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SOME COMBINATORIAL IDENTITIES OF THE r-WHITNEY-EULERIAN NUMBERS

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In this paper, we study further properties of a recently introduced generalized Eulerian number, denoted by $A_{m,r}(n,k)$, which reduces to the classical Eulerian number when $m=1$ and $r=0$. Among our results is a generalization of an earlier symmetric Eulerian number identity of Chung, Graham and Knuth. Using the row generating function for $A_{m,r}(n,k)$ for a fixed n, we introduce the r-Whitney-Euler-Frobenius fractions, which generalize the Euler-Frobenius fractions. Finally, we consider a further four-parameter combinatorial generalization of $A_{m,r}(n,k)$ and find a formula for its exponential generating function in terms of the Lambert-W function.

1. Introduction

Recall that the *r-Whitney numbers of the second kind* $W_{m,r}(n,k)$ (see, e.g., [**17**]) are defined as connection constants in the polynomial identities

$$(mx+r)^n = \sum_{k=0}^{n} m^k W_{m,r}(n,k) x^{\underline{k}}, \qquad n \geq 0, \tag{1}$$

where $x^{\underline{n}} = x(x-1)\cdots(x-n+1)$ if $n \geq 1$ with $x^{\underline{0}} = 1$. Note that when $m=1$ and $r=0$, the $W_{m,r}(n,k)$ reduce to the classical Stirling numbers of the second kind.

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Given variables m and r, let $A_{m,r}(n,k)$ denote the *r-Whitney-Eulerian numbers* defined by Mező and Ramírez [**19**] by the expression

$$A_{m,r}(n,k) = \sum_{j=0}^{k} (-1)^{k-j} m^j j! \binom{n-j}{k-j} W_{m,r}(n,j). \tag{2}$$

Observe that if $(m,r) = (1,0)$, we recover the classical Eulerian numbers $A(n,k)$. Additionally, if $(m,r) = (1,r)$, we obtain the cumulative numbers studied by Dwyer [**10, 11**], see also the Euler-Frobenius numbers considered by Gawronski and Neuschel [**13**]. If $(m,r) = (q+1,1)$, one obtains the q-Eulerian numbers studied by Brenti [**3**].

One can show that the r-Whitney-Eulerian numbers satisfy the recurrence relation

$$A_{m,r}(n,k) = (mk+r)A_{m,r}(n-1,k) + (m(n-k+1)-r)A_{m,r}(n-1,k-1), \tag{3}$$

with $A_{m,r}(0,0) = 1$ and $A_{m,r}(n,k) = 0$ if $k > n$ or $k < 0$. For a similar class of Eulerian numbers connected to the Whitney numbers, see the papers of Rahmani [**21**] and Mező [**18**]. Note that the $A_{m,r}(n,k)$ are given explicitly by (see [**22**, Theorem 6])

$$A_{m,r}(n,k) = \sum_{i=0}^{k} (-1)^i \left((k-i)m+r\right)^n \binom{n+1}{i}, \qquad 0 \le k \le n, \tag{4}$$

which reduces to the well-known Eulerian number identity (cf. [**8**, p. 243])

$$A(n,k) = \sum_{i=0}^{k} (-1)^i (k-i)^n \binom{n+1}{i}$$

when $m = 1$ and $r = 0$.

The Eulerian number $A(n,k)$ has a combinatorial interpretation in that it enumerates the subset of permutations $\pi = \pi_1\pi_2\cdots\pi_n$ of $[n] = \{1,2,\ldots,n\}$ whose members have exactly $k-1$ descents, where a descent corresponds to an index $i \in [n-1]$ such that $\pi_i > \pi_{i+1}$. Using this interpretation, one can show the recurrence (cf. [**8**])

$$A(n,k) = (n-k+1)A(n-1,k-1) + kA(n-1,k), \quad n,k \ge 2, \tag{5}$$

with $A(n,1) = 1$ for $n \ge 1$. Further properties of this sequence can be found in the texts [**8, 20**]. Note that in the final section of the current paper, we provide a combinatorial interpretation for a couple of polynomial generalizations of $A(n,k)$ in terms of statistics on certain classes of permutations.

The r-Whitney-Eulerian numbers are related to the generalized Eulerian numbers $\widehat{A}_{m,r}(n,k)$ defined by Xiong et al. [**25**] by the formula $A_{m,r}(n,n-k-1) =$

$\widehat{A}_{m,r}(n,k)$. This together with Lemmas 7 and 8 from [**25**] gives

$$\begin{aligned}(mx+r)^n &= \sum_{k=0}^{n} A_{m,r}(n,k)\binom{x+n-k}{n} \\ &= \sum_{k=1}^{n+1} A_{m,r}(n,n-k+1)\binom{x+k-1}{n}, \quad n \geq 0.\end{aligned}$$

These identities provide a generalization of the *Worpitzky's identity* for the Eulerian numbers, namely,

$$x^n = \sum_{k=0}^{n} \binom{x+k-1}{n} A(n,k), \quad n \geq 0,$$

see, e.g., [**14**, Eqn. 6.37].

The organization of this paper is as follows. In the next section, we find analogues of an Eulerian number identity originally shown by Chung et al. [**7**], where it was requested to seek extensions of their result. In the subsequent section, we establish connections between $A_{m,r}(n,k)$ and generalized harmonic numbers and Stirling numbers of the first kind. In the fourth section, we introduce the r-Whitney-Euler-Frobenius fractions, which are defined in terms of the generalized Eulerian polynomials $\sum_{k=0}^{n} A_{m,r}(n,k)x^k$, and establish some basic properties. In the final section, we consider a further four-parameter polynomial generalization, denoted by $A(n,k;a,b,c,d)$, of the Eulerian numbers and also a related sequence. The $A_{m,r}(n,k)$ will be seen to correspond to the special case of $A(n,k;a,b,c,d)$ when $a=m, b=m+r, c=m, d=m-r$. We provide a combinatorial interpretation for $A(n,k;a,b,c,d)$ and an expression for the exponential generating function (e.g.f.) is found in terms of the Lambert-W function by solving explicitly a certain multi-parameter linear partial differential equation.

2. A Special Quasi-Symmetric Identity

In this section, we generalize to $A_{m,r}(n,k)$ a previous identity for the classical Eulerian numbers. It was shown in [**7**] that for positive integers a and b,

$$\sum_{n=b-1}^{a+b} E(n,b-1)\binom{a+b}{n} = \sum_{n=a-1}^{a+b} E(n,a-1)\binom{a+b}{n}, \tag{6}$$

where $E(n,k)$ denotes the number of permutations of $[n]$ with k descents, with $E(0,0)=0$. A q-version of this identity was given by Han et al. [**15**] and here we establish an analogue of (6) for $A_{m,r}(n,k)$. To do so, we extend and modify a proof from [**7**] and consider a more general sum involving a third parameter c. First, we will need a formula for a certain type of binomial convolution.

Lemma 1. *If $a, b, c \in \mathbb{Z}$ with $|c| \leq a, b$, then*

$$\sum_{n=a-c}^{a+b} A_{m,r}(n, a-c)\binom{a+b}{n} = X - Y,$$

where

$$X := \sum_{\ell=0}^{b+c} \binom{a+b+1}{\ell} ((b+c+1-\ell)m - r)^{a+b-\ell} (1-(b+c+1-\ell)m + r)^{\ell},$$

$$Y := \sum_{\ell=0}^{b+c-1} \binom{a+b}{\ell} ((b+c-\ell)m - r)^{a+b-\ell-1} (1-(b+c-\ell)m + r)^{\ell}.$$

Proof. We have

$$\begin{aligned}\sum_{n=a-c}^{a+b} A_{m,r}(n, a-c)\binom{a+b}{n} &= \sum_{n=0}^{b+c} A_{m,r}(a+b-n, a-c)\binom{a+b}{a+b-n} \\ &= \sum_{n=0}^{b+c} \left(\binom{a+b+1}{a+b-n+1} - \binom{a+b}{a+b-n+1}\right) A_{m,r}(a+b-n, a-c).\end{aligned}$$

Let

$$\begin{aligned}X &= \sum_{n=0}^{b+c} \binom{a+b+1}{a+b-n+1} A_{m,r}(a+b-n, a-c) \\ &= \sum_{n=0}^{b+c} \binom{a+b+1}{a+b-n+1} \widehat{A}_{m,r}(a+b-n, b+c-1-n), \\ Y &= \sum_{n=0}^{b+c} \binom{a+b}{a+b-n+1} A_{m,r}(a+b-n, a-c) \\ &= \sum_{n=0}^{b+c} \binom{a+b}{a+b-n+1} \widehat{A}_{m,r}(a+b-n, b+c-1-n),\end{aligned}$$

where $\widehat{A}_{m,r}(n,k)$ is the generalized Eulerian number of Xiong et al. [**25**]. This sequence satisfies the following identity:

$$\widehat{A}_{m,r}(n,k) = \sum_{i=0}^{k+1} (-1)^i ((k+2-i)m - r)^n \binom{n+1}{i}. \tag{7}$$

From (7), the rearrangement of sums

$$\sum_{k=0}^{n}\sum_{j=0}^{n-k} f(k+j, j) = \sum_{p=0}^{n}\sum_{j=0}^{p} f(p, j),$$

and the fact $\binom{n}{k+j}\binom{k+j}{j} = \binom{n}{n-k}\binom{n-k}{j}$, we obtain

$$\begin{aligned}
X &= \sum_{n=0}^{b+c} \binom{a+b+1}{a+b-n+1} \sum_{j=0}^{b-n+c} (-1)^j \left((b-n+c+1-j)m-r\right)^{a+b-n} \binom{a+b-n+1}{j} \\
&= \sum_{n=0}^{b+c} \sum_{j=0}^{b+c-n} \binom{a+b+1}{n+j} \binom{n+j}{j} (-1)^j \left((b+c+1-(n+j))m-r\right)^{a+b-n} \\
&= \sum_{\ell=0}^{b+c} \sum_{j=0}^{\ell} \binom{a+b+1}{\ell} \binom{\ell}{j} (-1)^j \left((b+c+1-\ell)m-r\right)^{a+b-\ell+j} \\
&= \sum_{\ell=0}^{b+c} \binom{a+b+1}{\ell} \left((b+c+1-\ell)m-r\right)^{a+b-\ell} \sum_{j=0}^{\ell} \binom{\ell}{j} (-1)^j \left((b+c+1-\ell)m-r\right)^j \\
&= \sum_{\ell=0}^{b+c} \binom{a+b+1}{\ell} \left((b+c+1-\ell)m-r\right)^{a+b-\ell} \left(1-(b+c+1-\ell)m+r\right)^{\ell}
\end{aligned}$$

and

$$\begin{aligned}
Y &= \sum_{n=0}^{b+c} \sum_{j=0}^{b+c-n} \binom{a+b}{n+j-1} \binom{n+j-1}{j} (-1)^j \left((b+c+1-(n+j))m-r\right)^{a+b-n} \\
&= \sum_{\ell=0}^{b+c} \sum_{j=0}^{\ell} \binom{a+b}{\ell-1} \binom{\ell-1}{j} (-1)^j \left((b+c+1-\ell)m-r\right)^{a+b-\ell+j} \\
&= \sum_{\ell=1}^{b+c} \binom{a+b}{\ell-1} \left((b+c+1-\ell)m-r\right)^{a+b-\ell} \sum_{j=0}^{\ell-1} \binom{\ell-1}{j} (-1)^j \left((b+c+1-\ell)m-r\right)^j \\
&= \sum_{\ell=0}^{b+c-1} \binom{a+b}{\ell} \left((b+c-\ell)m-r\right)^{a+b-\ell-1} \left(1-(b+c-\ell)m+r\right)^{\ell}.
\end{aligned}$$

□

Lemma 2. *With notation as in the previous lemma, we have the identities*

$$\begin{aligned}
X &= \frac{-1}{r+m(a-c)} \\
&\quad - \sum_{\ell=0}^{a-c} \binom{a+b+1}{\ell} (1+r+m(a-c-\ell))^{a+b+1-\ell} (-r-m(a-c-\ell))^{\ell-1}, \\
Y &= \frac{-1}{r+m(a-c)} \\
&\quad - \sum_{\ell=0}^{a-c} \binom{a+b}{\ell} (1+r+m(a-c-\ell))^{a+b-\ell} (-r-m(a-c-\ell))^{\ell-1}.
\end{aligned}$$

Proof. We first recall the following binomial identity of Abel (cf. [**23**]): For $n \geq 1$, $x \neq 0$ and α real,

$$\frac{(x+y)^n}{x} = \sum_{k=0}^{n} \binom{n}{k} (y+\alpha k)^{n-k}(x-\alpha k)^{k-1}.$$

Substituting $n = a+b+1$, $\alpha = -m$, $x = -r-m(a-c)$, $y = 1-x = 1+r+m(a-c)$, we get

$$\begin{aligned}
\frac{-1}{r+m(a-c)} &= \sum_{\ell=0}^{a-c} \binom{a+b+1}{\ell} (1+r+m(a-c-\ell))^{a+b+1-\ell}(-r-m(a-c-\ell))^{\ell-1} \\
&+ \sum_{\ell=a-c+1}^{a+b+1} \binom{a+b+1}{\ell} (1+r+m(a-c-\ell))^{a+b+1-\ell}(-r-m(a-c-\ell))^{\ell-1} \\
&= \sum_{\ell=0}^{a-c} \binom{a+b+1}{\ell} (1+r+m(a-c-\ell))^{a+b+1-\ell}(-r-m(a-c-\ell))^{\ell-1} \\
&+ \sum_{\ell=0}^{b+c} \binom{a+b+1}{\ell} (1-m(b+c+1-\ell)+r)^{\ell}(m(b+c+1-\ell)-r)^{a+b-\ell}.
\end{aligned}$$

In the second sum, we have replaced the index ℓ with $a+b+1-\ell$ to obtain X, which implies the first equality. Similarly, considering $b-1$ in place of b in the preceding yields the equality for Y. □

We have the following extension of identity (6) for $A_{m,r}(n,k)$.

Theorem 3. *Let $a, b, c \in \mathbb{Z}$ with $|c| \leq a, b$ and suppose $2r+1 = (2c+1)m$. Let $\alpha(\ell) := r + m(a-c-\ell)$ and $\alpha^*(\ell) := \alpha(\ell+1)$. If $c \geq 0$, then*

$$\begin{aligned}
&\sum_{n=b-c}^{a+b} A_{m,r}(n, b-c)\binom{a+b}{n} - \sum_{n=a-c}^{a+b} A_{m,r}(n, a-c)\binom{a+b}{n} \\
&= \sum_{\ell=a-c+1}^{a+c} \binom{a+b+1}{\ell} (1+\alpha(\ell))^{a+b-\ell}(-\alpha(\ell))^{\ell} \\
&\qquad - \sum_{\ell=a-c}^{a+c-1} \binom{a+b}{\ell} (1+\alpha^*(\ell))^{a+b-\ell-1}(-\alpha^*(\ell))^{\ell}.
\end{aligned} \tag{8}$$

If $c < 0$, then

$$\begin{aligned}
&\sum_{n=b-c}^{a+b} A_{m,r}(n, b-c)\binom{a+b}{n} - \sum_{n=a-c}^{a+b} A_{m,r}(n, a-c)\binom{a+b}{n} \\
&= \sum_{\ell=a+c}^{a-c-1} \binom{a+b}{\ell} (1+\alpha^*(\ell))^{a+b-\ell-1}(-\alpha^*(\ell))^{\ell} \\
&\qquad - \sum_{\ell=a+c+1}^{a-c} \binom{a+b+1}{\ell} (1+\alpha(\ell))^{a+b-\ell}(-\alpha(\ell))^{\ell}.
\end{aligned} \tag{9}$$

Proof. From the previous lemmas, we have

$$\begin{aligned}
&\sum_{n=a-c}^{a+b} A_{m,r}(n,a-c)\binom{a+b}{n} \\
&\quad= -\sum_{\ell=0}^{a-c}\binom{a+b+1}{\ell}(1+\alpha(\ell))^{a+b+1-\ell}(-\alpha(\ell))^{\ell-1} \\
&\qquad\qquad +\sum_{\ell=0}^{a-c}\binom{a+b}{\ell}(1+\alpha(\ell))^{a+b-\ell}(-\alpha(\ell))^{\ell-1} \\
&\quad= \sum_{\ell=0}^{a-c}\binom{a+b+1}{\ell}(1+\alpha(\ell))^{a+b-\ell}(-\alpha(\ell))^{\ell-1}(-1-\alpha(\ell)) \\
&\qquad\qquad +\sum_{\ell=0}^{a-c}\binom{a+b}{\ell}(1+\alpha(\ell))^{a+b-\ell}(-\alpha(\ell))^{\ell-1} \\
&\quad= \sum_{\ell=0}^{a-c}\binom{a+b+1}{\ell}(1+\alpha(\ell))^{a+b-\ell}(-\alpha(\ell))^{\ell} \\
&\qquad\qquad -\sum_{\ell=1}^{a-c}\binom{a+b}{\ell-1}(1+\alpha(\ell))^{a+b-\ell}(-\alpha(\ell))^{\ell-1} \\
&\quad= \sum_{\ell=0}^{a-c}\binom{a+b+1}{\ell}(1+\alpha(\ell))^{a+b-\ell}(-\alpha(\ell))^{\ell} \\
&\qquad\qquad -\sum_{\ell=0}^{a-c-1}\binom{a+b}{\ell}(1+\alpha^*(\ell))^{a+b-\ell-1}(-\alpha^*(\ell))^{\ell}.
\end{aligned}$$

Interchanging the roles of a and b in Lemma 1 above, we obtain

$$\begin{aligned}
&\sum_{n=b-c}^{a+b} A_{m,r}(n,b-c)\binom{a+b}{n} \\
&\quad= \sum_{\ell=0}^{a+c}\binom{a+b+1}{\ell}((a+c+1-\ell)m-r)^{a+b-\ell}(1-(a+c+1-\ell)m+r)^{\ell} \\
&\qquad\qquad -\sum_{\ell=0}^{a+c-1}\binom{a+b}{\ell}((a+c-\ell)m-r)^{a+b-\ell-1}(1-(a+c-\ell)m+r)^{\ell}.
\end{aligned}$$

Since $2r+1=(2c+1)m$, we have $1+\alpha(\ell)=(a+c+1-\ell)m-r$ and $1+\alpha^*(\ell)=(a+c-\ell)m-r$. If $c\geq 0$, then the two sums on the right-hand side of the last equality contain $2c$ more terms than the corresponding sums in the equality preceding it, and subtracting implies (8). If $c<0$, then the sums on the right side of the equality preceding the last contain $-2c$ more terms and (9) follows. □

Taking $c = 0$ and $c = 1$ in Theorem 3 gives the following result.

Corollary 4. *Let $a, b \geq 1$. If $m = 2r + 1$, then*

$$\sum_{n=b}^{a+b} A_{m,r}(n,b)\binom{a+b}{n} = \sum_{n=a}^{a+b} A_{m,r}(n,a)\binom{a+b}{n}. \tag{10}$$

If $m = 2t + 1$ and $r = 3t + 1$, then

$$\begin{aligned}&\sum_{n=b-1}^{a+b} A_{m,r}(n,b-1)\binom{a+b}{n} - \sum_{n=a-1}^{a+b} A_{m,r}(n,a-1)\binom{a+b}{n}\\ &= (-1)^a(t+1)^b t^{a-1}\left(\binom{a+b+1}{a}t + \binom{a+b}{a-1}\right)\\ &\qquad - (-1)^b(t+1)^a t^{b-1}\left(\binom{a+b+1}{b}t + \binom{a+b}{b-1}\right).\end{aligned} \tag{11}$$

Remark: Note that (10) gives formula (6) above when $r = 0$ since $A_{1,0}(n,k) = E(n,k-1)$; hence it provides a polynomial generalization of (6). Formula (11) is seen to provide an analogue of (6) for $A_{m,r}(n,k)$ when $m = r = 1$ and $a, b > 1$.

Comparing (4) and (7), we have the relation $A_{m,r}(n,k) = \widehat{A}_{m,m-r}(n,k-1)$. From this, one can obtain a comparable version of Theorem 3 for $\widehat{A}_{m,r}(n,k)$. In particular, we have the following.

Corollary 5. *Let $a, b \geq 1$. If $m = 2r - 1$, then*

$$\sum_{n=b}^{a+b} \widehat{A}_{m,r}(n,b-1)\binom{a+b}{n} = \sum_{n=a}^{a+b} \widehat{A}_{m,r}(n,a-1)\binom{a+b}{n}. \tag{12}$$

If $m = -2r + 1$, then

$$\begin{aligned}&\sum_{n=b-1}^{a+b} \widehat{A}_{m,r}(n,b-2)\binom{a+b}{n} - \sum_{n=a-1}^{a+b} \widehat{A}_{m,r}(n,a-2)\binom{a+b}{n}\\ &= r^{b-1}(1-r)^a\left(\binom{a+b}{b-1} - r\binom{a+b+1}{b}\right)\\ &\qquad - r^{a-1}(1-r)^b\left(\binom{a+b}{a-1} - r\binom{a+b+1}{a}\right).\end{aligned} \tag{13}$$

Remark: Formula (12) gives (6) when $r = 1$. Taking $r = 0$ in (13) yields an analogue of (6) for $\widehat{A}_{m,r}(n,k)$ when $a, b > 1$. Letting $r = 1$ in (13), we obtain another symmetric identity:

$$\sum_{n=b-1}^{a+b} \widehat{A}_{-1,1}(n,b-2)\binom{a+b}{n} = \sum_{n=a-1}^{a+b} \widehat{A}_{-1,1}(n,a-2)\binom{a+b}{n}, \qquad a, b \geq 1.$$

3. Identities Involving Combinatorial Sequences

In this section, we find some identities relating generalized Eulerian and harmonic numbers and Stirling numbers of the first kind. We first recall some special functions. Let $\Gamma(z)$ denote the gamma function given by

$$\Gamma(z) := \int_0^\infty e^{-t} t^{z-1} dt, \qquad \mathrm{Re}(z) > 0.$$

Then the polygamma function $\psi^{(n)}(z)$ is defined as

$$\psi^{(n)}(z) := \frac{d^{n+1}}{dz^{n+1}}[\log \Gamma(z)] = \frac{d^n}{dz^n}\psi(z), \qquad n \geq 0,\ z \notin \mathbb{Z}_0^-,$$

where $\psi(z)$ denotes the psi (or digamma) function (cf. [**24**]) and $\mathbb{Z}_0^-$ is the set of non-positive integers. The Hurwitz zeta function $\zeta(s,a)$ is defined as

$$\zeta(s,a) := \sum_{k=0}^{\infty} \frac{1}{(k+a)^s}, \qquad \mathrm{Re}(s) > 1,\ a \notin \mathbb{Z}_0^-,$$

the $a = 1$ case of which corresponds to the usual zeta function, $\zeta(s) = \zeta(s,1)$. The polygamma and Hurwitz zeta functions are related by the following formula from [**24**]:

$$\psi^{(n)}(z) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(k+z)^{n+1}} = (-1)^{n+1} n! \zeta(n+1, z). \tag{14}$$

We will also make use of the following differentiation formula (see, e.g., [**6**]):

$$\frac{d}{dx}\binom{x+k}{n} = \binom{x+k}{n}[\psi(x+k+1) - \psi(x+k+1-n)]. \tag{15}$$

Further properties of the special functions given above can be found in [**24**].

Extending arguments from [**6**], we have the following more general identities involving $\widehat{A}_{m,r}(n,k)$.

Theorem 6. *If $n \geq 1$, then*

$$\begin{aligned} &nm(m(x-1)+r)^{n-1} \\ &= \sum_{k=-1}^{n-1} \widehat{A}_{m,r}(n,k)\binom{x+k}{n}(\psi(x+k+1) - \psi(x+k+1-n)) \end{aligned} \tag{16}$$

and

$$\begin{aligned} &nm^2(n-1)(m(x-1)+r)^{n-2} \\ &= \sum_{k=-1}^{n-1} \widehat{A}_{m,r}(n,k)\binom{x+k}{n}\big[(\psi(x+k+1) - \psi(x+k+1-n))^2 \\ &\qquad + \zeta(2, x+k+1) - \zeta(2, x+k+1-n)\big]. \end{aligned} \tag{17}$$

Proof. Recall that the $\widehat{A}_{m,r}(n,k)$ satisfy the following version of Worpitzky's identity:

$$(m(x-1)+r)^n = \sum_{k=-1}^{n-1} \widehat{A}_{m,r}(n,k)\binom{x+k}{n}, \qquad n \geq 0, \tag{18}$$

see [**25**, Lemma 7]. Differentiating both sides of (18) with respect to x, and using (15), we obtain the first equality. Noting (14) and differentiating (18) twice with respect to x yields the second equality. □

Let $\left[{n \atop k}\right]$ denote the (signless) Stirling number of the first kind, which can be defined via the relation $x^{\underline{n}} = \sum_{k=0}^{n} \left[{n \atop k}\right](-1)^{n-k}x^k$ for $n \geq 0$. We have the following identities involving $\left[{n \atop k}\right]$ and $A_{m,r}(n,k)$.

Theorem 7. *If $n \geq 0$, then*

$$(mx+r)^n = \frac{1}{n!}\sum_{k=0}^{n}\sum_{i=0}^{n}\sum_{\ell=i}^{n} A_{m,r}(n,k)\left[{n \atop \ell}\right]\binom{\ell}{i}(1-k)^i x^{\ell-i} \tag{19}$$

and

$$(mx+r)^n = \frac{1}{n!}\sum_{k=0}^{n}\sum_{i=0}^{n}\sum_{\ell=i}^{n}(-1)^{n-\ell} A_{m,r}(n,k)\left[{n \atop \ell}\right]\binom{\ell}{i}(n-k)^i x^{\ell-i}. \tag{20}$$

Proof. Recall the formula (cf. [**14**]): $\binom{x+k}{n} = (-1)^n \frac{(-x-k+n-1)^{\underline{n}}}{n!}$. From (18), we then have

$$\begin{aligned}
&(m(x-1)+r)^n \\
&= \frac{(-1)^n}{n!}\sum_{k=-1}^{n-1} \widehat{A}_{m,r}(n,k)(-x-k+n-1)^{\underline{n}} \\
&= \frac{(-1)^n}{n!}\sum_{k=0}^{n}\sum_{\ell=0}^{n} \widehat{A}_{m,r}(n,k-1)\left[{n \atop \ell}\right](-1)^{n-\ell}(-(x-1)+n-1-k)^{\ell} \\
&= \frac{1}{n!}\sum_{k=0}^{n}\sum_{\ell=0}^{n} \widehat{A}_{m,r}(n,k-1)\left[{n \atop \ell}\right]((x-1)+k-n+1)^{\ell} \\
&= \frac{1}{n!}\sum_{k=0}^{n}\sum_{i=0}^{n}\sum_{\ell=i}^{n} \widehat{A}_{m,r}(n,k-1)\left[{n \atop \ell}\right]\binom{\ell}{i}(k-n+1)^i(x-1)^{\ell-i},
\end{aligned}$$

where we have applied the binomial theorem and interchanged summation to obtain the last equality. Replacing k with $n-k$ in the outer sum, noting $\widehat{A}_{m,r}(n,n-k-1) = A_{m,r}(n,k)$ and changing $x-1$ to x then gives (19). To show (20), we proceed in a similar manner as before, but instead write $\binom{x+k}{n} = (-1)^n \frac{(-x-k)^{\overline{n}}}{n!}$ as the first step, where $y^{\overline{n}} = y(y+1)\cdots(y+n-1)$. □

Remark: The $m = 1$, $r = 0$ case of (19) is seen to be equivalent to the Eulerian number identity given in [**6**, Eqn. 2.16] (upon replacing k with $n - k$), while the comparable case of (20) seems to be new.

Let $H_n^{(m)}$ denote the generalized harmonic number of order m given by $H_n^{(m)} := \sum_{k=1}^{n} \frac{1}{k^m}$, the $m = 1$ case of which corresponds to the *harmonic number* H_n. We close this section by noting some interesting identities obtained by choosing certain values of x in the last two theorems.

Corollary 8. *We have the following identities:*

$$(i)\ nm(nm+r)^{n-1} = \sum_{k=0}^{n} \widehat{A}_{m,r}(n,k-1)\binom{n+k}{n}(H_{n+k} - H_k),$$

$$(ii)\ (n-1)nm^2(nm+r)^{n-2} = \sum_{k=0}^{n} \widehat{A}_{m,r}(n,k-1)\binom{n+k}{n}[(H_{n+k} - H_k)^2 + H_k^{(2)} - H_{n+k}^{(2)}],$$

$$(iii)\ (m+r)^n n! = \sum_{k=0}^{n}\sum_{i=0}^{n}\sum_{\ell=i}^{n} A_{m,r}(n,k)\begin{bmatrix} n \\ \ell \end{bmatrix}\binom{\ell}{i}(1-k)^i,$$

$$(iv)\ (r+1)^n n! = \sum_{k=0}^{n}\sum_{i=0}^{n}\sum_{\ell=i}^{n} A_{m,r}(n,k)\begin{bmatrix} n \\ \ell \end{bmatrix}\binom{\ell}{i}\frac{(m(1-k))^i}{m^\ell},$$

$$(v)\ (m+1)^n n! = \sum_{k=0}^{n}\sum_{i=0}^{n}\sum_{\ell=i}^{n} A_{m,r}(n,k)\begin{bmatrix} n \\ \ell \end{bmatrix}\binom{\ell}{i} r^{\ell-i-n}(1-k)^i.$$

Proof. The first two formulas above are obtained by replacing x by $n+1$ in Theorem 6 and referring to the identity [**6**]

$$\psi(z+n) - \psi(z) = \sum_{j=1}^{n} \frac{1}{z+j-1}, \qquad n \geq 1.$$

Formulas (iii), (iv) and (v) are obtained by taking $x = 1$, $x = \frac{1}{m}$ and $x = r$, respectively, in (19). □

4. Generalized Euler-Frobenius Fractions

The *Euler-Frobenius fractions* were defined by Euler in his treatise *Institutiones Calculi Differentialis* [**12**, Sect. 173, Chap. 7]. They have since been widely studied and bear a close relationship with the classical Eulerian polynomials. The *Euler-Frobenius fractions* $H_n(t)$ can be defined via the generating function

$$\frac{1-t}{e^x - t} = \sum_{n=0}^{\infty} H_n(t)\frac{x^n}{n!}, \quad t \neq 1.$$

Later, Carlitz introduced a generalization of the Euler-Frobenius fractions [**4**] given by

$$\frac{1-t}{e^x-t}e^{ux}=\sum_{n=0}^{\infty}H_n(u|t)\frac{x^n}{n!}.$$

Note that if $u=0$, then $H_n(0|t)=H_n(t)$. The *Eulerian polynomials* are defined by

$$A_n(x):=\sum_{k=1}^{n}A(n,k)x^k,\qquad n\geq 1,$$

with $A_0(x)=1$. These polynomials satisfy the following relation [**8**, p. 245]:

$$\frac{A_n(x)}{(1-x)^{n+1}}=\sum_{k=0}^{\infty}k^nx^k,\qquad n\geq 0.$$

We summarize some known properties of the Euler-Frobenius fractions.

Theorem 9 ([**16**])**.** *The following identities hold:*

$$\begin{aligned}A_n(t)&=(t-1)^nH_n(t),\\ \frac{d}{du}H_n(u|t)&=nH_{n-1}(u|t),\\ H_n(u|t)&=\sum_{k=0}^{n}\binom{n}{k}H_k(t)u^{n-k}.\end{aligned}$$

The *r-Whitney-Eulerian polynomials* are defined by

$$A_{n,m,r}(x):=\sum_{k=0}^{n}A_{m,r}(n,k)x^k.$$

These new polynomials satisfy the following relation [**19**]:

$$\sum_{i=0}^{\infty}(mi+r)^nx^i=\frac{A_{n,m,r}(x)}{(1-x)^{n+1}},\qquad n,r\geq 0,\ m\geq 1. \tag{21}$$

Different families of polynomials related to r-Whitney numbers have been recently studied in [**9**].

We now define the *r-Whitney-Euler-Frobenius fractions* $H_{n,m,r}(t)$ by means of the relation

$$A_{n,m,r}(t)=(t-1)^nH_{n,m,r}(t). \tag{22}$$

We have the following generating function formula.

Theorem 10. *The exponential generating function for the r-Whitney-Euler-Frobenius fractions is given by*

$$\sum_{n=0}^{\infty}H_{n,m,r}(t)\frac{x^n}{n!}=\frac{1-t}{e^{mx}-t}e^{(m-r)x}.$$

Proof. From (21) and (22), we have

$$H_{n,m,r}(t) = \frac{A_{n,m,r}(t)}{(t-1)^n} = (1-t)(-1)^n \sum_{i=0}^{\infty} (mi+r)^n t^i.$$

Therefore

$$\begin{aligned}\sum_{n=0}^{\infty} H_{n,m,r}(t)\frac{x^n}{n!} &= \sum_{n=0}^{\infty}\sum_{i=0}^{\infty}(1-t)(-1)^n(mi+r)^n t^i \frac{x^n}{n!} \\ &= \sum_{i=0}^{\infty}(1-t)t^i \sum_{n=0}^{\infty}\frac{(-x(mi+r))^n}{n!} \\ &= \sum_{i=0}^{\infty}(1-t)t^i e^{-x(mi+r)} = (1-t)e^{-xr}\frac{1}{1-te^{-xm}} \\ &= \frac{1-t}{e^{xm}-t}e^{(m-r)x}.\end{aligned}$$

□

This expression resembles that of the Carlitz generalization, actually they are almost the same. Observe that replacing x by x/m yields the expression

$$\frac{1-t}{e^x - t}e^{\frac{m-r}{m}x},$$

which is the generating function for $H_n(u|t)$ when $u = (m-r)/m$. Therefore, we have

$$\begin{aligned}H_{n,m,r}(t) &= m^n H_n\left(\frac{m-r}{m}|t\right), \\ H_{n,m,r}(t) &= \sum_{k=0}^{n}\binom{n}{k}H_k(t)(m-r)^{n-k}m^k.\end{aligned}$$

From the previous observations, one can derive further identities. For example, Carlitz [**5**] obtained a formula for the product of two generalized Euler-Frobenius fractions. In our case, this result is as follows.

Theorem 11. *For $\alpha, \beta, \alpha\beta \neq 1$, we have*

$$\begin{aligned}H_{i,m,r}(\alpha)H_{j,m,r}(\beta) &= H_{i+j,m,r}(\alpha\beta)\frac{(1-\alpha)(1-\beta)}{1-\alpha\beta} \\ &+ \frac{\alpha(1-\beta)}{1-\alpha\beta}\sum_{\ell=0}^{i}\binom{i}{\ell}H_\ell(\alpha)m^\ell H_{i+j-\ell,m,r}(\alpha\beta) \\ &+ \frac{\beta(1-\alpha)}{1-\alpha\beta}\sum_{k=0}^{j}\binom{j}{k}H_k(\beta)m^k H_{i+j-k,m,r}(\alpha\beta).\end{aligned}$$

5. Generalized Eulerian recurrences

In this section, we consider two kinds of more general Eulerian recurrences. The first will arise as a signed distribution of statistics on a set of marked permutations and will have $A_{m,r}(n,k)$ as a special case. The second recurrence, while it does not generalize $A_{m,r}(n,k)$ above, is perhaps more natural when considering distributions of statistics on permutations of $[n]$. Both sequences will be seen to reduce to the classical Eulerian numbers whenever all of the parameters are unity.

5.1 A generalization of $A_{m,r}(n,k)$

Given indeterminates a, b, c and d, let $A(n,k) = A(n,k;a,b,c,d)$ for $n \geq 0$ and $0 \leq k \leq n$ be defined by the recurrence

$$A(n,k) = (a(k-1)+b)A(n-1,k) + (c(n-k)+d)A(n-1,k-1), \tag{23}$$

for $n \geq 1$ and $1 \leq k \leq n$, with the initial condition $A(n,0) = (b-a)^n$ for all $n \geq 0$. Put $A(n,k) = 0$ if $k < 0$ or if $k > n \geq 0$. Note that $A(n,k;m,m+r,m,m-r) = A_{m,r}(n,k)$ for all n and k. We remark that (23) is a special case of the recurrence in Problem 6.94 of [**14**, p. 319], but with different initial conditions.

We first find a combinatorial interpretation for $A(n,k)$. Recall that if $\pi = \pi_1\pi_2\cdots\pi_n$, then a *descent* corresponds to an index $i \in [i-1]$ such that $\pi_i > \pi_{i+1}$. The letter π_i is referred to as a *descent top*.

Definition 12. Given $1 \leq k \leq n$ and $i \in [n]$, let $\mathcal{A}_{n,k,i}$ denote the set of "marked" permutations π of $[n]$ having $k-1$ descent tops in $[i+1,n]$ such that (I) the subsequence $1,2,\ldots,i-1$ occurs in π, (II) each element of $[i-1]$ is either circled or underlined and (III) no elements of $[i,n]$ are circled or underlined.

Definition 13. Let $\mathcal{A}_{n,k} = \cup_{i=1}^{n}\mathcal{A}_{n,k,i}$ if $k \geq 1$, with $\mathcal{A}_{n,0}$ consisting of the 2^n "marked" permutations obtained from the identity permutation $12\cdots n$ wherein each letter is either circled or underlined.

Note that $\mathcal{A}_{n,k,i}$ is empty if $i > n-k+1$. Let $\mathrm{cir}(\lambda)$ and $\mathrm{und}(\lambda)$ denote the statistics recording the number of elements that are either circled or underlined within $\lambda \in \mathcal{A}_{n,k}$.

We now define the statistics α_j for $1 \leq j \leq 4$ on $\mathcal{A}_{n,k}$ as follows. By a *left-right (lr) minimum (right-left (rl) minimum)* of π, we mean a letter π_i such that $\pi_j > \pi_i$ for all $j < i$ (respectively, $j > i$). Given $\lambda \in \mathcal{A}_{n,k,i}$ where $1 \leq k, i \leq n$, let

$$\alpha_1(\lambda) = n-k-\mathrm{cir}(\lambda) - (\#\text{ rl min in } [i+1,n]),$$
$$\alpha_2(\lambda) = \mathrm{cir}(\lambda) + (\#\text{ rl min in } [i+1,n]),$$
$$\alpha_3(\lambda) = \begin{cases} (\#\text{ descent tops in } [i+1,n]) - (\#\text{ lr min}) + 2, & \text{if } i > 1; \\ (\#\text{ descents}) - (\#\text{ lr min}) + 1, & \text{if } i = 1, \end{cases}$$
$$\alpha_4(\lambda) = \begin{cases} (\#\text{ lr min}) - 1, & \text{if } i > 1; \\ (\#\text{ lr min}), & \text{if } i = 1. \end{cases}$$

By combining a statistic defined on the various $\mathcal{A}_{n,k,i}$, one obtains the corresponding statistic on $\mathcal{A}_{n,k}$. Note that the α_1 statistic may also be written as

$$\alpha_1(\lambda) = \mathrm{und}(\lambda) + n - i - (k-1) - (\#\text{ rl min in } [i+1, n]),$$

while the first case of α_3 may be written as

$$\alpha_3(\lambda) = (\#\text{ descent tops in } [i+1, n]) - (\#\text{ lr min in } [i+1, n]) + 1 - [i \text{ is a lr min}],$$

where $\lambda \in \mathcal{A}_{n,k,i}$ for some i. To illustrate, let $n = 9$, $k = 5$, $i = 3$ and $\lambda = 968\underline{4}1\underline{5}237 \in \mathcal{A}_{9,5,3} \subseteq \mathcal{A}_{9,5}$. Then $\alpha_1(\lambda) = 9 - 5 - 0 - 1 = 3$, $\alpha_2(\lambda) = 0 + 1 = 1$, $\alpha_3(\lambda) = 4 - 4 + 2 = 2$ and $\alpha_4(\lambda) = 4 - 1 = 3$.

If $k = 0$, then we have $A(n, 0) = (b-a)^n = \sum_{\lambda \in \mathcal{A}_{n,0}} (-1)^{\mathrm{und}(\lambda)} a^{\mathrm{und}(\lambda)} b^{\mathrm{cir}(\lambda)}$. There is the following combinatorial interpretation for $A(n, k)$ when $k \geq 1$.

Theorem 14. *If $n \geq k \geq 1$, then*

$$A(n,k) = \sum_{\lambda \in \mathcal{A}_{n,k}} (-1)^{und(\lambda)} a^{\alpha_1(\lambda)} b^{\alpha_2(\lambda)} c^{\alpha_3(\lambda)} d^{\alpha_4(\lambda)}, \tag{24}$$

where und and the α_i statistics are as defined above.

Proof. We first verify (24) in the case when $k = 1$. Let $A'(n, k)$ denote the (signed) distribution on the right-hand side of (24). By the definition, upon considering $\lambda \in \mathcal{A}_{n,1,i}$ for the various i, we have

$$A'(n,1) = \sum_{i=1}^{n} (b-a)^{i-1} (c(i-1) + d) b^{n-i}.$$

One may then verify $bA'(n-1,1) + (c(n-1)+d)(b-a)^{n-1} = A'(n,1)$ for $n \geq 2$, with $A'(1,1) = d = A(1,1)$, whence $A'(n,1) = A(n,1)$ for all $n \geq 1$. So assume $k \geq 2$. Then the total weight is $(a(k-1)+b)A'(n-1,k)$ for all members of $\mathcal{A}_{n,k}$ where either (a) n separates two elements that form a descent within a member of $\mathcal{A}_{n-1,k,i}$ for some i where the descent top belongs to $[i+1, n-1]$ or (b) n occurs last. To see this, note that in the first case, only the α_1 statistic value is increased by one (as the first term in the definition of α_1 increases, while the other terms remain unchanged), whereas in the second case, only α_2 is increased by one (since the number of rl minima increases). On the other hand, if (i) n is neither first nor last and does not separate two elements forming a descent within a member of $\mathcal{A}_{n-1,k-1,i}$ for some i where the descent top lies in $[i+1, n-1]$ or if (ii) n is first, then it is seen that there are $(c(n-k)+d)A'(n-1,k-1)$ possibilities. Combining the previous cases implies $A'(n,k)$ satisfies recurrence (23) for $k \geq 2$. Since $A'(n,1) = A(n,1)$ and $A'(k,k) = d^k = A(k,k)$ for all k, it follows by induction that $A'(n,k) = A(n,k)$ for all $n, k \geq 1$, which completes the proof. □

Remark: When $a = m$, $b = m + r$, $c = m$ and $d = m - r$, one obtains a new combinatorial interpretation for $A_{m,r}(n,k)$ in terms of statistics on a discrete structure;

see [**22**, Theorem 1] for an interpretation of $A_{m,r}(n,k)$ as an enumerator of signed permutations that are colored subject to certain rules.

We now find a formula for the exponential generating function of $A(n,k)$. Define $A_k(x) = \sum_{n\geq k} A(n,k)\frac{x^n}{n!}$ for $k \geq 0$. Then (23) can be rewritten as

$$\frac{d}{dx}A_k(x) = (a(k-1)+b)A_k(x) + (c+d-ck)A_{k-1}(x) + cx\frac{d}{dx}A_{k-1}(x), \quad k \geq 1,$$

with $A_0(x) = e^{(b-a)x}$. Define $A(x,y) = A(x,y;a,b,c,d) = \sum_{k\geq 0} A_k(x)y^k$. Multiplying both sides of the above recurrence by y^k, and summing over $k \geq 1$, we obtain the linear first-order partial differential equation

$$(1-cxy)\frac{\partial}{\partial x}A(x,y) = (b-a+dy)A(x,y) + y(a-cy)\frac{\partial}{\partial y}A(x,y), \tag{25}$$

with $A(0,y) = 1$, which can be solved explicitly to yield the following result.

Theorem 15. *The generating function $A(x,y;a,b,c,d)$ is given by*

$$e^{-\int_0^x \frac{df(t)+ct(a-b)}{ct(f(t)+1)}dt},$$

where $f(t) = W(-cyte^{a(x-t)-cxy})$ and $W(t)$ denotes the Lambert-W function defined as the solution to $W(t)e^{W(t)} = t$.

We note that relatives of the p.d.e. (25) were considered earlier in [**1**]. It is well-known (see, e.g., [**14**, Eqn. 7.56]) that $A(x,y;1,1,1,1) = \frac{1-y}{1-ye^{x(1-y)}}$. Hence, we can state the following.

Corollary 16. *We have*

$$e^{-\int_0^x \frac{W(-yte^{x-t-xy})}{t(W(-yte^{x-t-xy})+1)}dt} = \frac{1-y}{1-ye^{x(1-y)}},$$

where $W(t)$ is the Lambert-W function.

Let $u = -yte^{x-t-xy}$. One may verify

$$\frac{\partial}{\partial y}\left(\frac{W(u)+t}{t(W(u)+1)}\right) = \frac{(t-1)(1-xy)e^{x-t-xy}W'(u)}{(W(u)+1)^2},$$

and hence

$$\begin{aligned}\int_0^x \frac{\partial}{\partial y}\left(\frac{W(u)+t}{t(W(u)+1)}\right)dt &= -\frac{1-xy}{y}\left[\frac{1}{W(u)+1}\right]_{t=0}^{t=x} \\ &= \frac{1-xy}{y}\left(1-\frac{1}{1-xy}\right) = -x,\end{aligned}$$

where we have used the fact $W(ze^z) = z$ if $z \geq -1$. Thus, we have

$$\int_0^x \frac{W(-yte^{x-t-xy})+t}{t(W(-yte^{x-t-xy})+1)}dt = x - xy,$$

as this integral is seen to equal x when $y = 0$, whence

$$e^{\frac{r}{m}\int_0^x \frac{W(-yte^{x-t-xy})+t}{t(W(-yte^{x-t-xy})+1)}dt} = e^{\frac{rx(1-y)}{m}}.$$

Corollary 16 then implies

$$e^{-\int_0^x \frac{W(-yte^{x-t-xy})}{t(W(-yte^{x-t-xy})+1)}dt} e^{\int_0^x \frac{rW(-yte^{x-t-xy})+rt}{mt(W(-yte^{x-t-xy})+1)}dt} = \frac{(1-y)e^{\frac{rx(1-y)}{m}}}{1-ye^{x(1-y)}},$$

which may be rewritten as

$$e^{-\int_0^x \frac{(m-r)W(-yte^{x-t-xy})-rt}{mt(W(-yte^{x-t-xy})+1)}dt} = \frac{(1-y)e^{\frac{rx(1-y)}{m}}}{1-ye^{x(1-y)}}.$$

Therefore, by Theorem 15, we get the following result.

Corollary 17. *We have*

$$\begin{aligned} A(x,y;m,m+r,m,m-r) &= e^{-\int_0^x \frac{(m-r)W(-myte^{m(x-t)-mxy})-rmt}{mt(W(-myte^{m(x-t)-mxy})+1)}dt} \\ &= \frac{(1-y)e^{rx(1-y)}}{1-ye^{mx(1-y)}}. \end{aligned}$$

Remark: The second formula above for $A(x,y;m,m+r,m,m-r)$ coincides with the e.g.f. formula from [**19, 22**] for $A_{m,r}(n,k)$. We also note the following further special cases of $A(x,y;a,b,c,d)$:

$$\begin{aligned} A(x,y;1,1,1,d) &= A(x,y;1,1,1,1)^d, \\ A(x,y;1,1/c,c,1) &= A(x,cy;1,1,1,1)^{2/c-1}e^{x(1-c)(1/c-y)}, \\ A(x,y;a,b,c,c-bc/a) &= e^{x(b-a)(1-cy/a)}. \end{aligned}$$

5.2 A related distribution on $\mathcal{S}_n$

Note that $A(n,k;a,b,c,d)$, while it does reduce to the classical Eulerian number when $a = b = c = d = 1$, is not a joint distribution polynomial in general for statistics on permutations of $[n]$ having $k-1$ descents since it contains terms with negative coefficients if $a \neq b$. One may wish to consider alternatively the following variant, which we will denote by $B(n,k) = B(n,k;a,b,c,d)$, that is indeed a joint

distribution of statistics. Let $B(n,k)$ for $n \geq 0$ and $0 \leq k \leq n$ be defined by the same recurrence

$$B(n,k) = (a(k-1)+b)B(n-1,k) + (c(n-k)+d)B(n-1,k-1), \tag{26}$$

for $n \geq 1$ and $1 \leq k \leq n$, but instead with the initial condition $B(n,0) = \delta_{n,0}$.

Let $\mathcal{B}_{n,k}$ denote the set of permutations of $[n]$ having $k-1$ descents for $n \geq k \geq 1$. Then we have the following combinatorial interpretation for $B(n,k)$.

Theorem 18. *The polynomial $B(n,k;a,b,c,d)$ is the joint distribution on $\mathcal{B}_{n,k}$ for the following four respective permutation statistics: (a) ascents − rl minima + 1, (b) rl minima − 1, (c) descents − lr minima + 1, (d) lr minima.*

Proof. Let $B'(n,k)$ denote the joint distribution of the four statistics on $\mathcal{B}_{n,k}$. We will show that $B'(n,k)$ satisfies recurrence (26). In the $k=1$ and $k=n$ cases, we have $B'(n,1) = b^{n-1}d$ and $B'(n,n) = d^n$ and the recurrence is clear in these cases, so assume $1 < k < n$. To show (26), we consider the placement of the element n within $\lambda \in \mathcal{B}_{n,k}$. If n occurs at the very end of λ, then n contributes a right-left minima greater than one, while if n is inserted between two elements forming a descent within a member of $\mathcal{B}_{n-1,k}$, then n fails to be a right-left minima, and in both cases the number of descents remains unchanged. This yields contributions of $bB'(n-1,k)$ and $a(k-1)B'(n-1,k)$ in the respective cases. On the other hand, if n occurs at the very beginning of λ, then it contributes a left-right minima as well as a descent and hence there are $dB'(n-1,k-1)$ possibilities. Finally, if n is inserted between two elements forming an ascent within a member of $\mathcal{B}_{n-1,k-1}$, then there are $(n-2)-(k-2) = n-k$ choices for the position of n and hence there are $c(n-k)B'(n-1,k-1)$ possibilities, which completes the proof. □

We now find a formula for the e.g.f. of $B(n,k)$. Define

$$B_k(x) = \sum_{n \geq k} B(n,k)\frac{x^n}{n!}.$$

Then (26) can be written as

$$\frac{d}{dx}B_k(x) = (a(k-1)+b)B_k(x) + (c+d-ck)B_{k-1}(x) + cx\frac{d}{dx}B_{k-1}(x), \quad k \geq 1,$$

with $B_0(x) = 1$. Define $B(x,y) = B(x,y;a,b,c,d) = \sum_{k \geq 0} B_k(x)y^k$. Multiplying both sides of the above recurrence by y^k and summing over $k \geq 1$, we obtain

$$(1-cxy)\frac{\partial}{\partial x}B(x,y) = (b-a+dy)B(x,y) + a - b + y(a-cy)\frac{\partial}{\partial y}B(x,y),$$

with $B(0,y) = 1$. Again, it is possible to express the solution to the p.d.e. in terms of the Lambert-W.

Theorem 19. *The generating function $B(x,y;a,b,c,d)$ is given by*

$$\left(1+(a-b)\int_0^x \frac{e^{\frac{1}{c}\int_0^t \frac{df(r)+cr(a-b)}{r(f(r)+1)}dr}}{f(t)+1}dt\right) e^{-\int_0^x \frac{df(t)+ct(a-b)}{ct(f(t)+1)}dt},$$

where $f(t)=W(-cyte^{a(x-t)-cxy})$.

Corollary 20. *The generating function $B(x,y;b,b,c,d)$ is given by*

$$B(x,y;b,b,c,d)=\left(\frac{b-cy}{b-cye^{x(b-cy)}}\right)^{d/c}.$$

Proof. Taking $a=b$ in Theorem 19 yields

$$B(x,y;b,b,c,d)=e^{-\frac{d}{c}\int_0^x \frac{W(-cyte^{b(x-t)-cxy})}{t(W(-cyte^{b(x-t)-cxy})+1)}dt}.$$

Thus, by Corollary 16, we complete the proof. □

For example, Corollary 20 with $d=c$ gives

$$B(x,y;b,b,c,c)=\frac{1-cy/b}{1-cy/be^{bx(1-cy/b)}}=\sum_{n\geq 0}\sum_{k=0}^{n} A_{n,k}\frac{b^{n-k}c^k x^n y^k}{n!},$$

where $A_{n,k}$ denotes here the classical Eulerian number. This implies that the coefficient of $\frac{x^n y^k}{n!}$ in $B(x,y;b,b,c,c)$ is given by $b^{n-k}c^k A_{n,k}$, which is also easily realized combinatorially.

As another example, taking $a=c=1$ in Corollary 20 implies

$$B(x,y;1,1,1,z)=\left(\frac{1-y}{1-ye^{x(1-y)}}\right)^z.$$

Note that this yields a formula for the e.g.f. of the joint distribution for the statistics on $\mathcal{S}_n$ recording the number of descents and lr minima (marked here by y and z, respectively). From this, we have

$$\frac{d}{dz}B(x,y;1,1,1,z)=\left(\frac{1-y}{1-ye^{x(1-y)}}\right)^z \ln\left(\frac{1-y}{1-ye^{x(1-y)}}\right),$$

which leads to

$$\frac{d}{dz}B(x,1;1,1,1,z)\mid_{z=1}=\frac{-\ln(1-x)}{1-x}=\sum_{n\geq 1}H_n x^n,$$

where H_n is the harmonic number.

Remark: From the final equality, one concludes that there are $n!H_n$ lr minima altogether in $\mathcal{S}_n$. Since the statistics recording the number of lr minima and cycles are

equally distributed on $\mathcal{S}_n$, this reaffirms the well-known fact (see, e.g., [**2**, Theorem 12]) that there are on average H_n cycles within a randomly chosen permutation of $[n]$.

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