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CONTINOUS ANALOGUES FOR THE BINOMIAL COEFFICIENTS AND THE CATALAN NUMBERS

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Using techniques from the theories of convex polytopes, lattice paths, and indirect influences on directed manifolds, we construct continuous analogues for the binomial coefficients and the Catalan numbers. Our approach for constructing these analogues can be applied to a wide variety of combinatorial sequences. As an application we develop a continuous analogue for the binomial distribution.

1. INTRODUCTION

In this work we construct continuous analogues for the binomial coefficients and the Catalan numbers. Our constructions are based on the theory of convex polytopes, the theory of lattice paths, and the theory of indirect influences on directed manifolds. We introduce our methodology for finding continuous analogues – applicable to many kinds of combinatorial objects – through the following table:

Combinatorial Object	Continuous Analogue
Lattice $\mathbb{Z}^d$	Smooth manifold $\mathbb{R}^d$
Lattice step vector $v \in \mathbb{Z}^d$	Constant vector field v on $\mathbb{R}^d$
Lattice step vectors $v_1, ..., v_k \in \mathbb{Z}^d$	Directed manifold $(\mathbb{R}^d, v_1, ..., v_k)$
Lattice paths	Directed paths
Finite pattern decomposition	Countable pattern decomposition
$\sharp$ Integer points in interior of polytopes	Volume of polytopes
Binomial coefficients $\binom{n}{k}$	Continuous binomial coefficients $\left\{ {x \atop s} \right\}$
Catalan numbers c_n	Continuous Calatan numbers $C(x)$

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A polytope in $\mathbb{R}^d \supseteq \mathbb{Z}^d$ gives rise to the weighted poset of its faces (ordered by inclusion,) with the weight of a face being the number of integer points in its relative interior. Restricting attention to the lowest and highest elements of this poset, a couple of combinatorial problems arise whenever we are given a convex polytope $P \subseteq \mathbb{R}^d$: count the number of vertices of P, and count the number of integer points in P°, the relative interior of P. Accordingly, a couple of different meanings can be given to the problem of finding a convex polytopal interpretation, or realization, of a sequence a_n of natural numbers:

I. Find a sequence $P_n \subseteq \mathbb{R}^{d_n}$ of polytopes such that $a_n = |\text{vertices}(P_n)|$.

II. Find a sequence $P_n \subseteq \mathbb{R}^{d_n}$ of polytopes such that $a_n = |P_n^\circ \cap \mathbb{Z}^{d_n}|$.

Clearly, in both cases, one can always find a (non-unique) sequence of polytopes with the required property, just as it happens when we consider interpretations of the natural numbers as the cardinality of arbitrary finite sets. Thus, we are actually interested in finding nice polytopal interpretations having additional properties. The reader may wonder why we count points in the interior of polytopes, and not in the whole polytope. To a great extent both choices are equally valid, and indeed they are tightly related by the Möbius inversion formula, and the Ehrhart reciprocity theorem [**21**]. We give preponderance to interior integral points because that is what arises in our general constructions in Sections 2 and 3.

For the Catalan numbers $c_n = \frac{1}{n+1}\binom{2n}{n}$ problem I admits a nice answer in terms of the Stasheff's associahedra, which play a prominent role in the study of algebras associative up to homotopy, and particularly in the construction of the operad for A_∞-algebras [**34**]. The associahedra were first constructed by Tamari, coming from a different viewpoint, who gave a combinatorial description of the poset of its faces [**35**].

Solutions to problem II lead naturally to the construction of continuous analogues for the sequence of natural numbers a_n as follows: the numbers a_n count the integral points in the interior of the polytopes $P_n \subseteq \mathbb{R}^{d_n}$, and we can think of the volume $\text{vol}(P_n)$ as counting – actually measuring – points in P_n after the integrality restrictions are lifted. Therefore, one feels entitled to regard the real numbers $\text{vol}(P_n)$ as being continuous analogues for the natural numbers $a_n = |P_n^\circ \cap \mathbb{Z}^{d_n}|$. Although a bit vague for the moment, this analogy will become much clearer when applied to the polytopes coming from the theory of lattice paths studied in this work. In this case our analogy simply amounts to replacing lattice paths by directed paths (i.e. polygonal paths with specified tangent vectors), a process that can be intuitively grasped by comparing Figures 5 and 6. For more on the theory of lattice paths the reader may consult Banderier and Flajolet [**2**], Humphreys [**26**], Krattenthaler [**31**], Mohanty [**35**], and Narayana [**37**].

The relation between the numbers $|P^\circ \cap \mathbb{Z}^d|$ and $\text{vol}(P)$ for a convex polytope P is much deeper than what one might naively think. Let us highlight a few points that show the depth of this relationship:

- Consider the poset of subfaces of P and its associated Möbius function μ [**30**]. The following identities hold:

$$|P\cap\mathbb{Z}^d| \;=\; \sum_{F \text{ subface of } P} |F^\circ\cap\mathbb{Z}^d| \quad \text{and}$$

$$|P^\circ\cap\mathbb{Z}^d| \;=\; \sum_{F \text{ subface of } P} \mu(F,P)|F\cap\mathbb{Z}^d|.$$

- Assume that P has dimension k and its vertices lie in $\mathbb{Z}^d$. Erhart's theory, see Diaz and Robins [**16**], Erhart [**21**], and Macdonald [**33**], tell us that the functions

$$P(n) \;=\; |nP\cap\mathbb{Z}^d| \qquad \text{and} \qquad Q(n) \;=\; |nP^\circ\cap\mathbb{Z}^d|$$

  are polynomials of degree k such that $P(n) = (-1)^kQ(-n)$. Moreover, the degree k coefficients of both $P(n)$ and $Q(n)$ are equal to the volume of P:

$$\lim_{n\to\infty}\frac{P(n)}{n^k} \;=\; \lim_{n\to\infty}\frac{Q(n)}{n^k} \;=\; \mathrm{vol}(P).$$

- Suppose that the convex polytope $P\subseteq\mathbb{Z}^d$ has dimension d, integer vertices, and d-edges emanating from each vertex of P which generate $\mathbb{Z}^d$. To this data one associates a toric variety X_P and an holomorphic line bundle $L_P\to X_P$ such that:

$$|P\cap\mathbb{Z}^d| \;=\; \chi(L_P) \;=\; \dim\, H^0(X_P,L_P).$$

  Thus $|P\cap\mathbb{Z}^d|$ counts independent sections of the line bundle $L_P\to X_P$. The standard reference for this result is Danilov [**12**].

- The construction above can be understood in terms of symplectic manifolds and geometric quantization, see Guillemin [**23**], Guillemin, Ginzburg, and Karshon [**24**], Hamilton [**25**]. Deltzant [**19**] constructed a toric symplectic manifold X_P, via symplectic reduction, which comes with a Kähler structure and a pre-quantum line bundle L_P, in the sense that first Chern class of L_P is the symplectic form of X_P. The holomorphic structure on X_P give rise to a polarization on L, therefore $H^0(X_P,L_P)$ is the Hilbert space associated to X_P in the geometric quantization approach, see Śniatycki [**38**] and Woodhouse [**41**].

- It follows from the Duistermaat-Heckman theorem, see [**20**] and Guillemin [**23**], that the phase space symplectic manifold X_P and the convex polytope P have the same volume. Therefore, in this case, the transition

$$\mathrm{vol}(P)=\mathrm{vol}(X_P) \qquad\longrightarrow\qquad |P\cap\mathbb{Z}^d| = \dim\, H^0(X_P,L_P)$$

  is a numerical manifestation of the classical-to-quantum transition:

$$X_P \qquad\longrightarrow\qquad H^0(X_P,L_P).$$

- A different sort of relation between $|P\cap\mathbb{Z}^d|$ and the volume of polytopes arises by considering the polytopal deformation P_h of P defined by deforming the equations defining P by adding a small number h to the constant term of each equation. Still with the same conditions on the polytope P as above we have that:

$$|P\cap\mathbb{Z}^d| \;=\; \mathrm{Todd}(P,\frac{\partial}{\partial h})\mathrm{vol}(P_h)\Big|_{h=0} \qquad \text{and}$$

$$|P^\circ\cap\mathbb{Z}^d| \;=\; \mathrm{Todd}(P,-\frac{\partial}{\partial h})\mathrm{vol}(P_h)\Big|_{h=0},$$

  where $\mathrm{Todd}(P,\frac{\partial}{\partial h})$ is the operator obtained by substituting $\frac{\partial}{\partial h}$ into an explicitly defined formal power series introduced by Todd. This result is due to Khovanskii and Pukhlikov [**29**] and has been extended, using different techniques, to the case of simple polytopes (d-edges emanating from each vertex) by Brion and Vergne [**7**], Cappell and Shaneson [**8**], Guillemin, Ginzburg, and Karshon [**24**], and Karshon, Sternberg, and Weitsman. [**28**].

- Another approach – applicable for a rational convex polytope P – relates $|P\cap\mathbb{Z}^d|$ with the volume of the various faces of P:

$$|P\cap\mathbb{Z}^d| \;=\; \sum_{F \text{ subface of } P} c(F,P)\mathrm{vol}(F)$$

  where the coefficients $c(F,P)$ are rational numbers which satisfy the properties of being local and computable, with the measure on faces defined in terms of the lattice generators of the affine extension of each face. This result is due to Berline and Vergne [**4**].

- The counting of lattice points inside a polytope has a long history which we do not attempt to summarize, the interested reader may consult Brion [**6**], De Loera [**13**], Lagarias and Ziegler [**32**], and the references therein. We remark that a polynomial time algorithm for such counting was introduced by Barvinok [**1**].

Our construction of continuous analogues for certain type of combinatorial objects is best explained via the following flow diagram:

combinatorial object $\longrightarrow$ lattice path reformulation $\longrightarrow$

finite decomposition over patterns $\longrightarrow$ volume of polytopes $\longrightarrow$

countable decomposition over patterns $\longrightarrow$ continuous object.

In this work we consider problem II for the binomial and Catalan numbers applying the methodology outlined above and described in details in Sections 2 and 3. In both cases we begin by decomposing the given sequence of numbers as finite sums over time and patterns, where each summand counts the interior points of a lattice polytope. The starting point to achieve this decomposition is to describe our given sequence of numbers as counting lattice paths, e.g. the Catalan numbers count Dyck paths. Once we have an interpretation of each summand as counting interior points of convex polytopes, we define our continuous analogous by removing the integrality restrictions, i.e. we compute volume of polytopes and replace finite sums by countable sums. The construction of continuous analogues for the binomial coefficients leads to the development of a continuous analogue for the discrete binomial distribution whose density is shown in Figures 2 and 3.

Our constructions can be motivated from a physical point of view as follows. Ever since Feynman reformulated quantum mechanics in terms of path integrals [**22**], constructing a rigorous theory for such integrals has been a major challenge for mathematicians. Counting (weighted) lattice paths may be regarded as a fully discretized version of this problem. Our proposal – from this viewpoint – is to extend the domain of allowed paths:

$$\text{lattice paths} \quad \subseteq \quad \text{directed paths} \quad \subseteq \quad \text{continuous paths},$$

from lattice paths to directed paths, which form a moduli space and yet by construction retain a strong combinatorial flavor.

We stress that we are after continuous analogues rather than extensions to continuous variables. It is well-known that the latter can be achieved for the binomials – and thus for the Catalan numbers – with the help of the classical gamma and beta functions. This work takes part in our program aimed to bring geometric methods to the study of problems arising from the theory of complex networks [**9, 11, 14, 15, 18**].

2. LATTICE PATHS AND PATTERNS

Let us recall the settings upon which the theory of lattice paths is built [**2**]. We fix throughout Sections and the following data: a dimension $d \in \mathbb{N}_{>0}$, and step vectors $V = \{v_1, ..., v_k\} \subseteq \mathbb{Z}^d \subseteq \mathbb{R}^d$ with index set $[k] = \{1, .., k\}$.

A lattice path (with steps in V) from p to q in $\mathbb{Z}^d$ is given by a tuple of points $(p_0, ..., p_n)$ in $\mathbb{Z}^d$, with $n \geq 1$, such that, for $i \in [n]$, we have $p_0 = p$, $p_n = q$, $p_i - p_{i-1} \in V$. Thinking of time as a discrete variable, we identify the parameter n with the travel time for a particle starting at p and moving towards q trough the path $(p_0, ..., p_n)$. We add a zero time path from each lattice point to itself; this convention turns the set of lattice points and lattice paths among them into the objects and morphisms, respectively, of a category with composition given by concatenation.

The set $\mathrm{L}_{p,q}$ of lattice paths from p to q, can be described as

$$\mathrm{L}_{p,q} \;=\; \bigsqcup_{l=0}^{\infty} \mathrm{L}_{p,q}(l) \;=\; \bigsqcup_{l=0}^{\infty} \big\{(a_1,...,a_l) \in [k]^l \;\;\big|\;\; p + v_{a_1} + \cdots + v_{a_l} = \; q\big\},$$

where $\mathrm{L}_{p,q}(l)$ is the set of lattice paths from p to q displayed in time l.

A pattern (of directions) of length $n+1$ on the set of indexes $[k]$ is given by a $(n+1)$-tuple $c = (c_0, c_1, ..., c_n)$ such that $c_i \in [k]$ and $c_i \neq c_{i+1}$. Let $D(n,k)$ be the set of all patterns of length $n+1$. We have a natural map that associates a pattern to each lattice path by contracting contiguous repeated indices. For example set $n = 8$, $k = 3$, and consider a lattice path $(2,2,1,3,3,3,1,1) \in \mathrm{L}_{p,q}(8)$. The associated pattern is $(2,1,3,1) \in D(3,3)$. We formally add a pattern of length 0.

Going back to our general settings, we have that

$$\mathrm{L}_{p,q} \;=\; \bigsqcup_{l=0}^{\infty} \bigsqcup_{n=0}^{l-1} \bigsqcup_{c \in D(n,k)} \mathrm{L}^{c}_{p,q}(l),$$

where $\mathrm{L}^{c}_{p,q}(l)$ is the set of lattice paths from p to q displayed in time l and with associated pattern equal to $c \in D(n,k)$.

Proposition 1.

1. The number of lattice paths from p to q displayed in time l is given by

$$|\mathrm{L}_{p,q}(l)| \;=\; \sum_{n=0}^{l-1} \sum_{c \in D(n,k)} |\mathrm{L}^{c}_{p,q}(l)|,$$

   where $\mathrm{L}^{c}_{p,q}(l)$ is the set of tuples $(s_0, ..., s_n) \in \mathbb{N}^{n+1}_{>0}$ such that

$$p + s_0 v_{c_0} + \cdots + s_n v_{c_n} = q \qquad \text{and} \qquad s_0 + \cdots + s_n = l.$$

2. The number of lattice paths from p to q, if finite, is given by

$$|\mathrm{L}_{p,q}| \;=\; \sum_{l=0}^{\infty} \sum_{n=0}^{l-1} \sum_{c \in D(n,k)} |\mathrm{L}^{c}_{p,q}(l)|.$$

The main problem in lattice path theory is to count the number of lattice paths joining a pair of lattice points. Usually further restrictions are imposed on the allowed paths. For example, one may want to count lattice paths that are restricted to visiting points in a subset of $\mathbb{Z}^d$, which we assume to be the set of integral points of a convex polyhedron $H \subseteq \mathbb{R}^d$. let $\mathrm{L}^{H}_{p,q}(l)$ be the set of time l lattice paths lying in H. Also let $\mathrm{L}^{c,H}_{p,q}(l)$ be the set of lattice paths fully included in H of time l, and pattern c.

Proposition 2. Let $p, q \in H$.

1. The number of lattice paths from p to q fully included in the convex polyhedron H and displayed in time l is given by

$$|\mathrm{L}^{H}_{p,q}(l)| \;=\; \sum_{n=0}^{l} \sum_{c\in D(n,k)} |\mathrm{L}^{c,H}_{p,q}(l)|,$$

   where $\mathrm{L}^{c,H}_{p,q}(l)$ is the set of tuples $(s_0, ..., s_n) \in \mathbb{N}^{n+1}_{>0}$ such that the following conditions hold for $0 \leq i \leq n-1$:

$$p+s_0v_{c_0}+\cdots+s_iv_{c_i} \in H, \qquad p+s_0v_{c_0}+\cdots+s_nv_{c_n} = q, \qquad s_0+\cdots+s_n = l.$$

2. The number of lattice paths from p to q fully included in the convex polyhedron H, if finite, is given by

$$|\mathrm{L}^{H}_{p,q}| \;=\; \sum_{l=0}^{\infty}\sum_{n=0}^{l-1} \sum_{c\in D(n,k)} |\mathrm{L}^{c,H}_{p,q}(l)|.$$

3. FROM LATTICE PATHS TO DIRECTED PATHS

Our next goal is to provide a suitable setting for "counting" directed paths, for which we keep the same set of allowed directions $V = \{v_1, ..., v_k\} \subseteq \mathbb{Z}^k$ as for lattice paths, while lifting the discrete time restriction, i.e. we consider tuples $(p_0, ..., p_n) \in \mathbb{R}^d$ such that

$$p_0 \;=\; p, \qquad p_n \;=\; q, \qquad p_i \;=\; p_{i-1} + s_i v, \qquad \text{with } v \in V, \text{ and } s_i \in \mathbb{R}_{\geq 0}.$$

The total travel time $t = s_1 + \cdots + s_n$ no longer has to be an integer; hence we are facing a moduli space of paths rather than a discrete set of paths.

To formalize the "counting" of such paths we turn to our work on indirect influences on directed manifolds [**9**]. Essentially this approach give us a way to put measures on the various components of the space of directed paths on a directed manifold. Fortunately, for our present purposes, we can proceed quite independently in an essentially self-contained fashion.

Recall from [**9**] that a directed manifold is a smooth manifold together with a tuple of vector fields on it. We are going to work with the directed manifold $(\mathbb{R}^d, v_1, ..., v_k)$. A directed path from $p \in \mathbb{R}^d$ to $q \in \mathbb{R}^d$ displayed in time $t > 0$ and going through $n \geq 0$ changes of directions is parameterized by a pair (c, s) with the following properties:

- $c = (c_0, ..., c_n)$ is a pattern in $D(n, k)$.
- $s = (s_0, ..., s_n)$ is a $(n+1)$-tuple such that $s_0 + \cdots + s_n = t$, with $s_i \in \mathbb{R}_{\geq 0}$. We say that s defines the time distribution of the directed path associated to (c, s), and let Δ_n^t be the n-simplex of all such tuples. We regard Δ_n^t as a subset of the space $\mathbb{R}^{n+1}$ endowed with its canonical inner product.
- (c, s) determines a $(n+2)$-tuple of points $(p_0, \ldots, p_{n+1}) \in (\mathbb{R}^d)^{n+2}$ given by:

$$p_0 \ = \ p \qquad \text{and} \qquad p_i \ = \ p_{i-1} \ + \ s_{i-1} v_{c_{i-1}} \qquad \text{for} \qquad 1 \leq i \leq n+1.$$

- (c, s) must be such that $p_{n+1}(c, s) \ = \ q$.

The pair (c, s) determines the directed polygon path

$$\varphi_{c,s} : [0, s_0 + \cdots + s_n] \ \simeq \ [0, s_0] \bigsqcup_{s_0, 0} \cdots \bigsqcup_{s_{n-1}, 0} [0, s_n] \ \longrightarrow \ M$$

from p to q where the restriction of $\varphi_{c,s}$ to the interval $[0, s_i]$ is given by

$$\varphi_{c,s}|_{[0,s_i]}(r) \ = \ p_i \ + \ r v_{c_i} \qquad \text{for} \qquad r \in [0, s_i].$$

We say that the points $p_0, ..., p_n$ are the peaks of the path $\varphi_{c,s}$.

The moduli space $\Gamma_{p,q}(t)$ of directed paths from p to q displayed in time $t > 0$ is given by

$$\Gamma_{p,q}(t) \ = \ \coprod_{n=0}^{\infty} \coprod_{c \in D(n,k)} \{s \in \Delta_n^t \mid p_{n+1}(c, s) = q\} \ = \ \coprod_{n=0}^{\infty} \coprod_{c \in D(n,k)} \Gamma_{p,q}^c(t).$$

In addition we formally set

$$\Gamma_{p,q}(0) \ = \ \Gamma_{p,q}^{\emptyset}(0) \ = \ \begin{cases} \{p\} & \text{if } p = q, \\ \emptyset & \text{otherwise.} \end{cases}$$

The unique path in the latter set has the empty pattern.

Our guiding principle in this work is that one can think of the space $\Gamma_{p,q}(l)$ as being a continuous analogue of the set $\mathrm{L}_{p,q}(l)$ of lattice paths from p to q displayed in time $l \geq 0$. Note that in $\Gamma_{p,q}(t)$ neither p nor q nor t are restricted to be integers points. Even if they are integers $\Gamma_{p,q}(t)$ is still a larger space than $\mathrm{L}_{p,q}(t)$ since in $\Gamma_{p,q}(t)$ the intermediary peaks are not restricted to be integer points.

Proposition 3. Consider the directed manifold $(\mathbb{R}^d, v_1, ..., v_k)$, let $H \subseteq \mathbb{R}^d$ be a convex polyhedron, let $p, q \in H$, and $t \in \mathbb{R}_{>0}$.

1. For $c = (c_0, ..., c_n) \in D(n,k)$, the space $\Gamma^c_{p,q}(t)$ is the convex polytope given by:

$$\Gamma^c_{p,q}(t) = \left\{ (s_0, ..., s_n) \in \mathbb{R}^{n+1}_{\geq 0} \;\middle|\; p + s_0 v_{c_0} + \cdots + s_n v_{c_n} = q, \ \ s_0 + \cdots + s_n = t \right\}.$$

2. $\mathrm{L}^c_{p,q}(t)$ is the set of integer points in the interior of $\Gamma^c_{p,q}(t)$.

3. For $c \in D(n,k)$, let $\Gamma^{c,H}_{p,q}(t)$ be the subset of $\Gamma^c_{p,q}(t)$ consisting points whose associated path $\varphi_{c,s}$ lies entirely in H. The moduli space $\Gamma^{c,H}_{p,q}(t)$ is the convex polytope consisting of all tuples $(s_0, ..., s_n) \in \mathbb{R}^{n+1}_{\geq 0}$ such that the following conditions hold for $0 \leq i \leq n-1$:

$$p + s_0 v_{c_0} + \cdots + s_i v_{c_i} \in H, \qquad p + s_0 v_{c_0} + \cdots + s_n v_{c_n} = q, \qquad s_0 + \cdots + s_n = l.$$

4. $\mathrm{L}^{c,H}_{p,q}(t)$ is the set of integer points in the interior of $\Gamma^{c,H}_{p,q}(t)$.

Next we introduce our main definition in this work.

Definition 4. The volume of the moduli space of directed paths $\Gamma_{p,q}(t)$ is given by:

$$\mathrm{vol}(\Gamma_{p,q}(t)) = \sum_{n=0}^{\infty} \sum_{c \in D(n,k)} \mathrm{vol}(\Gamma^c_{p,q}(t)).$$

Remark 5. To compute the volume of a polytope $P \subseteq \Delta^t_n$ we regard it as a top dimensional subset of its affine linear span $\widehat{P} \subseteq \mathbb{R}^{n+1}$, and compute its volume with respect to the Lebesgue measure on $\widehat{\widehat{P}}$ induced by the inner product on $\mathbb{R}^{n+1}$. As it stands, there is no guarantee that the infinite sum above is convergent. Nevertheless, it turns out to be convergent in the examples developed in Sections 4 and 5.

4. CONTINUOUS BINOMIALS COEFFICIENTS

The binomial coefficient $\binom{m}{n}$ counts sets of cardinality n within a set of cardinality m. To construct continuous analogues for the binomial coefficients we need a lattice path representation for them. Consider the step vectors

$$V = \{(1,0), (0,1)\} \subseteq \mathbb{Z}^2 \subseteq \mathbb{R}^2.$$

It is well-known that the binomial coefficient $\binom{m}{n}$ is such that

$$\binom{m}{n} = \left|\{\text{lattice paths from } (0,0) \text{ to } (n, m-n)\}\right|.$$

Notice that such a lattice path is displayed in time m. Thus according to our general methodology for constructing continuous analogues we should consider the directed manifold $(\mathbb{R}^2, (1,0), (0,1))$, and compute the volume of the moduli spaces of directed paths $\Gamma_{(0,0),(x,y)}(t)$; we denote the latter space by $\Gamma(x,y)$ as it is empty unless $t = x + y$.

Definition 6. For $0 < s < x$ in $\mathbb{R}$, the continuous binomial coefficient $\left\{ {x \atop s} \right\}$ is given by

$$\left\{ {x \atop s} \right\} = \mathrm{vol}(\Gamma(s, x-s)) = \sum_{n=0}^{\infty} \sum_{c \in D(n,k)} \mathrm{vol}(\Gamma^c(s, x-s)).$$

The domain of the symbol $\left\{ {x \atop s} \right\}$ is extended by continuity to $0 \leq s \leq x$.

Intuitively, $\left\{ {x \atop s} \right\}$ is the total measure of the set of paths starting at the origin and built with horizontal and vertical moves, with travelling time in the horizontal direction of s, and travelling time in the vertical direction of $x - s$. Figure 1 shows a couple of directed paths accounted for by the continuous binomial coefficient $\left\{ {\pi \atop e} \right\}$.

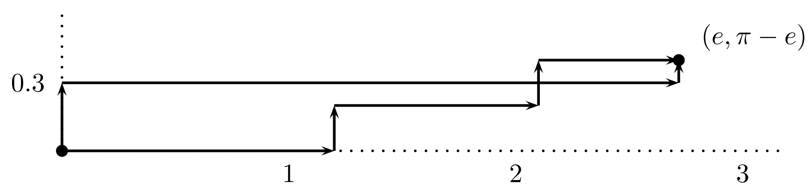

Figure 1. Directed paths in $\Gamma(e, \pi - e)$ with patterns $(1,2,1,2,1)$ and $(2,1,2)$.

To compute explicitly the continuous binomial coefficients we used the following identity shown in [**9**].

Identity 7. The volume $\mathrm{vol}(\Gamma(x,y))$ of the convex polytope $\Gamma(x,y)$ is given by

$$\sum_{n=0}^{\infty} \left(2\frac{x^n y^n}{n!n!} + \frac{x^{n+1} y^n}{(n+1)!n!} + \frac{x^n y^{n+1}}{n!(n+1)!} \right) = 2\sum_{n=0}^{\infty} \frac{x^n y^n}{n!n!} + (x+y)\sum_{n=0}^{\infty} \frac{x^n y^n}{n!(n+1)!}.$$

Remark 8. The presence of a couple of factorials in the denominators of the summands in formula for $\mathrm{vol}(\Gamma(x,y))$ from the proof of Theorem 9 guarantees uninform convergency. Similar remarks will apply for all the power series appearing in this section.

Theorem 9. For $0 \leq s \leq x$ the continuous binomial function $\left\{ {x \atop s} \right\}$ is given by

$$2 \sum_{0 \leq a \leq b,\ a+b \text{ even}} (-1)^{\frac{b-a}{2}} \binom{b}{\frac{a+b}{2}} \frac{x^a}{a!} \frac{s^b}{b!} \quad +$$

$$\sum_{0 \leq a \leq b-1,\ a+b \text{ odd}} (-1)^{\frac{b-a-1}{2}} \binom{b}{\frac{a+b+1}{2}} \frac{x^a}{a!} \frac{s^b}{b!} \quad +$$

$$\sum_{0 \leq a \leq b+1,\ a+b \text{ odd}} (-1)^{\frac{b-a+1}{2}} \binom{b}{\frac{b-a+1}{2}} \frac{x^a}{a!} \frac{s^b}{b!} \quad =$$

$$2\sum_{n=0}^{\infty} \frac{s^n (x-s)^n}{n!n!} \; + \; x \sum_{n=0}^{\infty} \frac{s^n (x-s)^n}{n!(n+1)!} \quad = \quad \sum_{n=0}^{\infty} (x+2n+2) \frac{s^n (x-s)^n}{n!(n+1)!}.$$

Proof. Follows from Identity 7 using that $\left\{ {x \atop s} \right\} = \operatorname{vol}(\Gamma(s, x-s))$. □

The following result gives continuous analogues for a couple of well-known properties of the binomial coefficients.

Corollary 10. For $0 \leq s \leq x$, we have that $\left\{ {x \atop 0} \right\} = \left\{ {x \atop x} \right\} = 2 + x$, and

$$\left\{ {x \atop s} \right\} = \left\{ {x \atop x-s} \right\}.$$

Remark 11. We have chosen to set the value of $\left\{ {x \atop 0} \right\}$ by continuity. The increment of the weight of the unique directed path joining $(0,0)$ and $(x,0)$ from 1 to $2+x$ is reminiscent of the process by which a particle acquires an higher effective mass – compared to its bare mass – by being surrounded by other massive particles.

Corollary 12. For $s \geq 0$, we have that:

$$\left\{ {1+s \atop s} \right\} \quad = \quad 3 \; + \; \sum_{n=1}^{\infty} \left(n^2 + 3n + 3\right) \frac{s^n}{n!(n+1)!}.$$

Our next results shows that the binomial coefficients are an eigenfunction, with eigenvalue 1, of an hyperbolic partial differential equation.

Theorem 13. The continuous binomial coefficients $\left\{ {x \atop s} \right\}$ satisfy the following partial differential equation:

$$\left(\frac{\partial^2}{\partial x^2} \; + \; \frac{\partial^2}{\partial x \partial s} \right) \left\{ {x \atop s} \right\} \quad = \quad \left\{ {x \atop s} \right\}.$$

Proof. We know from [**9**] that the function $\text{vol}(\Gamma) = \text{vol}(\Gamma(x,y))$, given explicitly in the proof of Theorem 9, satisfies the following partial differential equation:

$$\frac{\partial^2 \text{vol}(\Gamma)}{\partial x \partial y} \;=\; \text{vol}(\Gamma).$$

Therefore we have that:

$$\Big(\frac{\partial^2}{\partial x^2} \;+\; \frac{\partial^2}{\partial x \partial s}\Big)\begin{Bmatrix} x \\ s \end{Bmatrix} \;=\; \frac{\partial^2 \text{vol}(\Gamma(s,x-s))}{\partial x^2} \;+\; \frac{\partial^2 \text{vol}(\Gamma(s,x-s))}{\partial x \partial s} \;=$$

$$\frac{\partial^2 \text{vol}(\Gamma)}{\partial y^2}(s,x-s) \;+\; \frac{\partial}{\partial x}\Big[\frac{\partial \text{vol}(\Gamma)}{\partial x}(s,x-s) \;-\; \frac{\partial \text{vol}(\Gamma)}{\partial y}(s,x-s)\Big] \;=$$

$$\frac{\partial^2 \text{vol}(\Gamma)}{\partial y^2}(s,x-s) \;+\; \frac{\partial^2 \text{vol}(\Gamma)}{\partial x \partial y}(s,x-s) \;-\; \frac{\partial^2 \text{vol}(\Gamma)}{\partial y^2}(s,x-s) \;=$$

$$\text{vol}(\Gamma(s,x-s)) \;=\; \begin{Bmatrix} x \\ s \end{Bmatrix}.$$

□

The following result, due to Tom Koornwinder, gives an explicit formula for the continuous binomial coefficients in terms of the modified Bessel functions I_0 and I_1.

Theorem 14. For $0 \le s \le x$, we have that

$$\begin{Bmatrix} x \\ s \end{Bmatrix} \;=\; 2\mathrm{I}_0(2s^{1/2}(x-s)^{1/2}) \;+\; \frac{x}{s^{1/2}(x-s)^{1/2}}\mathrm{I}_1(2s^{1/2}(x-s)^{1/2}).$$

Proof. The result follows from Identity 7 and the defining expressions

$$\mathrm{I}_0(z) \;=\; \sum_{k=0}^{\infty} \frac{(\frac{1}{4}z^2)^k}{k!k!} \qquad \text{and} \qquad \mathrm{I}_1(z) \;=\; \frac{z}{2}\sum_{k=0}^{\infty} \frac{(\frac{1}{4}z^2)^k}{k!(k+1)!}$$

for the modified Bessel functions I_0 and I_1.

□

Fix $0 < p < 1$ and $x > 0$. A continuous analogue of the discrete binomial distribution may be defined via the density function on the interval $[0,x]$ given by

$$\frac{1}{b_p(x)}\begin{Bmatrix} x \\ s \end{Bmatrix} p^s(1-p)^{x-s} \qquad \text{where} \qquad b_p(x) = \int_0^x \begin{Bmatrix} x \\ s \end{Bmatrix} p^s(1-p)^{x-s}ds$$

which, intuitively, measures the probability of the motion of a particle in such that:

- The particle starts at $(0,0)$ and moves with speed 1 for x units of times.

- The particle moves with probability p in the horizontal direction, and with probability $1-p$ in the vertical direction.
- The particle ends up at the point $(s, x-s)$.

For $p = \frac{1}{2}$ the normalization constant, up to a factor of 2^{-x}, is given by

$$\int_0^x \left\{ \begin{matrix} x \\ s \end{matrix} \right\} ds,$$

and measures the number of paths starting at the origin and traveling for x units of time either in the horizontal or in the vertical direction. Note that $\int_0^x \left\{ \begin{smallmatrix} x \\ s \end{smallmatrix} \right\} ds$ plays, for the continuous binomial coefficients, the role that 2^n plays for the binomial coefficients.

Below we are going to use the following identity, valid for $n, m \in \mathbb{N}$, involving the classical beta B and gamma Γ functions:

$$\int_0^1 s^n(1-s)^m ds \;=\; \mathrm{B}(n+1, m+1) \;=\; \frac{\Gamma(n+1)\Gamma(m+1)}{\Gamma(n+m+2)} \;=\; \frac{n!m!}{(n+m+1)!}.$$

Also we are going to use the falling factorials $(a)_n \;=\; a(a-1)\cdots(a-n+1)$ for $a \geq n$. Note that the notation $a^{\underline{n}}$ for the falling factorial is also quite common.

Theorem 15.

1. For $x \in \mathbb{R}_{\geq 0}$ we have that

$$\int_0^x \left\{ \begin{matrix} x \\ s \end{matrix} \right\} ds = 2(e^x - 1) \qquad \text{and thus} \qquad b_{\frac{1}{2}}(x) = (e^x - 1)2^{1-x}.$$

2. The following identities hold:

$$e \;=\; 1 \;+\; \frac{1}{2}\int_0^1 \left\{ \begin{matrix} 1 \\ s \end{matrix} \right\} ds \qquad \text{and} \qquad \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n \;=\; 1 \;+\; \frac{1}{2}\int_0^x \left\{ \begin{matrix} x \\ s \end{matrix} \right\} ds.$$

3. For $0 < p < 1$ we have that $b_p(x)$ is given by

$$2(1-p)^x \sum_{k,n=0}^{\infty} \ln^k\left(\frac{p}{1-p}\right)\binom{n+k}{k}\left[\frac{x^{2n+k+1}}{(2n+k+1)!} \;+\; \frac{x^{2n+k+2}}{(2n+2)(2n+k+1)!}\right].$$

4. For $0 < p < 1$, the function $b_p(x)$ can be written as

$$\pi^{\frac{1}{2}}(p(1-p))^{\frac{x}{2}} \sum_{n=0}^{\infty} \frac{(x+2n+2)}{(n+1)!}\left(\frac{x}{\ln(\frac{p}{1-p})}\right)^{n+\frac{1}{2}} I_{n+\frac{1}{2}}\left(\frac{x}{2}\ln\left(\frac{p}{1-p}\right)\right).$$

Proof. Item 2 follows directly from item 1, which is shown as follows:

$$\int_0^x \left\{ {x \atop s} \right\} ds \ = \ 2\sum_{n=0}^{\infty}\frac{1}{n!n!}\int_0^x s^n(x-s)^n ds \ + \ \sum_{n=0}^{\infty}\frac{x}{(n+1)!n!}\int_0^x s^n(x-s)^n ds \ =$$

$$2\sum_{n=0}^{\infty}\frac{x^{2n+1}}{n!n!}B(n+1,n+1) \ + \ \sum_{n=0}^{\infty}\frac{x^{2n+2}}{(n+1)!n!}B(n+1,n+1) \ =$$

$$2\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!} \ + \ 2\sum_{n=0}^{\infty}\frac{x^{2n+2}}{(2n+2)!} \ = \ 2\sinh(x)+2\cosh(x)-2 \ =$$

$$e^x - e^{-x} + e^x + e^{-x} - 2 \ = \ 2(e^x-1).$$

The proof of item 3 is omitted as it can be deduced just as item 2 of Theorem 16 below with $l=0$. Item 4 follows by termwise integration making the change of variable $s=\frac{x}{2}(t+1)$, using the integral representation for the modified Bessel functions given by the identity

$$\int_{-1}^1 (1-t^2)^{v-\frac{1}{2}}e^{zt}dt \ = \ \pi^{\frac{1}{2}}\Gamma(v+\frac{1}{2})(\frac{z}{2})^{-v}I_v(z).$$

Explicitly, we have that

$$b_p(x) \ = \ \int_0^x \left\{ {x \atop s} \right\} p^s(1-p)^{x-s}ds \ = \ (1-p)^x\int_0^x \left\{ {x \atop s} \right\} e^{\ln(\frac{p}{1-p})s}ds \ =$$

$$(1-p)^x\sum_{n=0}^{\infty}\frac{x+2n+2}{n!(n+1)!}\int_0^x s^n(x-s)^n e^{\ln(\frac{p}{1-p})s}ds \ =$$

$$(p(1-p))^{\frac{x}{2}}\sum_{n=0}^{\infty}\frac{x+2n+2}{n!(n+1)!}(\frac{x}{2})^{2n+1}\int_{-1}^1(1-t^2)^n e^{\frac{x}{2}\ln(\frac{p}{1-p})t}dt \ =$$

$$\pi^{\frac{1}{2}}(p(1-p))^{\frac{x}{2}}\sum_{n=0}^{\infty}\frac{x+2n+2}{(n+1)!}\Big(\frac{x}{\ln(\frac{p}{1-p})}\Big)^{n+\frac{1}{2}}I_{n+\frac{1}{2}}(\frac{x}{2}\ln(\frac{p}{1-p})).$$

□

We define the centered continuous binomial distribution, for $p=\frac{1}{2}$, via its density d_x, see Figure 2, which has support in the interval $[-\frac{x}{2},\frac{x}{2}]$ where it is given by

$$d_x(s) = \frac{1}{b(x)}\left\{ {x \atop \frac{x}{2}+s} \right\}.$$

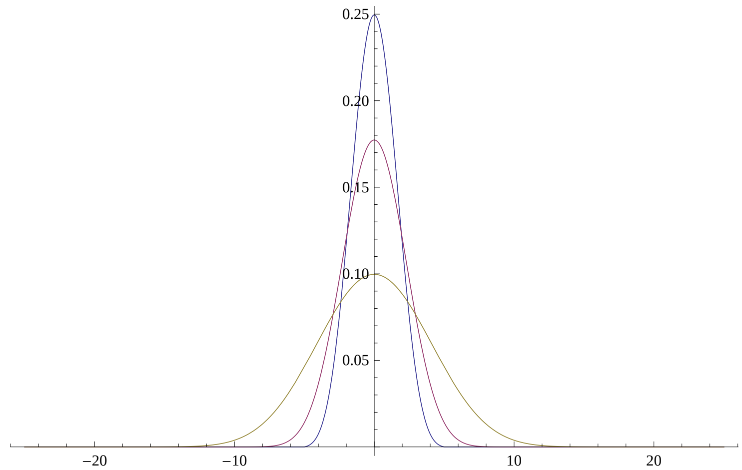

Figure 2: Continuous binomial density d_x for $x = 10, 20, 50$.

Theorem 16.

1. The moments $E_{\frac{1}{2}}(s^l)$, for $l \geq 1$, of the continuous binomial distribution, for $p = \frac{1}{2}$, are given by

$$E_{\frac{1}{2}}(s^l) \;=\; \frac{1}{b_{\frac{1}{2}}(x)}\sum_{n=0}^{\infty}(n+l)_l\frac{x^{2n+l+1}}{(2n+l+l)!} + \frac{1}{b_{\frac{1}{2}}(x)}\sum_{n=0}^{\infty}(n+l)_{(l-1)}\frac{x^{2n+l+2}}{(2n+l+1)!}.$$

2. The moments $E_p(s^l)$, for $l \geq 1$, of the continuous binomial distribution, for $0 < p < 1$, are given by

$$\frac{2(1-p)^x}{b_p(x)}\sum_{k,n=0}^{\infty}\ln^k\Big(\frac{p}{1-p}\Big)\binom{n+k+l}{n}(k+l)_l\left(\frac{x^{2n+k+l+1}}{(2n+k+l+1)!} + \frac{x^{2n+k+l+2}}{(2n+2)(2n+k+l+1)!}\right).$$

Proof. Item 1 follows the same pattern as the proof of Theorem 15. We show item 2:

$$b_p(x)E_p(s^l) \;=\; \int_0^x s^l\left\{\begin{matrix}x\\ s\end{matrix}\right\}p^s(1-p)^{x-s}ds \;=$$

$$2\sum_{n=0}^{\infty}\int_0^x s^l\frac{s^n(x-s)^n}{k!n!n!}p^s(1-p)^{x-s}ds \;+\; \sum_{n=0}^{\infty}x\int_0^x s^l\frac{s^n(x-s)^n}{k!(n+1)!n!}p^s(1-p)^{x-s}ds \;=$$

$$2(1-p)^x\sum_{k,n=0}^{\infty}\ln^k\Big(\frac{p}{1-p}\Big)\int_0^x\frac{s^{n+k+l}(x-s)^n}{k!n!n!}ds \quad +$$

$$(1-p)^x\sum_{k,n=0}^{\infty}\ln^k\Big(\frac{p}{1-p}\Big)x\int_0^x\frac{s^{n+k+l}(x-s)^n}{k!(n+1)!n!}ds \quad =$$

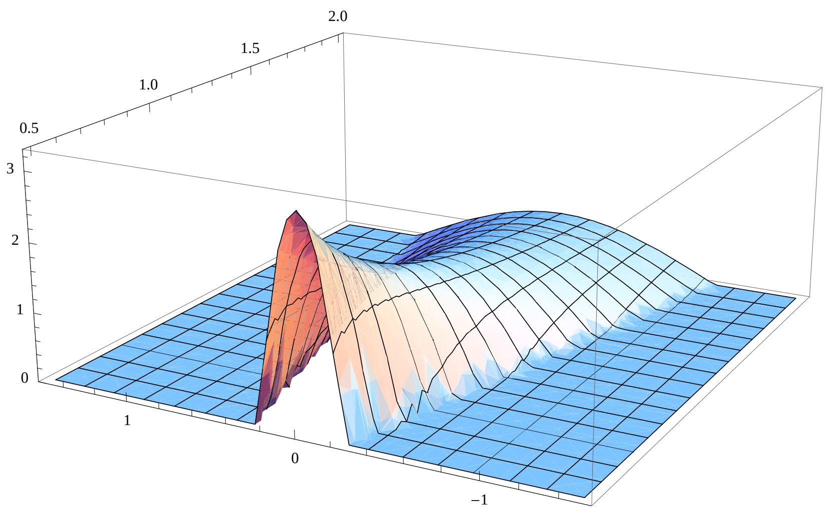

Figure 3: Continuous binomial density $d_x(s)$ for $x \in [0.7, 2]$ and $s \in [-1.5, 1.5]$.

$$2(1-p)^x \sum_{k,n=0}^{\infty} \ln^k\left(\frac{p}{1-p}\right) \frac{(n+k+l)!}{k!n!} \frac{x^{2n+k+l+1}}{(2n+k+l+1)!} \quad +$$

$$(1-p)^x \sum_{k,n=0}^{\infty} \ln^k\left(\frac{p}{1-p}\right) \frac{(n+k+l)!}{k!(n+1)!} \frac{x^{2n+k+l+2}}{(2n+k+l+1)!}.$$

□

Proposition 17.

1. The odd moments of d_x vanish: $E(s^{2k+1}) = 0$.
2. The moment $E(s^{2k})$ of d_x is given by

$$E(s^{2k}) = \frac{1}{b_{\frac{1}{2}}(x)} \sum_{l=0}^{2k} \binom{2k}{l} \left(-\frac{x}{2}\right)^{2k-l} \int_0^x t^l \left\{ \begin{matrix} x \\ t \end{matrix} \right\} dt,$$

   where $\int_0^x t^l \left\{ \begin{matrix} x \\ t \end{matrix} \right\} dt$ has been given explicitly in Theorem 16.

3. Let δ be the Dirac's delta function centered at 0. We have that

$$\lim_{x \to 0} d_x = \delta \qquad \text{in the weak topology.}$$

Proof. To show property 1 it is enough to check that d_x is an even function:

$$d_x(-s) \;=\; \frac{1}{b(x)}\left\{\begin{matrix} x \\ \frac{x}{2}-s \end{matrix}\right\} \;=\; \frac{1}{b(x)}\left\{\begin{matrix} x \\ x-(\frac{x}{2}+s) \end{matrix}\right\} \;=\; \frac{1}{b(x)}\left\{\begin{matrix} x \\ \frac{x}{2}+s \end{matrix}\right\} \;=\; d_x(s),$$

where we have used Corollary 10. The moment of order $2k$ is given

$$E(s^{2k}) \;=\; \frac{1}{b(x)}\int_{-\frac{x}{2}}^{\frac{x}{2}} s^{2k}\left\{\begin{matrix} x \\ \frac{x}{2}+s \end{matrix}\right\}ds.$$

Making the change of variable $t=\frac{x}{2}+s$ we obtain that:

$$E(s^{2k}) \;=\; \frac{1}{b(x)}\sum_{l=0}^{2k}\binom{2k}{l}(-\frac{x}{2})^{2k-l}\int_0^x t^l\left\{\begin{matrix} x \\ t \end{matrix}\right\}dt.$$

Property 2 follows from this identity using Theorem 16.

Consider Property 3. We showed in [**9**] that $\left\{\begin{smallmatrix} x \\ s \end{smallmatrix}\right\} = \mathrm{vol}(\Gamma(s,x-s))$, for $0 \le s \le x$, achieves its maximum at $s=\frac{x}{2}$. Thus, d_x achieves its maximum at $s=0$.

Property 4 follows since the functions d_x are non-negative, almost continuous (with discontinuity points $-\frac{x}{2}$ and $\frac{x}{2}$), of total mass 1, and support in the interval $[-\frac{x}{2},\frac{x}{2}]$. Let $f(s)$ be a continuous function on $\mathbb{R}$. Given $\epsilon>0$ choose x small enough such that $|f(s)-f(0)|<\epsilon$ for all $s\in[-\frac{x}{2},\frac{x}{2}]$. Under this conditions we have that

$$\left|\int_{-\infty}^{\infty} f(s)d_x(s)ds \;-\; f(0)\right| \;\le\; \int_{-\frac{x}{2}}^{\frac{x}{2}}\big|f(s)-f(0)\big|d_x(s)ds \;<\; \epsilon.$$

□

Remark 18. The continuous binomial density $d_x(s)$ is plotted in Figure 3. The reader should note the remarkable similarity with the plots for the Brownian motion density, a subject that deserves further study.

Although Theorem 9 already shows the combinatorial nature of the continuous binomial coefficients $\left\{\begin{smallmatrix} x \\ s \end{smallmatrix}\right\}$, the combinatorial interpretation is somewhat obscure due to the presence of negative signs. This problem, as shown below, can be easily overcome by performing the change of variables $(s,x)\longrightarrow(s,(t+1)s)$.

Theorem 19. For $t\ge 0$ and $s\ge 0$ we have that:

$$\left\{\begin{matrix} (t+1)s \\ s \end{matrix}\right\} \;=\; 2+s+\sum_{n=1}^{\infty}\frac{t^n}{n!}\Big(2(2n)_n\frac{s^{2n}}{(2n)!}+(2n+1)_n\frac{s^{2n+1}}{(2n+1)!}+(2n-1)_n\frac{s^{2n-1}}{(2n-1)!}\Big).$$

Proof.

$$\left\{\begin{matrix} (t+1)s \\ s \end{matrix}\right\} \;=\; \mathrm{vol}(\Gamma(s,ts)) \;=\; \sum_{n=0}^{\infty}\Big(2\frac{s^ns^nt^n}{n!n!} \;+\; \frac{s^{n+1}s^nt^n}{(n+1)!n!} \;+\; \frac{s^ns^{n+1}t^{n+1}}{n!(n+1)!}\Big) \;=$$

$$\sum_{n=0}^{\infty}\Big(2(2n)_n\frac{s^{2n}t^n}{(2n)!n!} \;+\; (2n+1)_n\frac{s^{2n+1}t^n}{(2n+1)!n!} \;+\; (2n+1)_{n+1}\frac{s^{2n+1}t^{n+1}}{(2n+1)!(n+1)!}\Big) \;=$$

$$2+s+\sum_{n=1}^{\infty}\Big(2(2n)_n\frac{s^{2n}t^n}{(2n)!n!} \;+\; (2n+1)_n\frac{s^{2n+1}t^n}{(2n+1)!n!} \;+\; (2n-1)_n\frac{s^{2n-1}t^n}{(2n-1)!n!}\Big).$$

□

Theorem 19 can be understood in terms of combinatorial species [**3, 5, 10, 17, 27**] as follows. Let $\mathbb{B}$ be the category of finite sets and bijections, and set be the category of finite sets and maps. Let $\mathrm{Inj}:\mathbb{B}\times\mathbb{B}\longrightarrow \mathrm{set}$ be the functor sending a pair of finite sets (a,b) to the set $\mathrm{Inj}(a,b)$ of injective maps from a to b.

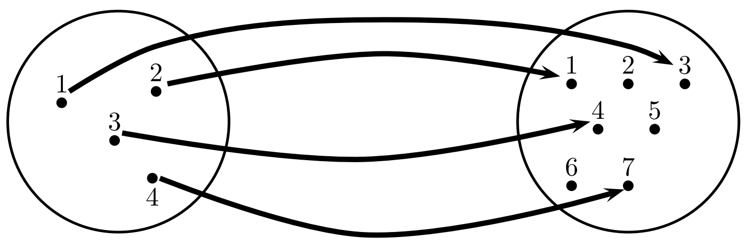

Figure 4. A map in $B([4],[7])$ according to condition 5.

Consider the functor $B:\mathbb{B}\times\mathbb{B}\longrightarrow \mathrm{set}$ defined on $(a,b)\in\mathbb{B}\times\mathbb{B}$ by

1) $[2]$ if $a=b=\emptyset$; 2) $[1]$ if $a=\emptyset$, $|b|=1$; 3) $[2]\times\mathrm{Inj}(a,b)$ if $|b|=2|a|$;

4) $\mathrm{Inj}(a,b)$ if $|b|=2|a|+1$; 5) $\mathrm{Inj}(a,b)$ if $|b|=2|a|-1$; 6) $\emptyset$ otherwise.

Figure 4 shows a map contributing to B trough condition 5 above.

Corollary 20. The generating function of B is $\left\{\begin{matrix}(t+1)s\\ s\end{matrix}\right\}$.

Proof. The result follows from the definition of B, Theorem 19, the definition of the generating function

$$\sum_{n,m=0}^{\infty}\big|B([n],[m])\big|\frac{t^n s^m}{n!m!},$$

and the fact that $(a)_n$ counts injective functions from $[n]$ to $[a]$. □

Proposition 21. The following identity holds

$$\left\{\begin{matrix}2s\\ s\end{matrix}\right\} \;=\; 2\sum_{n=0}^{\infty}\binom{n}{\lfloor n/2\rfloor}\frac{s^n}{n!} \;=\; 2\big(I_0(2s)+I_1(2s)\big).$$

Thus, quite pleasantly, the midpoint continuous binomial $\left\{ {2s \atop s} \right\}$ is twice the generating function of the midpoint binomial coefficients.

To obtain a continuous analogue for the binomial coefficients we used their combinatorial interpretation as paths in a suitable lattice, and thus our interpretation for the continuous binomial coefficients counts directed paths in the corresponding direct manifold. The usefulness of the interpretation of the binomials coefficients as counting subsets of a fixed cardinality can hardly be overstated, so it is natural to ponder whether an analogue interpretation is available for the continuous binomial coefficients.

Let $\mathrm{U}[x,s]$ be the family of subsets $S \subseteq [0,x]$ such that:

- S is a finite disjoint union of closed subintervals of $[0,x]$.
- The sum of the lengths of the closed subintervals defining S is equal to s.

The linear order on $[0,x]$ induces a linear order on the closed subintervals defining a set $S \in \mathrm{U}[x,s]$. Consider the map

$$\mathrm{path} : \mathrm{U}[x,s] \longrightarrow \Gamma(s, x-s)$$

sending $S = [a_1,b_1] \sqcup [a_2,b_2] \sqcup \cdots \sqcup [a_n,b_n]$ in $\mathrm{U}[x,s]$, written in the linear order, to the directed path in $\Gamma(s,x-s)$ constructed as follows (see Figure 5 where S is the union of the marked subintervals on the left):

- For $S = \emptyset$, a valid choice if and only if $s = 0$, the associated path has format (2) and time distribution (x).
- If $a_1 = 0$ and $b_n = x$, then the associated path has format $(1,2,...,2,1)$ of length $2n-1$ and time distribution $(b_1, a_2-b_1, b_2-a_2, ..., x-a_n)$.
- If $a_1 = 0$ and $b_n < x$, then the associated path has format $(1,2,...,1,2)$ of length $2n$ and time distribution $(b_1, a_2-b_1, b_2-a_2, ..., b_n-a_n, x-b_n)$.
- If $a_1 > 0$ and $b_n = x$, then the associated path has format $(2,1,...,2,1)$ of length $2n$ and time distribution $(a_1, b_1-a_1, a_2-b_1, ..., x-a_n)$.
- If $a_1 > 0$ and $b_n < x$, then the associated path has format $(2,1,...,1,2)$ of length $2n+1$ and time distribution $(a_1, b_1-a_1, a_2-b_1, ..., b_n-a_n, x-b_n)$.

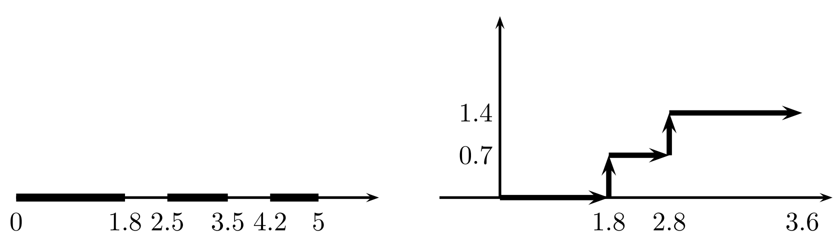

Figure 5. Set S in $\mathrm{U}[0,5]$ and its associated directed path.

It is easy to see that the map path is injective. Moreover the map path is essentially surjective, i.e. the image of path has full measure. Indeed, to show that

$$\mathrm{vol}\big(\Gamma(s, x-s) \setminus \mathrm{path}(\mathrm{U}[x,s])\big) \;=\; 0$$

one simply notes that a path in $\Gamma(s, x-s) \setminus \mathrm{path}(\mathrm{U}[x,s])$ must have at least one coordinate equal to zero, and therefore the later set is included in a finite union of codimension one subsets, which implies that it has to be a set of measure zero.

Since path is a bijection onto its image, a set of full measure, one obtains by pull-back a measure-procedure on $\mathrm{U}[x,s]$. Thus we have shown the following result.

Proposition 22. For $0 \le s \le x$, we have that:

$$\left\{ \begin{matrix} x \\ s \end{matrix} \right\} \;=\; \mathrm{vol}\big(\mathrm{U}[x,s]\big) \;=\; \sum_{n=0}^{\infty} \mathrm{vol}\big(\mathrm{U}_n[x,s]\big),$$

where $\mathrm{U}_n[x,s] \subseteq \mathrm{U}[x,s]$ consists of sets which are the union of n closed subintervals.

5. CONTINUOUS CATALAN NUMBERS

We proceed to construct continuous analogues for the Catalan numbers $c_n = \frac{1}{n+1}\binom{2n}{n}$. The Catalan numbers admit a myriad of interesting combinatorial interpretations, see Stanley's book [**39**], among those we work with a lattice path interpretation, see Figure 6.

Consider step vectors $V = \{(1,1),(1,-1)\} \subseteq \mathbb{Z}^2 \subseteq \mathbb{R}^2$. It is well-known that the Catalan numbers count Dyck paths, i.e.:

$$c_n \;=\; \big|\{\text{lattice paths in } \mathbb{Z}^2_{\ge 0} \text{ from } (0,0) \text{ to } (2n,0)\}\big|.$$

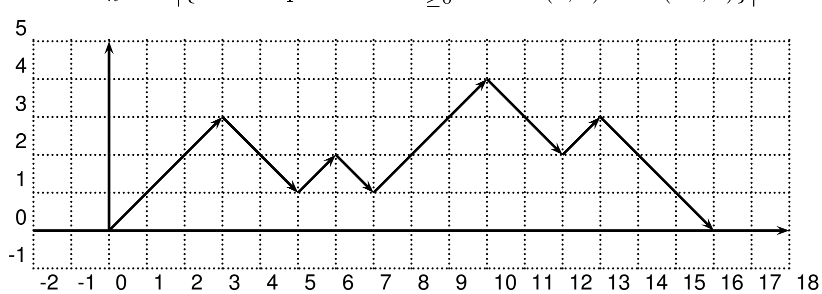

Figure 6. A Dyck path from 0 to 16.

If a Dyck path has a pattern of length $2k$, then it has k peaks and $k-1$ valley points. Therefore pattern decomposition induces the counting of Dyck paths by the number of peaks [**36, 37**], i.e. it leads to the Narayana identity

$$\frac{1}{n+1}\binom{2n}{n} \;=\; \sum_{k=1}^{n}\frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

To construct continuous analogues for the Catalan numbers we consider the directed manifold $(\mathbb{R}^2,(1,1),(1,-1))$ and measure directed paths from $(0,0)$ to $(x,0)$ fully included in $\mathbb{R}^2_{\geq 0}$, see Figure 7.

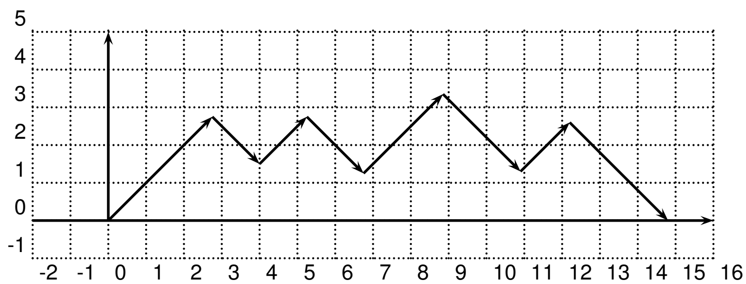

Figure 7. A Dyck directed path from 0 to 14.8.

Given $(x,y)\in\mathbb{R}^2_{\geq 0}$, we let $\Lambda(x,y)$ be the moduli space of directed paths from $(0,0)$ to (x,y) included in $\mathbb{R}^2_{\geq 0}$ with patterns of the form $(1,2,\ldots,1,2)$; we let $\Lambda^n(x,y)\subseteq\Lambda(x,y)$ be the set of directed paths with pattern of length $2n+2$. By construction $\mathrm{vol}(\Lambda^k(2n,0))$ is a continuous analogue of the Narayana number $\frac{1}{n}\binom{n}{k}\binom{n}{k-1}$, for $0\leq k<n$, i.e. there are exactly $\frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ integer points in the interior of $\Lambda^k(2n,0)$.

Definition 23. For $0<y<x$, in $\mathbb{R}$ the two-variables continuous Catalan function $C(x,y)$ is given by

$$C(x,y) \;=\; \sum_{n=0}^{\infty}\mathrm{vol}(\Lambda^n(x,y)).$$

The domain of C is extended to $0\leq y\leq x$ by continuity. The one-variable continuous Catalan function is given by $C(x)=C(x,0)$ for $x\in\mathbb{R}_{\geq 0}$.

Proposition 24. Consider the moduli space $\Lambda^n(x,y)$ for $0\leq y\leq x$ and $n\in\mathbb{N}$.

1. $\Lambda^n(x,y)$ is the convex polytope given in simplicial coordinates by:

$$s_0+\cdots+s_n \;=\; \frac{x+y}{2}, \qquad t_0+\cdots+t_n \;=\; \frac{x-y}{2},$$

$$s_0 \ \geq \ t_0, \quad s_0+s_1 \ \geq \ t_0+t_1, \quad \cdots \quad s_0+\cdots+s_{n-1} \ \geq \ t_0+\cdots+t_{n-1}.$$

In particular $\Lambda^0(x,y) \ = \ \{(\frac{x+y}{2}, \frac{x-y}{2})\}$ and thus $\mathrm{vol}(\Lambda^0(x,y)) = 1$.

2. For $n \geq 1$, the convex polytope $\Lambda^n(x,y)$ is given in Cartesian coordinates by:

$$0 \leq x_1 \leq \cdots \leq x_n \leq \frac{x+y}{2}, \qquad 0 \leq y_1 \leq \cdots \leq y_n \leq \frac{x-y}{2}, \qquad x_i \geq y_i.$$

3. For $n \geq 1$, $\Lambda^n(x,y)$ is given in terms of valley points coordinates by:

$$0 \ \leq \ a_1+b_1 \ \leq \ \cdots \ \leq \ a_n+b_n \ \leq \ x+y,$$

$$0 \ \leq \ a_1-b_1 \ \leq \ \cdots \ \leq \ a_n-b_n \ \leq \ x-y.$$

Proof. Item 1 follows directly from Proposition 3. Item 2 follows from item 1 making the change of variables $x_i \ = \ s_0+\cdots+t_{i-1}$ and $y_i \ = \ t_0+\cdots+s_{i-1}$ for $1 \leq i \leq n$. Item 3 follows from item 2 making the change of variables $a_i = x_i + y_i$ and $b_i = x_i - y_i$. □

Corollary 25. The infinite sum defining $C(x,y)$ is convergent and uniformly convergent on bounded sets.

Proof. From Proposition 24 we have that

$$0 \ \leq \ \sum_{n=0}^{\infty} \mathrm{vol}(\Lambda^n(x,y)) \ \leq \ \sum_{n=0}^{\infty} \mathrm{vol}(\Delta_n^{\frac{x+y}{2}})\mathrm{vol}(\Delta_n^{\frac{x-y}{2}}) \ = \ \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!}\frac{(x-y)^n}{n!}.$$

The later series has the desired properties. □

Example 26. For $0 \leq y \leq x$, the polytope $\Lambda^1(x,y)$ is given by

$$0 \leq x_1 \leq \frac{x+y}{2}, \qquad 0 \leq y_1 \leq \frac{x-y}{2}, \qquad y_1 \leq x_1.$$

Applying the change of variables $(a,b) = (x_1 - y_1, y_1)$ (with Jacobian determinant 1) we obtain the polytope given by

$$0 \ \leq b \ \leq \ \frac{x-y}{2}, \qquad 0 \ \leq a \ \leq \ \frac{x+y}{2} - b.$$

Thus we have that:

$$\mathrm{vol}(\Lambda^1(x,y)) \ = \ \int_0^{\frac{x-y}{2}} \int_0^{\frac{x+y}{2}-b} 1 da db \ = \ \frac{1}{8}(x-y)(x+3y) \quad \text{and} \quad \mathrm{vol}(\Lambda^1(x,0)) \ = \ \frac{1}{8}x^2.$$

Proposition 27. For $0 \leq y \leq x$, and $n \in \mathbb{N}$ the following recursive formula holds:

$$\mathrm{vol}(\Lambda^{n+1}(x,y)) = \int_0^{\frac{x-y}{2}} \int_0^{\frac{x+y}{2}-b} \mathrm{vol}(\Lambda^n(a+2b,a))\,da\,db.$$

Proof. Consider Cartesian coordinates on $\Gamma^{n+1}(x,y)$:

$$0 \leq x_1 \leq \cdots \leq x_n \leq x_{n+1} \leq \frac{x+y}{2}, \quad 0 \leq y_1 \leq \cdots \leq y_n \leq y_{n+1} \leq \frac{x-y}{2}, \quad x_i \geq y_i.$$

Making the change of variables $(a,b) = (x_{n+1} - y_{n+1}, y_{n+1})$ with Jacobian 1, we obtain the polytope given by:

$$0 \leq b \leq \frac{x-y}{2}, \qquad 0 \leq a \leq \frac{x+y}{2} - b,$$

$$0 \leq x_1 \leq \cdots \leq x_n \leq a+b, \qquad 0 \leq y_1 \leq \cdots \leq y_n \leq b, \qquad x_i \geq y_i.$$

Therefore we have that:

$$\mathrm{vol}(\Lambda^{n+1}(x,y)) = \int_{\Lambda^{n+1}(x,y)} 1\,dx_1 \cdots dx_{n+1}dy_1 \cdots dy_{n+1} =$$

$$\int_0^{\frac{x-y}{2}} \int_0^{\frac{x+y}{2}-b} \int_{\Lambda^n(a+2b,a)} 1\,dx_1 \cdots dx_n dy_1 \cdots dy_n\,da\,db =$$

$$\int_0^{\frac{x-y}{2}} \int_0^{\frac{x+y}{2}-b} \mathrm{vol}(\Lambda^n(a+2b,a))\,da\,db.$$

□

Corollary 28. For $0 \leq y \leq x$, and $n \in \mathbb{N}_{>0}$ the function $\mathrm{vol}(\Lambda^n(x,y))$ is given by:

$$\int_0^{\frac{x-y}{2}} \int_0^{\frac{x+y}{2}-b_1} \int_0^{b_1} \int_0^{a_1+b_1-b_2} \cdots \int_0^{b_{n-1}} \int_0^{a_{n-1}+b_{n-1}-b_n} 1\,da_n db_n da_{n-1} db_{n-1} \cdots da_1 db_1.$$

Proof. Follows iterating Proposition 27. □

Proposition 29. The Catalan function $C(x,y)$ satisfies the integral equation:

$$C(x,y) = 1 + \int_0^{\frac{x-y}{2}} \int_0^{\frac{x+y}{2}-b} C(a+2b,a)\,da\,db.$$

Proof. Follows from Propositions 24 and 27. □

Vignat and Wakhare [**40**] have established, using the recursive formula from Proposition 27, the following results. Note that item 4 below identifies the Catalan function $C(2x)$ with the even degree divided power generating function of the Catalan numbers.

Theorem 30. For $n \in \mathbb{N}$, and $0 \le y \le x$, we have that:

1. $\mathrm{vol}(\Lambda^n(x,y)) \;=\; \dfrac{(x-y)^n(x+y)^{n-1}(x+(2n+1)y)}{2^{2n}n!(n+1)!}.$

2. $C(x,y) \;=\; I_0(\sqrt{x^2-y^2}) \;-\; \dfrac{x-y}{x+y}I_2(\sqrt{x^2-y^2}).$

3. $C(2x) \;=\; \dfrac{I_1(2x)}{x} \;=\; \displaystyle\sum_{n=0}^{\infty} c_n \frac{x^{2n}}{2n!}.$

Since $I_1(2x) = xC(2x)$ and it is well-known that

$$x^2 I_1''(x) \;+\; xI_1'(x) \;-\; (x^2+1)I_1(x) \;=\; 0,$$

we obtain the following result.

Corollary 31. The Catalan function $C(x)$ satisfies the differential equation

$$xC''(x) \;+\; 3C'(x) \;-\; xC(x) \;=\; 0.$$

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REFERENCES

1. A. Barvinok: *Integer Points in Polyhedra.* Euro. Math. Soc, Zürich, 2008.
2. C. Banderier, P. Flajolet: *Basic Analytic Combinatorics of Directed Lattice Paths.* Theoret. Comput. Sci. **281** (2002), 37–80.
3. F. Bergeron, G, Labelle and P. Leroux: *Combinatorial species and tree like structures.* Cambridge Univ. Press, Cambridge, 1998.
4. N. Berline, M. Vergne: *Local Euler-Maclaurin formula for polytopes.* Moscow Math. J. **7** (2007), 355–386.
5. H. Blandín, R. Díaz: *Rational Combinatorics.* Adv. Appl. Math. **40** (2008), 107–126.

6. M. Brion: *Points entiers dans les polytopes convexes Séminaire Bourbaki.* Astérisque **227** (1995), 145–169.
7. M. Brion, M. Vergne: *Lattice Points in Simple Polytopes.* J. Amer. Math. Soc. **10** (1997), 371–392.
8. S. Cappell, J. Shaneson: *Genera of algebraic varieties and counting lattice points.* Bull. Amer. Math. Soc. **30** (1994), 62–69.
9. L. Cano, R. Díaz: *Indirect Influences on Directed Manifolds.* Adv. Stud. Contemp. Math. **28** (2018), 93–114.
10. E. Castillo, R. Díaz: *Rota-Baxter Categories.* Int. Electron. J. Algebra **5** (2009), 27–57.
11. J. Catumba, R. Diaz, A. Vargas: *Indirect Influences, Links Ranking, and Deconstruction of Networks.* Pre-print, arXiv:1803.07582.
12. V. Danilov: *The geometry of toric varieties.* Russian Math. Surveys **33** (1978), 7–154.
13. J. De Loera: *The many aspects of counting lattice points in polytopes.* Mathematische Semesterberichte **52** (2005), 175-195.
14. R. Díaz: *Indirect Influences.* Adv. Stud. Contemp. Math. **23** (2013), 29–41.
15. R. Díaz, L. Gómez:*Indirect Influences in International Trade.* Netw. Heterog. Media **10** (2015), 149–165.
16. R. Diaz, S. Robins: *The Ehrhart Polynomial of a Lattice Polytope.* Ann. Math. **145** (1997), 503–518.
17. R. Díaz, E. Pariguan:*Super, Quantum and Non-Commutative Species.* Afr. Diaspora J. Math. **8** (2009), 90–130.
18. R. Díaz, A. Vargas:*On the Stability of the PWP method.* In *Advances in Mathematics Research* **26**, Nova Science, Hauppauge, 2019.
19. T. Delzant:*Hamiltoniens périodiques et images convexes de l'application moment.* Bull. Soc. Math. France **116** (1988), 315–339.
20. J. Duistermaat, G. Heckman: *On the Variation in the Cohomology of the Symplectic Form of the Reduced Phase Space.* Invent. Math.**69** (1982), 259–268.
21. E. Ehrhart: *Démonstration de la loi de réciprocité pour un polyèdre entier.* C. R. Acad. Sci. Paris **265** (1967), 5–7.
22. R. Feynman: *Space-Time Approach to Non-Relativistic Quantum Mechanics.* Rev. Modern Phys. **20** (1948), 367.
23. V. Guillemin:*Moment Maps and Combinatorial Invariants of Hamiltonian T^n-spaces.* Birkhäusser Boston, Boston, 1994.
24. V. Guillemin, V. Ginzburg, Y. Karshon:*Moment Maps, Cobordisms, and Hamiltonian Group Actions.* Math. Surv. & Mono. **98**, Amer. Math. Soc., Providence, 2002.
25. M. Hamilton: *The quantization of a toric manifold is given by the integer lattice points in the moment polytope.* In M. Harada, Y. Karshon, M. Masuda, T. Panov (Eds.), *Toric topology*, Contemp. Math. **460**, pp 131–140, Amer. Math. Soc., Providence, 2008.
26. K. Humphreys:*A history and a survey of lattice path enumeration.* J. Stat. Plan. Inference **140** (2010), 2237–2254.

27. A. Joyal: *Une théorie combinatoire des séries formelles.* Adv. in Math. **42** (1981), 1–82.

28. Y. Karshon, S. Sternberg, J. Weitsman: *A Riemann-Roch theorem for integrals and sums of quasipolynomials over virtual polytopes.* St. Petersburg Math. J. **4** (1993), 789–812.

29. A. Khovanskii, A. Pukhlikov: *The Riemann-Roch theorem for integrals and sum of quasipolynomials on virtual polytopes.* Algebra i Analiz **4** (1992), 188–216, translation in *St. Petersburg Math. J.* **4** (1993), 789–812.

30. J. Kung (Ed.): *Gian-Carlo Rota on Combinatorics.* Birkhäuser, Boston and Basel, 1995.

31. C. Krattenthaler: *Lattice Path Enumeration.* In M. Bóna (Ed.), Handbook of Enumerative Combinatorics, pp 589–678, CRC Press, Chapman and Hall, 2015.

32. J. Lagarias, G. Ziegler: *Bounds for Lattice Polytopes containing a fixed number of Interior Points in a Sublattice.* Can. J. Math. Vol. **43** (1991), 1022–1035.

33. I. G. Macdonald: *Polynomials associated with finite cell-complexes.* J. London Math. Soc. **2** (1971), 181–192.

34. M. Markl, S. Shnider, J. Stasheff: *Operads in Algebra, Topology and Physics.* Amer. Math. Soc., Providence, 2002.

35. S. Mohanty: *Associahedra, Tamari Lattices and Related Structures.* Birkhäusser, Bassel, 2012.

36. T. Narayana: *Sur les treillis formes par les partitions d'un en tier et leurs applications a la theorie des probabilites.* C. R. Acad. Sci. **240** (1955), 1188–1189.

37. T. Narayana: *Lattice path combinatorics with statistical applications.* Univ. Toronto Press, Toronto, 1979.

38. J. Śniatycki: *Geometric Quantization and Quantum Mechanics.* Springer-Verlag, New York, 1980.

39. R. Stanley: *Catalan Numbers.* Cambridge Univ. Press, Cambridge, 2015.

40. C. Vignat, T. Wakhare: *A Continuous Analogue of Lattice Path Enumeration: Part II.* Preprint, arXiv:1606.01986.

41. N. Woodhouse: *Geometric Quantization.* Clarendon Press, Oxford, 1991.

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