal{P}_m$ satisfies the recurrence formula which is determined by the characteristic equation of the matrix $A_m B_m$. Now, since for any two arbitrary square matrices A and B the matrices AB and BA have the same characteristic polynomial (Theorem 1.3.22 in [**3**]), the statement of the lemma follows. □

Note that $\widehat{K}_m^B(n) \stackrel{\text{def}}{=} \widehat{K}_m^{(1)}(2n+1) = \dfrac{1}{m} \sum\limits_{b \in \mathcal{B}_m^*} f^b(2n) = \dfrac{1}{m} \sum\limits_{b \in \mathcal{B}_m^*} g^b(n)$ $(n \geq 0)$, whereas the sequence $\widehat{K}_m^P(n) \stackrel{\text{def}}{=} K_m(2n) = \sum\limits_{p \in \mathcal{P}_m^*} f^p(2n-1) = \sum\limits_{p \in \mathcal{P}_m^*} g^p(n-1)$ $(n \geq 1)$. If two sequences satisfy the same linear, homogeneous difference equation with constant coefficients, then their linear combination also satisfies the very same recurrence formula. Thus it holds that:

Corollary 1. *The sequences $\widehat{K}_{\underline{m}}^B(n)$ $(n \geq 0)$ and $\widehat{K}_m^P(n)$ $(n \geq 1)$, satisfy the same difference equation of order at most $|\, V(\mathcal{D}_m)\,|\,/2$.*

In case of $m = 3$ we have $A_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $B_3 = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$, whereas in case of $m = 5$ we have $A_5 = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ and $B_5 = \begin{bmatrix} 5 & 5 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

Characteristic polynomial both for matrix A_3B_3 and for matrix B_3A_3 is $1-5z+z^2$ whereas for matrices A_5B_5 and B_5A_5 that polynomial is in fact $1-19z+41z^2-19z^3+z^4$. Now applying the standard approach we obtain $\mathcal{G}_3^B(z) = \dfrac{1}{1-5z+z^2}$ and $\mathcal{G}_3^P(z) = \dfrac{1-z}{1-5z+z^2}$, whereas in case of $m = 5$ we have $\mathcal{G}_5^B(z) = \dfrac{1-z^2}{1-19z+41z^2-19z^3+z^4}$ and $\mathcal{G}_5^P(z) = \dfrac{(1-z)(1-7z+z^2)}{1-19z+41z^2-19z^3+z^4}$, which is in agreement with the results obtained in [**6**].

Let us now turn our attention to the case in which the vacancy v of some near-perfect matching of the graph $G_{m,2n+1}$ appears on the cycle $C_m^{(j)}$ where $2 \leq j \leq 2n$.

The Proof of Theorem 1

Without loss of generality, we shall assume that $v \equiv v_1^{(j)}$ is a vacancy.

I case: j is an even integer

Let $j = 2k$ for some k $(1 \leq k \leq n)$. The existence of a near-perfect matching implies that the state of the vertices belonging to the cycle $C_m^{(j)}$, denoted by p, contains an even number of letters L and an odd number of letters R, i.e. $p \in \mathcal{P}_m$. The number od near-perfect matchings in the graph $G_{m,2n+1}$ with vacancy $v \equiv v_1^{(j)}$ which generate the state p on the j-th cycle is, according to the multiplication law, equal to the number $f^p(2(n-k)+1) f^{\hat{p}}(2k-1)$, where

$\hat{p}$ denotes the state obtained from p when all the letters L, with the exception of the first appearance, have been replaced with R and all letters R with L. Note that $\hat{p} \in \mathcal{P}_m$, too. The total number of near-perfect matchings in the graph $G_{m,2n+1}$ with vacancy $v \equiv v_1^{(j)}$ is equal to

$$\widehat{K}_m^{(2k)}(2n+1) = \sum_{p \in \mathcal{P}_m^1} f^p(2(n-k)+1) f^{\hat{p}}(2k-1), \tag{5}$$

where$\mathcal{P}_m^1$ denotes the set of all the words from $\mathcal{P}_m$ which start with the letter L.

II case: j is an odd integer

If $j = 2k+1$ for some k $(1 \leq k \leq n-1)$, then the state of vertices belonging to the cycle $C_m^{(j)}$, denoted by b, belongs to the set $\mathcal{B}_m$ because it contains an odd number od letters L and an even number of letters R. The number of near-perfect matchings of the graph $G_{m,2n+1}$ with vacancy $v \equiv v_1^{(j)}$ which generate the state b on the j-th cycle is equal to $f^b(2(n-k)) f^{\hat{b}}(2k)$, where $\hat{b}$ represents the state obtained from b when all the letters L, except for its first occurrence, have been replaced with R and by the substitution of all the letters R with L. The total number of near-perfect matchings in the graph $G_{m,2n+1}$ with vacancy $v \equiv v_1^{(j)}$ is now equal to

$$\widehat{K}_m^{(2k+1)}(2n+1) = \sum_{b \in \mathcal{B}_m^1} f^b(2(n-k)) f^{\hat{b}}(2k), \tag{6}$$

where $\mathcal{B}_m^1$ denotes the set of all the words from $\mathcal{B}_m$ which begin with the letter L.

Now, utilizing (5), (6), (1) and (4) we have

$$\widehat{K}_m^N(n) \stackrel{\text{def}}{=} m \cdot \sum_{j=1}^{2n+1} \widehat{K}_m^{(j)}(2n+1) = m \cdot \left[\sum_{p \in \mathcal{P}_m^1} \left(\sum_{k=1}^{n} f^p(2(n-k)+1) f^{\hat{p}}(2k-1) \right) + \right.$$
$$\left. \sum_{b \in \mathcal{B}_m^1} \left(2 f^b(2n) f^{\hat{b}}(0) + \sum_{k=1}^{n-1} f^b(2(n-k)) f^{\hat{b}}(2k) \right) \right], \text{ i.e.}$$

$$\widehat{K}_m^N(n) = m \cdot \left[\sum_{p \in \mathcal{P}_m^1} \left(\sum_{k=0}^{n-1} g^p((n-1)-k) g^{\hat{p}}(k) \right) + \sum_{b \in \mathcal{B}_m^1} \left(\sum_{k=0}^{n} g^b(n-k) g^{\hat{b}}(k) \right) \right]. \tag{7}$$

Let $\alpha^p(n)$ be the sequence $\alpha^p(n) \stackrel{\text{def}}{=} \sum_{k=0}^{n} g^p(n-k) g^{\hat{p}}(k)$, where $p \in \mathcal{P}_m \cup \mathcal{B}_m$ i $n \geq 0$. From (7) we have

$$\widehat{K}_m^N(n) = m \cdot \left[\sum_{p \in \mathcal{P}_m^1} \alpha^p(n-1) + \sum_{b \in \mathcal{B}_m^1} \alpha^b(n) \right]. \tag{8}$$

Since the generating function of sequence $\alpha^p(n)$ is equal to the product of generating functions of the sequences $g^p(n)$ and $g^{\hat{p}}(n)$, thus based upon Lemma 7 the statement of Theorem 1 follows, which was our goal. □

Let us now derive the formula for $\widehat{K}_5^N(n)$.

The set $\mathcal{B}_5^1$ comprises five vertices (*LMMMM*, *LMMLL*, *LLMML*, *LLLMM* and *LLLLL*) equivalent to $b^{(1)}$ (whereas the vertices corresponding to them are *LMMMM*, *LMMRR*, *LRMMR*, *LRRMM* i *LRRRR*, respectively, which are equivalent to $b^{(1)}$, $b^{(2)}$, $b^{(3)}$, $b^{(2)}$ and $b^{(4)}$), five vertices (*LMMRR*, *LRRMM*, *LRRLL*, *LLRRL* i *LLLRR*) equivalent to $b^{(2)}$ (for two of them the corresponding vertices (the ones "with a hat") are equivalent to $b^{(1)}$, two to $b^{(2)}$ and one to $b^{(3)}$), four vertices (*LRMMR*, *LLRLR*, *LRLLR* i *LRLRL*) equivalent to $b^{(3)}$ (for two of them the corresponding vertices are equivalent to $b^{(3)}$, one to $b^{(1)}$ and one to $b^{(2)}$) and vertex $b^{(4)}$ (*LRRRR*) (whose corresponding vertex $\hat{b}^{(4)} = b^{(8)}$ is equivalent to $b^{(1)}$).

The set $\mathcal{P}_5^1$ contains four vertices of the same type as $p^{(5)}$ and two vertices of the same type as $p^{(7)}$ (all of which are equivalent to $p^{(1)}$ and for each such vertex p, the vertex corresponding to it $\hat{p}$ is equivalent to $p^{(1)}$), four vertices of the same type as $p^{(8)}$ (all equivalent to $p^{(1)}$, for two of which the corresponding vertices are equivalent to $p^{(2)}$ and for the other two to $p^{(4)}$), two vertices of the same type as $p^{(4)}$ (the vertices corresponding to them are equivalent to $p^{(1)}$) and two vertices of the same type as $p^{(6)}$ (equivalent to $p^{(2)}$ and the vertices corresponding to them are equivalent to $p^{(1)}$).

From the initial conditions for sequences g^p, where $p \in \overline{\mathcal{P}}_5 \cup \overline{\mathcal{B}}_5$, obtained by the exponentiation of matrix $\overline{M}_5$, we derive their generating functions:

$$g^{p_1}(z) = \frac{1-z^2}{Q(z)}, \quad g^{p_2}(z) = \frac{1-5z+3z^2}{Q(z)}, \quad g^{p_3}(z) = g^{b_1}(z) = \frac{1-8z+8z^2-z^3}{Q(z)},$$

$$g^{p_4}(z) = g^{b_3}(z) = \frac{z+z^2}{Q(z)}, \quad g^{b_2}(z) = \frac{3z-5z^2+z^3}{Q(z)}, \quad g^{b_4}(z) = \frac{z-z^3}{Q(z)},$$

where $Q(z) \stackrel{\text{def}}{=} 1-19z+41z^2-19z^3+z^4$.

Applying the formula (8) we get

$$\mathcal{G}_5^N(z) = 5z\left[6g^{p_1}(z)g^{p_1}(z)+4g^{p_1}(z)g^{p_2}(z)+4g^{p_1}(z)g^{p_4}(z)\right]+5\Big[g^{b_1}(z)g^{b_1}(z)+ 4g^{b_1}(z)g^{b_2}(z)+2g^{b_1}(z)g^{b_3}(z)+2g^{b_1}(z)g^{b_4}(z)+2g^{b_2}(z)g^{b_2}(z)+2g^{b_2}(z)g^{b_3}(z)+ 2g^{b_3}(z)g^{b_3}(z)\Big] = 5\frac{(1+10z-56z^2+84z^3-24z^4-10z^5+z^6)}{(1-19z+41z^2-19z^3+z^4)^2},$$

which has been obtained in [**6**].

The derivation of the formula $\mathcal{G}_3^N(z) = 3\frac{(1+2z-z^2)}{(1-5z+z^2)^2}$ is fairly analogous and is hence left as an exercise for the readers.

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