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JACHYMSKI-MATKOWSKI-ŚWIĄTKOWSKI'S FIXED POINT THEOREM

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We improve Jachymski-Matkowski-Świątkowski's fixed point theorem for contractions in semimetric spaces with some additional assumption. We prove another fixed point theorem for contractions.

1. Introduction and Preliminaries

We begin with the definition of metric space, though it is well known.

Definition 1. Let X be a set and let p be a function from $X \times X$ into $[0, \infty)$. Then (X, p) is called a *metric space* if the following hold:

(D1) $p(x, x) = 0$

(D2) $p(x, y) = 0 \Rightarrow x = y$

(D3) $p(x, y) = p(y, x)$ (symmetry)

(D4) $p(x, z) \leq p(x, y) + p(y, z)$ (subadditivity or triangle inequality)

Remark. (X, p) is said to be a *semimetric space* if (D1)–(D3) hold.

Definition 2. Let (X, p) be a metric space. Then X is *complete* if the following holds:

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(D5) If a sequence $\{z_n\}$ in X is *Cauchy*, that is,

$$\lim_{n\to\infty} \sup_{m>n} p(z_n, z_m) = 0 \tag{1}$$

holds, then there exists $z \in X$ satisfying $\lim_n p(z_n, z) = 0$. (completeness)

The Banach contraction principle tells that every contraction T on X has a unique fixed point under (D1)–(D5); see [**1, 3**]. Jachymski, Matkowski and Świątkowski in [**8**] proved a very splendid generalization, assuming (D6) and (D7) below instead of (D4).

Theorem 3 (a corollary of Theorem 1 in [**8**]). *Let (X, p) satisfy* (D1)–(D3), (D5) *and the following*:

(D6) *If a sequence $\{z_n\}$ satisfies* $\lim_n p(z_n, x) = \lim_n p(z_n, y) = 0$, *then* $x = y$ *holds.* (Hausdorffness)

(D7) *There exist $\delta > 0$ and $\varepsilon > 0$ such that $p(x, y) < \delta$ and $p(y, z) < \delta$ imply $p(x, z) < \varepsilon$.*

Let T be a contraction on X, that is, there exists $r \in [0, 1)$ satisfying

$$p(Tx, Ty) \le r\, p(x, y) \tag{2}$$

for all $x, y \in X$. Then T has a unique fixed point z. Moreover, $\lim_n p(T^n x, z) = 0$ *holds for any $x \in X$.*

Remark. It is obvious that (D7) and the following are equivalent:

(D7)' There exist $\delta > 0$ and $\varepsilon > 0$ such that $p(x, y) + p(y, z) < \delta$ implies $p(x, z) < \varepsilon$.

The assumption on the space is a little bit strong in Theorem 3. Indeed, Theorem 3 is not a generalization of fixed point theorems proved in [**2, 11**] (Theorems 18 and 23 below). Motivated by this fact, in this paper, we improve Theorem 3 (see Theorem 6 below). We also prove another fixed point theorem (Theorem 7 below).

Throughout this paper we denote by $\mathbb{N}$ the set of all positive integers. Let X be a set. Then we denote by $\#X$ the cardinal number of X. We define a subset $X^{(k)}$ of X^k as follows: $(x_1, x_2, \cdots, x_k) \in X^{(k)}$ iff $(x_1, x_2, \cdots, x_k) \in X^k$ and $x_1, x_2, \cdots, x_k$ are all different.

2. Fixed Point Theorems

In this section, we improve Theorem 3. Also, we prove another fixed point theorem.

Lemma 4. *Let (X, p) satisfy the following*:

(D8) (1) *and* $\lim_n p(z_n, x) = \lim_n p(z_n, y) = 0$ *imply* $x = y$. (Hausdorffness for Cauchy and convergent sequences)

If $z, x \in X$ *satisfy* $p(z, z) = p(z, x) = 0$, *then* $z = x$ *holds.*

Proof. Define a sequence $\{z_n\}$ in X by $z_n = z$. Then

$$\lim_{n\to\infty} \sup_{m>n} p(z_n, z_m) = \lim_{n\to\infty} p(z_n, z) = \lim_{n\to\infty} p(z_n, x) = 0$$

holds. So, by (D8), we obtain the desired result. □

Remark. The proof employs the method in the proof of Lemma 1.1 in [**5**].

Theorem 5. *Let* (X, p) *satisfy* (D5) *and* (D8). *Let* S *be a contraction on* X. *Assume that there exists* $u \in X$ *satisfying*

$$\sup\{p(u, S^n u) : n \in \mathbb{N}\} < \infty. \tag{3}$$

Then S *has a unique fixed point* z. *Moreover* $p(z, z) = 0$ *and* $\lim_n p(S^n x, z) = 0$ *hold for any* $x \in X$.

Proof. We let $r \in [0, 1)$ satisfy (2) with $T := S$. We set $\varepsilon \in [0, \infty)$ to the left-hand side of (3). We have

$$\lim_{n\to\infty} \sup_{m>n} p(S^n u, S^m u) \le \lim_{n\to\infty} \sup_{m\in\mathbb{N}} r^n\, p(u, S^m u) = \lim_{n\to\infty} r^n\, \varepsilon = 0,$$

thus, $\{S^n u\}$ satisfies (1) with $z_n := S^n u$. By (D5), we have $\lim_n p(S^n u, z) = 0$ for some $z \in X$. We have

$$\lim_{n\to\infty} p(S^n u, Sz) \le \lim_{n\to\infty} r\, p(S^{n-1} u, z) = 0.$$

By (D8), we obtain $Sz = z$. We note

$$p(z, z) = \lim_{n\to\infty} p(S^n z, S^n z) \le \lim_{n\to\infty} r^n\, p(z, z) = 0.$$

Let w be a fixed point of S. Then we have

$$p(z, w) = p(Sz, Sw) \le r\, p(z, w).$$

Since $r < 1$, we have $p(z, w) = 0$. By Lemma 4, we obtain $z = w$. Thus the fixed point z of S is unique. For any $x \in X$, we have by $Sz = z$

$$\lim_{n\to\infty} p(S^n x, z) = \lim_{n\to\infty} p(S^n x, S^n z) \le \lim_{n\to\infty} r^n\, p(x, z) = 0$$

holds. □

Now we improve Theorem 3.

Theorem 6. *Let (X,p) satisfy* (D5), (D8) *and the following*:

(D9:ν) *There exist $\nu \in \mathbb{N}$, $\delta > 0$ and $\varepsilon > 0$ such that*

$$\max\{p(u_j,u_{j+1}) : j = 1,2,\cdots,\nu+1\} < \delta \quad \textit{implies} \quad p(u_1,u_{\nu+2}) < \varepsilon$$

for any $u_1,u_2,\cdots,u_{\nu+2} \in X^{(\nu+2)}$.

Let T be a contraction on X. Then T has a unique fixed point z. Moreover $p(z,z) = 0$ and $\lim_n p(T^n x, z) = 0$ hold for any $x \in X$.

Remark. It is obvious that (D8) and (D9:1) are weaker than (D6) and (D7), respectively. We do not need (D1)–(D3).

Proof. We let $r \in [0,1)$ satisfy (2). We choose $\delta > 0$ and $\varepsilon > 0$ appearing in (D9:ν). Choose $\kappa \in \mathbb{N}$ satisfying $\varepsilon\, r^{\kappa} < \delta$. Define a mapping S on X by $S = T^{\kappa}$. For any $x,y \in X$, we have

$$\lim_{n\to\infty} p(S^n x, S^n y) \le \lim_{n\to\infty} r^{\kappa\, n}\, p(x,y) = 0. \tag{4}$$

Fix $x \in X$. We will show that there exists $\mu \in \mathbb{N}\cup\{0\}$ satisfying (3) with $u := T^{\mu}x$, dividing the following two cases.

- There exist $k,\ell \in \mathbb{N}$ satisfying $k < \ell$ and $S^k x = S^{\ell} x$.
- $S^n x$ $(n = 1,2,\cdots)$ are all different.

In the first case, we note $\#\big\{T^n x : n \in \mathbb{N}\cup\{0\}\big\} < \infty$. So, putting $u = x$, we have (3). In the second case, by (4), there exists $\mu \in \mathbb{N}$ satisfying

$$\max\big\{p(S^{\mu}x, S^{\mu+j}x) : j \in \{1,2,\cdots,\nu\}\big\} < \min\{\delta,\varepsilon\}.$$

We have

$$\max\big\{p(S^{\mu+j}x, S^{\mu+j+1}x) : j \in \{0,1,\cdots,\nu-1\}\big\} = p(S^{\mu}x, S^{\mu+1}x) < \delta. \tag{5}$$

We will show the following by induction:

$$\max\big\{p(S^{\mu}x, S^{\mu+i}x) : i \in \{1,2,\cdots,j\}\big\} < \varepsilon \tag{6}$$

for $j \in \mathbb{N}$. It is obvious that (6) holds for $j \in \{1,2,\cdots,\nu\}$. Assume (6) for some $j \in \mathbb{N}$ with $j \ge \nu$. Then we have

$$p(S^{\mu+\nu}x, S^{\mu+j+1}x) \le r^{\kappa\,\nu}\, p(S^{\mu}x, S^{\mu-\nu+j+1}x) \le r^{\kappa\,\nu}\,\varepsilon < \delta. \tag{7}$$

By (5), (7) and (D9:ν), we obtain (6) for $j := j+1$. By induction, (6) holds for any $j \in \mathbb{N}$. Putting $u = S^{\mu}x$, we have (3). We have shown (3) in all cases. By Theorem 5, S has a unique fixed point z. Moreover $p(z,z) = 0$ holds. We have

$$STz = T^{\kappa+1}z = TSz = Tz.$$

Hence Tz is a fixed point of S. Since z is the unique fixed point of S, we obtain $Tz = z$. We note that every fixed point of T is a fixed point of S. So T has at most one fixed point. Therefore the fixed point z of T is unique. By $Tz = z$,

$$\lim_{n\to\infty} p(T^n x, z) = \lim_{n\to\infty} p(T^n x, T^n z) \le \lim_{n\to\infty} r^n\, p(x, z) = 0$$

holds. □

We prove another fixed point theorem.

Theorem 7. *Let (X, p) satisfy* (D5), (D8) *and the following*:

(D10) *There exist a function q from $X \times X$ into $[0, \infty)$ and $\alpha > 1$ such that for any $\beta > 0$, there exists $M > 0$ such that*

$$p(x, z) < \alpha\, p(y, z)$$

holds for any $x, y, z \in X$ satisfying $p(x, z) > M$ and $q(x, y) \le \beta$.

Let T be a contraction on X. Then T has a unique fixed point z. Moreover $p(z, z) = 0$ and $\lim_n p(T^n x, z) = 0$ hold for any $x \in X$.

Proof. We let $r \in [0, 1)$ satisfy (2). We choose q and α appearing in (D10). Choose $\kappa \in \mathbb{N}$ satisfying $\alpha\, r^\kappa < 1$. Define a mapping S on X by $S = T^\kappa$. Fix $u \in X$. Arguing by contradiction, we assume that (3) does not hold, that is,

$$\sup\{p(u, S^n u) : n \in \mathbb{N}\} = \infty \tag{8}$$

holds. Put $\beta = q(u, Su) + 1$. Then we can choose $M > 0$ appearing in (D10). By (8), we can choose $\ell \in \mathbb{N}$ satisfying

$$p(u, S^\ell u) > \max\{p(u, S^{\ell-1} u), M\}.$$

Then we have

$$p(u, S^\ell u) < \alpha\, p(Su, S^\ell u) \le \alpha\, r^\kappa\, p(u, S^{\ell-1} u) \le p(u, S^{\ell-1} u) < p(u, S^\ell u),$$

which implies a contradiction. Therefore we have shown (3). By Theorem 5, S has a unique fixed point z. Moreover $p(z, z) = 0$ holds. As in the proof of Theorem 6, we obtain the desired result. □

3. Examples

In this section, we will show many examples satisfying the assumption of Theorem 6 (or Theorem 7).

Lemma 8. *Let (X, p) satisfy* (D4). *Then* (D9:1) *and* (D10) *hold with $q := p$.*

Proof. (D9:1) obviously holds. Let $\alpha > 1$ and $\beta > 0$ be arbitrary. Choose $M > 0$ satisfying

$$(\alpha - 1)\, M - \alpha\, \beta > 0.$$

We fix $x, y, z \in X$ satisfying $p(x, z) > M$ and $p(x, y) \leq \beta$. Then we have by (D4)

$$p(x, z) < p(x, z) + (\alpha - 1)\, M - \alpha\, \beta < \alpha\, \big(p(x, z) - p(x, y)\big) \leq \alpha\, p(y, z).$$

Therefore we obtain (D10). □

Proposition 9. *Let* (X, p) *be a complete metric space. Then* (D5)*,* (D8)*,* (D9:1) *and* (D10) *hold.*

Remark. Both Theorems 6 and 7 are generalizations of the Banach contraction principle [**1, 3**].

Proof. (D5), (D8) and (D9:1) obviously hold. By Lemma 8, we obtain (D10). □

Proposition 10. *Let* (X, d) *be a complete metric space and let* η *be a function from* $[0, \infty)$ *into itself satisfying the following*:

- *For any sequence* $\{a_n\}$ *in* $[0, \infty)$*,* $\lim_n \eta(a_n) = 0$ *iff* $\lim_n a_n = 0$*.*

Define a function p *from* $X \times X$ *into* $[0, \infty)$ *by* $p(x, y) = \eta\big(d(x, y)\big)$*. Then* (X, p) *satisfies* (D5)*,* (D8) *and* (D9:1)*.*

Proof. Jachymski proved (D5), (D6) and (D7) in Section 3 in [**7**]. So, (D8) and (D9:1) hold. □

In [**4**], Czerwik introduced the concept of b-metric space.

Definition 11 (Czerwik [**4**])**.** Let X be a set and let p be a function from $X \times X$ into $[0, \infty)$. Then (X, p) is said to be a *b-metric space* if (D1)–(D3) and the following hold:

(b3) There exists $K \geq 1$ satisfying $p(x, z) \leq K\, \big(p(x, y) + p(y, z)\big)$ for any $x, y, z \in X$. (K-relaxed triangle inequality)

The concepts of Cauchyness and completeness in b-metric spaces are defined by (1) and (D5), respectively.

Proposition 12. *Let* (X, p) *be a complete b-metric space. Then* (D5)*,* (D8) *and* (D9:1) *hold.*

Proof. (D5) and (D9:1) obviously hold. Let us prove (D8). Assume $\lim_n p(z_n, x) = \lim_n p(z_n, y) = 0$. Then we have

$$p(x, y) \leq K \lim_{n \to \infty} \big(p(z_n, x) + p(z_n, y)\big) = 0.$$

Thus, (D8) holds. □

Remark. From the definition and the proof above, we obtain (D1)–(D3), (D6) and (D7).

In [**6**], Hitzler and Seda introduced the concept of dislocated metric space.

Definition 13 (Hitzler and Seda [**6**])**.** Let X be a set and let p be a function from $X \times X$ into $[0, \infty)$. Then (X, p) is said to be a *dislocated metric space* if (D2)–(D4).

The concepts of Cauchyness and completeness in dislocated metric spaces are defined by (1) and (D5), respectively.

Proposition 14. *Let* (X, p) *be a complete dislocated metric space. Then* (D5), (D8), (D9:1) *and* (D10) *hold.*

Proof. (D5) and (D9:1) obviously hold. By Lemma 8, we obtain (D10). We can prove (D8) as in the proof of Proposition 12. □

Remark. From the definition, we can easily prove (D6) and (D7).

In [**2**], Branciari introduced the concept of ν-generalized metric space. See also [**12, 14**] and references therein.

Definition 15 (Branciari [**2**])**.** Let X be a set, let p be a function from $X \times X$ into $[0, \infty)$ and let $\nu \in \mathbb{N}$. Then (X, p) is said to be a *ν-generalized metric space* if (D1)–(D3) and the following hold:

(N3:ν) $p(u_1, u_{\nu+2}) \leq p(u_1, u_2) + p(u_2, u_3) + \cdots + p(u_{\nu+1}, u_{\nu+2})$ for any $(u_1, u_2, \cdots, u_{\nu+2}) \in X^{(\nu+2)}$.

The concepts of Cauchyness and completeness in ν-generalized metric spaces are defined by (1) and (D5), respectively.

Lemma 16 (Proposition 2.7 in [**16**], Proposition 13 in [**14**])**.** *Let* (X, p) *be a ν-generalized metric space. Let* $\{x_n\}$ *and* $\{y_n\}$ *be Cauchy sequences in* X *satisfying* $\lim_n p(x_n, x) = \lim_n p(y_n, y) = 0$ *for some* $x, y \in X$. *Then*

$$p(x, y) = \lim_{n\to\infty} p(x_n, y_n)$$

holds.

Proposition 17. *Let* (X, p) *be a complete ν-generalized metric space. Then* (D5), (D8), (D9:ν) *and* (D10) *hold.*

Proof. In the case of $\nu = 1$, X is a metric space. So by Proposition 9, we obtain the desired result. So we assume $\nu \geq 2$. (D5) and (D9:ν) obviously hold. In order to show (D8), we assume (1) and $\lim_n p(z_n, x) = \lim_n p(z_n, y) = 0$. We have by Lemma 16 and (D1)

$$p(x, y) = \lim_{n\to\infty} p(z_n, z_n) = 0.$$

So by (D2), $x = y$ holds. We have shown (D8). We will show (D10), dividing the following two cases:

(i) $\#X < \infty$

(ii) $\#X = \infty$

In the first case, putting $M = \max\{p(x,y) : x,y \in X\} + 1$, we can prove (D10). In the second case, we define a function q from $X \times X$ into $[0,\infty)$ by $q(x,x) = 0$ and

$$q(x,y) = \inf\Big\{ \sum_{j<k} p(u_j,u_k) : (u_1,\cdots,u_{\nu+1}) \in X^{(\nu+1)}, u_1 = x, u_{\nu+1} = y \Big\} + 1$$

for $x,y \in X$ with $x \neq y$. Let $\alpha > 1$ and $\beta > 0$ be arbitrary. Choose $M > 0$ satisfying

$$(\alpha - 1)\, M - \alpha\, \beta > 0.$$

We note $M > \beta$. We let $x,y,z \in X$ satisfy

$$q(x,y) \le \beta < M < p(x,z). \tag{9}$$

Then it is obvious that $x \neq z$ holds. We consider the following two cases:

(ii-1) $x = y$

(ii-2) $x \neq y$

In the case of (ii-1), we have

$$p(x,z) = p(y,z) < \alpha\, p(y,z).$$

In the case of (ii-2), from the definition of q, there exist $u_2,\cdots,u_\nu \in X$ satisfying

$$\sum_{j<k} p(u_j,u_k) < q(x,y) \quad \text{and} \quad (u_1,\cdots,u_{\nu+1}) \in X^{(\nu+1)},$$

where we put $u_1 = x$ and $u_{\nu+1} = y$. Then we have by (9),

$$\sum_{j=2}^{\nu+1} p(x,u_j) < q(x,y) < p(x,z)$$

and hence $z \notin \{u_2,\cdots,u_{\nu+1}\}$ holds. So we have by (N3:ν)

$$p(x,z) \le \sum_{j=1}^{\nu} p(u_j,u_{j+1}) + p(y,z) < q(x,y) + p(y,z)$$

and hence

$$p(x,z) < p(x,z) + (\alpha-1)\, M - \alpha\,\beta < \alpha\,\big(p(x,z) - q(x,y)\big) \le \alpha\, p(y,z).$$

Therefore we obtain (D10). □

By Proposition 17, we can consider that both Theorems 6 and 7 are generalizations of the following. Also, see Remark 16 in [**12**] and the proof of Theorem 9 in [**15**].

Theorem 18 (Branciari [**2**]). *Let (X, p) be a complete ν-generalized metric space and let T be a contraction on X. Then T has a unique fixed point z.*

In 2001, the concept of τ-distance was introduced. There are many examples of τ-distances. See also [**9, 10, 13**] and references therein.

Definition 19 ([**11**]). Let (X, d) be a metric space. Then a function p from $X \times X$ into $[0, \infty)$ is called a *τ-distance* on X with η if there exists a function η from $X \times [0, \infty)$ into $[0, \infty)$ and the following are satisfied:

(τ1) $p(x, z) \leq p(x, y) + p(y, z)$ for all $x, y, z \in X$.

(τ2) $\eta(x, 0) = 0$ and $\eta(x, t) \geq t$ for all $x \in X$ and $t \in [0, \infty)$, and η is concave and continuous in its second variable.

(τ3) $\lim_n x_n = x$ and $\lim_n \sup\{\eta(z_n, p(z_n, x_m)) : m \geq n\} = 0$ imply $p(w, x) \leq \liminf_n p(w, x_n)$ for all $w \in X$.

(τ4) $\lim_n \sup\{p(x_n, y_m) : m \geq n\} = 0$ and $\lim_n \eta(x_n, t_n) = 0$ imply $\lim_n \eta(y_n, t_n) = 0$.

(τ5) $\lim_n \eta(z_n, p(z_n, x_n)) = 0$ and $\lim_n \eta(z_n, p(z_n, y_n)) = 0$ imply $\lim_n d(x_n, y_n) = 0$.

Lemma 20 (Lemma 1 in [**11**]). *Let (X, d) be a metric space and let p be a τ-distance on X with η. Assume that $\{x_n\}$ is a p-Cauchy sequence, that is, there exists a sequence $\{z_n\}$ in X such that* $\lim_n \sup\{\eta(z_n, p(z_n, x_m)) : m \geq n\} = 0$. *Then $\{x_n\}$ is a Cauchy sequence in (X, d).*

Lemma 21 (Lemma 3 in [**11**]). *Let (X, d) be a metric space with a τ-distance p. If a sequence $\{z_n\}$ in X satisfies* (1), *then $\{z_n\}$ is p-Cauchy. Moreover if a sequence $\{x_n\}$ in X satisfies* $\lim_n p(z_n, x_n) = 0$, *then $\{x_n\}$ is also p-Cauchy and* $\lim_n d(z_n, x_n) = 0$.

Proposition 22. *Let (X, d) be a complete metric space with a τ-distance p. Then* (D5), (D8), (D9:1) *and* (D10) *hold.*

Proof. By Lemma 8, (D9:1) and (D10) hold. Let $\{z_n\}$ be a sequence in X satisfying (1). Then by Lemma 21, $\{z_n\}$ is p-Cauchy. So by Lemma 20, $\{z_n\}$ is Cauchy in (X, d). Since X is complete, $\{z_n\}$ converges to some z in (X, d). Then from (τ3), we have

$$\limsup_{n\to\infty} p(z_n, z) \leq \limsup_{n\to\infty} \liminf_{m\to\infty} p(z_n, z_m) \leq \lim_{n\to\infty} \sup_{m>n} p(z_n, z_m) = 0.$$

Thus, (D5) holds. Assume additionally $\lim_n p(z_n, x) = \lim_n p(z_n, y) = 0$. Then by Lemma 21, we have $\lim_n d(z_n, x) = \lim_n d(z_n, y) = 0$ and hence $x = y$. Thus, (D8) holds. □

By Proposition 22, we can consider that both Theorems 6 and 7 are generalizations of the following.

Theorem 23 (Theorem 2 in [**11**]). *Let (X, d) be a complete metric space and let T be a contraction on X with respect to a τ-distance p, that is, there exist a τ-distance p on X and $r \in [0, 1)$ satisfying (2) for all $x, y \in X$. Then T has a unique fixed point.*

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