

## HIGHER INTEGRABILITY THEOREMS ON TIME SCALES FROM REVERSE HÖLDER'S INEQUALITIES

*Dedicated to Academician Professor Gradimir Milovanović  
on the occasion of his 70th birthday.*

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In this paper, we establish some new reverse dynamic inequalities and use them to prove some higher integrability theorems for decreasing functions on time scales. In order to derive our main results, we first prove a new dynamic inequality for convex functions related to the inequality of Hardy, Littlewood and Pólya, known from the literature. Then, we prove a refinement of the famous Hardy inequality on time scales for a class of decreasing functions. As an application, our results are utilized to formulate the corresponding reverse integral and discrete inequalities, which are essentially new.

### 1. INTRODUCTION

The origin of reverse Hölder's inequality can be traced back to the work of Muckenhoupt in 1972, who proved a higher integrability result for decreasing functions from reverse mean value integral inequalities. We say that  $f$  satisfies a reverse Hölder's inequality if for some constants  $p > q$  holds the inequality

$$\left[ \int_Q |f(t)|^p dt \right]^{\frac{1}{p}} \leq K \left[ \int_Q |f(t)|^q dt \right]^{\frac{1}{q}}, \quad K > 0,$$

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2010 Mathematics Subject Classification. 26D15, 34N05, 39A12.

Keywords and Phrases. Reverse Hölder's inequality, Hardy's inequality, higher integrability, Muckenhoupt-type inequality, time scales.

where  $f$  is a measurable function defined on  $Q$  for any  $Q \subset \mathbb{R}^n$  with sides parallel to the coordinate axes. In [15], Muckenhoupt proved that if  $f$  is a non-negative and decreasing function on  $Q$  and there exists a constant  $A > 1$  such that

$$(1) \quad \frac{1}{|Q|} \int_Q f(x) dx \leq Af(x), \text{ for all } x \in Q,$$

then the function  $f$  belongs to  $L^p(Q)$  for every  $p \in [1, A/(A-1)]$ , and

$$(2) \quad \frac{1}{|Q|} \int_Q f^p(x) dx \leq \frac{A}{A-p(A-1)} \left( \frac{1}{|Q|} \int_Q f(x) dx \right)^p.$$

Bojarski et al. [6], improved the result of Muckenhoupt by replacing the constant in inequality (2) with a smaller one. Gehring [9, 10], extended the work of Muckenhoupt and proved that if  $1 < q < \infty$  and the function  $f \in L^q(Q)$  satisfies the condition

$$(3) \quad \frac{1}{|Q|} \int_Q f^q(x) dx \leq C_q \left( \frac{1}{|Q|} \int_Q f(x) dx \right)^q,$$

for all cubes  $Q$  with sides parallel to coordinate axes, then there exists  $\epsilon = \epsilon(n, q, C_q) > 0$  such that  $f \in L^p(Q)$  for  $p < q + \epsilon$ , while for each  $p$  there exists a new constant  $A_p = A_p(n, q, C_q, p)$  such that

$$(4) \quad \frac{1}{|Q|} \int_Q f^p(x) dx \leq A_p \left( \frac{1}{|Q|} \int_Q f(x) dx \right)^p.$$

This result has numerous applications in theory of weighted spaces, quasiconformal mappings and partial differential equations. It should be noticed here that the proof of Gehring's inequality has been established by employing the Calderón-Zygmund decomposition theorem and the scale structure of the  $L^p$ -space.

Franciosi and Moscarillo [8], improved and extended the results of Gehring by using decreasing rearrangement  $f^*$  of the function  $f$  (for more details, see [12]), and proved some higher integrability results from reverse integral inequalities. Their results are based on the application of the famous Hardy inequality

$$(5) \quad \int_0^a \left( \frac{1}{x} \int_0^x f(t) dt \right)^q dx \leq \left( \frac{q}{q-1} \right)^q \int_0^a f^q(x) dx,$$

where  $a \in (0, \infty)$  and  $f$  is a non-negative function in  $L^q(0, a)$ . In particular, they proved that if  $q > 1$  and  $f \in L^1(0, a)$ ,  $a \in (0, \infty)$ , is a decreasing function such that

$$\frac{1}{t} \int_0^t f^q(x) dx \leq C_q \left( \frac{1}{t} \int_0^t f(x) dx \right)^q, \text{ for } t \in (0, a),$$

then  $f \in L^p(0, a)$  and

$$\frac{1}{t} \int_0^t f^p(x) dx \leq A_p \left( \frac{1}{t} \int_0^t f(x) dx \right)^p, \text{ for } t \in (0, a).$$

Here,  $p \in [q, p_0)$ , where  $p_0$  is the unique positive root of the equation

$$\left(-\sqrt[q]{C_q}\right)^x - \frac{x-q}{x} \left(\frac{x}{x-1}\right) = 0,$$

and the constant  $A_p$  is defined by

$$A_p = \frac{q/p}{\left(\frac{1}{C_q}\right)^{p/q} - \frac{p-q}{p} \left(\frac{p}{p-1}\right)^p}.$$

Further, D'Apuzzo and Sbordone [7], extended the results of Franciosi and Moscarillo [8], and proved that if  $q > 1$  and a decreasing function  $f \in L^q(a, b)$  satisfies

$$(6) \quad \frac{1}{b-a} \int_a^b f^q(x) dx \leq K \left( \frac{1}{b-a} \int_a^b f(x) dx \right)^q,$$

then  $f \in L^p(a, b)$  and

$$(7) \quad \left( \frac{1}{b-a} \int_a^b f^p(x) dx \right)^{\frac{q}{p}} \leq \frac{q}{p\gamma(p)(b-a)} \int_a^b f^q(x) dx,$$

for any  $q \leq p < q_0$ , where  $q_0$  is the unique positive root of the equation

$$(8) \quad \gamma(x) = 1 - K^q \frac{x-q}{x} \left(\frac{x}{x-1}\right)^q = 0, \quad K > 1.$$

It should be noticed here that it is impossible to increase the value  $q_0$ .

After the paper of Bojarski et al. [6], it was natural to look for an improvement of the constants in Gehring's inequality. This problem has been solved by Korenovskii [13], who proved that the results can be extended to cover the gap in [8] and to improve the corresponding results from [7] by removing the monotonicity condition of the function  $f$  via the properties of the decreasing rearrangement  $f^*$ . In particular, he proved that if  $f \in L^q[a, b]$  satisfies (3) with  $Q_0 = [a, b]$ , then  $f \in L^p[a, b]$  and  $f$  satisfies the inequality (6), where  $q < p < p_0$ , with  $p_0$  defined by (8).

Popoli [16], unified the above results of D'Apuzzo, Sbordone and Korenovskii, and proved that if  $p < q$ ,  $pq > 0$  and  $f \in L^q[a, b]$  is decreasing function satisfying

$$(9) \quad \left( \frac{1}{b-a} \int_a^b f^q(x) dx \right)^{\frac{1}{q}} \leq K \left( \frac{1}{b-a} \int_a^b f^p(x) dx \right)^{\frac{1}{p}},$$

then  $f \in L^s[a, b]$  and

$$(10) \quad \left( \frac{1}{b-a} \int_a^b f^s(x) dx \right)^{\frac{q}{s}} \leq \frac{q}{s\gamma_K(p, q, q/s)} \left( \frac{1}{b-a} \int_a^b f^q(x) dx \right),$$

for  $q \leq s < q_0$ , where  $q_0$  is the unique solution of the equation

$$(11) \quad \left( \frac{x}{x-q} \right)^{\frac{1}{q}} = K \left( \frac{x}{x-p} \right)^{\frac{1}{p}},$$

and  $\gamma_C$  is defined by

$$(12) \quad \gamma_C(a, b, x) = 1 - C^b (1-x) \left( \frac{b}{b-ax} \right)^{\frac{b}{a}}, \quad \text{for } C > 1.$$

In addition, if  $pq < 0$  and  $f \in L^q[a, b]$  is an increasing function satisfying (9), then  $f \in L^s[a, b]$  and

$$(13) \quad \left( \frac{1}{b-a} \int_a^b f^s(x) dx \right)^{\frac{p}{s}} \leq \frac{p}{s\gamma_{1/K}(q, p, p/s)} \left( \frac{1}{b-a} \int_a^b f^p(x) dx \right),$$

for  $p_0 < s \leq p$ , where  $p_0$  is the unique solution of the equation (11). The proofs of the main results in [7], [13] and [16] rely on an elementary inequality for convex functions proved by Hardy et al. [11], which will be discussed in the main section.

In the last decades several authors have been interested in finding some discrete results on  $l^p$ -analogues for  $L^p$ -bounds in harmonic analysis and as a result this subject became a topic of ongoing research. One of the reasons for this upsurge of interest in a discrete case is also due to the fact that discrete operators may even behave differently from their continuous counterparts as is exhibited by a discrete spherical maximal operator (see e.g. [14]). But due to the lack of calculus in a discrete space, since there are no power rules or even chain rules, which are the main tools for proving the corresponding results in a continuous setting, there is a big challenge in proving such results in  $l^p$ , in comparison to  $L^p$ . The main objective of the present paper is to overcome the above discussed problems on discrete spaces. More precisely, we introduce a new approach on time scales to develop a study of boundedness and integrability of mappings, with weights of the form  $(x-a)^{\alpha-1}$  where  $\alpha \leq 1$ , which will cover both continuous and discrete results as special cases.

The study of dynamic inequalities on time scales has been received a lot of attention in recent years and became a major field in pure and applied mathematics. The general idea is to prove the corresponding result for a dynamic inequality where the domain of a function is a so-called time scale  $\mathbb{T}$ , which may be an arbitrary closed subset of real numbers  $\mathbb{R}$ . These dynamic inequalities cover the classical continuous and discrete inequalities as special cases when  $\mathbb{T} = \mathbb{R}$  and  $\mathbb{T} = \mathbb{N}$ , and furthermore, they can be extended to different types of inequalities on various time scales such as  $\mathbb{T} = h\mathbb{N}$ ,  $h > 0$ ,  $\mathbb{T} = q^{\mathbb{N}}$  for  $q > 1$ , etc. For this topic, the reader is referred to monographs [1, 2], papers [3, 17, 18] and references therein. In particular, our aim in this paper is to extend the results due to D'Apuzzo and Sbordone [7], and Popoli [16], to time scales and derive the corresponding discrete results which will be essentially new.

The paper is organized as follows. After this Introduction, in Section 2 we recall some basic notation, definitions and facts on time scales. In Section 3, we

first prove a new dynamic inequality via convexity, which contains as a special case the elementary inequality proved by Hardy et al. [11]. Then, we prove a new refinement of Hardy-type inequality on time scales which holds for a class of decreasing functions. Finally, we prove a higher integrability theorem for decreasing functions on time scales which covers the corresponding results established in [7] and [16]. The main contributions of the present article are the reverse discrete inequalities and the higher summability theorem on discrete space  $l^p(\mathbb{N})$ . In fact, we obtain an upper bound for the norm on space  $l^p(\mathbb{N})$  expressed via the norm on  $l^q(\mathbb{N})$ , when  $p > q$ .

## 2. PRELIMINARIES ON TIME SCALES

In this section, we present some basic notation, definitions and properties concerning the calculus on time scales, for more details the reader is referred to [4].

A time scale  $\mathbb{T}$  is an arbitrary nonempty closed subset of real numbers  $\mathbb{R}$ . For  $t \in \mathbb{T}$ , the forward jump operator  $\sigma : \mathbb{T} \rightarrow \mathbb{T}$  is defined by  $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ , while the backward jump operator  $\rho : \mathbb{T} \rightarrow \mathbb{T}$  is defined by  $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$ . The point  $t$  is said to be right-scattered if  $\sigma(t) > t$ , respectively left-scattered if  $\rho(t) < t$ . The point  $t$  is called right-dense if  $t < \sup \mathbb{T}$  and  $\sigma(t) = t$ , respectively left-dense if  $t > \inf \mathbb{T}$  and  $\rho(t) = t$ . The graininess function  $\mu : \mathbb{T} \rightarrow [0, \infty)$  is defined by  $\mu(t) = \sigma(t) - t$ . For an arbitrary function  $f : \mathbb{T} \rightarrow \mathbb{R}$ ,  $f^\sigma(t)$  stands for a composition  $f(\sigma(t))$ . A function  $f : \mathbb{T} \rightarrow \mathbb{R}$  is called rd-continuous if it is continuous at each right-dense point or maximal point in  $\mathbb{T}$  and if its left-sided limits exist at each left-dense point in  $\mathbb{T}$ . Without loss of generality, we assume that  $\sup \mathbb{T} = \infty$ , and we define the time scale interval  $[a, b]_{\mathbb{T}}$  by  $[a, b]_{\mathbb{T}} := [a, b] \cap \mathbb{T}$ .

Now, we recall the product and quotient rules for delta derivative  $f^\Delta$  of the function  $f$  (for more details, see [4]). Namely, if  $f$  and  $g$  are delta differentiable functions on  $\mathbb{T}$ , then the product  $fg$  is delta differentiable on  $\mathbb{T}$ . Moreover, if  $gg^\sigma \neq 0$ , then the quotient  $f/g$  is also delta differentiable on  $\mathbb{T}$ , so we have

$$(14) \quad (fg)^\Delta = f^\Delta g + f^\sigma g^\Delta = fg^\Delta + f^\Delta g^\sigma \quad \text{and} \quad \left(\frac{f}{g}\right)^\Delta = \frac{f^\Delta g - fg^\Delta}{gg^\sigma}.$$

The first chain rule that will be utilized in this paper asserts that

$$(15) \quad (f^\gamma(t))^\Delta = \gamma \int_0^1 [hf^\sigma + (1-h)f]^\gamma f^{\gamma-1} dh f^\Delta(t), \quad \gamma \in \mathbb{R},$$

which is a simple consequence of Keller's chain rule [4, Theorem 1.90]. The second chain rule asserts that if the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuously differentiable and  $g : \mathbb{T} \rightarrow \mathbb{R}$  is delta differentiable, then the composition  $f \circ g : \mathbb{T} \rightarrow \mathbb{R}$  is delta differentiable and

$$(16) \quad f^\Delta(g(t)) = f'(g(d))g^\Delta(t), \quad \text{for } d \in [t, \sigma(t)].$$

If  $F^\Delta(t) = f(t)$ , then the Cauchy delta integral of  $f$  is defined by  $\int_{t_0}^t f(s)\Delta s := F(t) - F(t_0)$ . It is well known that if  $f$  is rd-continuous on  $\mathbb{T}$ , then the Cauchy integral  $F(t) := \int_{t_0}^t f(s)\Delta s$ ,  $t_0 \in \mathbb{T}$ , exists and satisfies  $F^\Delta(t) = f(t)$  for  $t \in \mathbb{T}$ . Integration on a discrete time scale is defined by  $\int_a^b f(t)\Delta t = \sum_{t \in [a,b)} \mu(t)f(t)$ . Further, the integration by parts formula on time scales reads

$$(17) \quad \int_a^b u(t)v^\Delta(t)\Delta t = [u(t)v(t)]_a^b - \int_a^b u^\Delta(t)v^\sigma(t)\Delta t.$$

Now, we introduce some basic concepts from general measure theory and integration applied to the measurable space obtained in [5, chapter 5], to define the Lebesgue  $\Delta$ -measure on  $\mathbb{T}$ . We say that  $f : \mathbb{T} \rightarrow \mathbb{R}$  is  $\Delta$ -measurable if for every  $\alpha \in \mathbb{R}$ , the set

$$f^{-1}([-\infty, \alpha)) = \{t \in \mathbb{T} : f(t) < \alpha\}$$

is  $\Delta$ -measurable. The Lebesgue integral associated with the measure  $\Delta$  on  $I$  is called the Lebesgue  $\Delta$ -integral. For a  $\Delta$ -measurable set  $I \subset \mathbb{T}$  and  $\Delta$ -measurable function  $f : I \rightarrow \mathbb{R}$ , the corresponding  $\Delta$ -integral of  $f$  over  $I$  is denoted by  $\int_I f(t)\Delta t$ .

Now, let  $I \subset \mathbb{T}$  be  $\Delta$ -measurable such that  $|I|$  is the corresponding Lebesgue  $\Delta$ -measure of  $I$ . We say that  $f : I \rightarrow \mathbb{R}$  belongs to  $L_\Delta^p(I)$ ,  $p > 1$ , provided that either  $\|f\|_p = \left(\int_I |f(t)|^p \Delta t\right)^{\frac{1}{p}} < \infty$ , for  $1 < p < \infty$ , or there exists a constant  $C \in \mathbb{R}^+$  such that  $\|f\|_\infty = \sup_I |f(t)| \leq C$ , for  $p = +\infty$ .

Finally, we will utilize the well-known Hölder's inequality in a time scale setting (see, [4, Theorem 6.13]), which asserts that if  $1/p + 1/q = 1$ ,  $p > 1$ , and  $a, b \in \mathbb{T}$ , then the inequality

$$(18) \quad \int_a^b |f(t)g(t)| \Delta t \leq \left[ \int_a^b |f(t)|^p \Delta t \right]^{\frac{1}{p}} \left[ \int_a^b |g(t)|^q \Delta t \right]^{\frac{1}{q}}$$

holds for any pair of rd-continuous functions  $f, g : \mathbb{T} \rightarrow \mathbb{R}$ .

Throughout this paper we assume that the functions in the statements of the theorems are non-negative and rd-continuous, while the integrals considered are assumed to exist and be finite. Therefore, these conditions will be omitted, for brevity.

### 3. MAIN RESULTS

In this section, we prove our main results. First, we prove a time scale version of an integral inequality due to Hardy, Littlewood and Pólya [11] and the new refinement of Hardy-type inequality on time scales that will play important roles in the proof of the main results of higher integrability.

**Theorem 1.** *Let  $\mathbb{T}$  be a time scale with  $a, b \in \mathbb{T}$  and assume that  $\phi : [0, \infty) \rightarrow \mathbb{R}$  is a differentiable convex function. If  $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$  is a non-negative decreasing*

function, then holds the inequality

$$(19) \quad \phi(0) + \int_a^b \phi'((x-a)f(x))f(x)\Delta x \leq \phi\left(\int_a^b f(x)\Delta x\right).$$

*Proof.* Let  $x \in [a, b]_{\mathbb{T}}$ . Since  $f$  is decreasing function, it follows that  $(x-a)f(x) \leq \int_a^x f(t)\Delta t$ . On the other hand, by defining  $F(x) = \int_a^x f(t)\Delta t$ , we see that  $F^\Delta(x) = f(x) \geq 0$ , which implies that  $F$  is increasing function. Now, due to convexity of the function  $\phi$ , we have

$$(20) \quad \phi'((x-a)f(x))f(x) \leq \phi'\left(\int_a^x f(t)\Delta t\right)f(x) = \phi'(F(x))f(x).$$

Further, taking into account the chain rule (16), it follows that

$$\phi^\Delta(F(x)) = \phi'(F(\zeta))F^\Delta(x), \text{ where } \zeta \in [x, \sigma(x)],$$

and consequently,

$$(21) \quad \phi^\Delta(F(x)) \geq \phi'(F(x))F^\Delta(x) = \phi'(F(x))f(x),$$

since  $F$  is increasing function. Now, considering relations (20) and (21), we obtain the inequality

$$\phi'((x-a)f(x))f(x) \leq \phi^\Delta(F(x)).$$

Finally, integrating the last inequality from  $a$  to  $b$  yields relation

$$\int_a^b \phi'((x-a)f(x))f(x)\Delta x \leq \int_a^b (\phi(F(x)))^\Delta \Delta x = \phi(F(b)) - \phi(0),$$

which proves our assertion. The proof is complete.  $\square$

In the sequel, we consider a special case of Theorem 1, when the function  $\phi : [0, \infty) \rightarrow \mathbb{R}$  is defined by  $\phi(u) = u^p$ , for  $p \geq 1$ . Clearly, this function is differentiable and convex, so we have the following consequence.

**Corollary 1.** *Let  $\mathbb{T}$  be a time scale with  $a, b \in \mathbb{T}$ , and let  $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$  be a non-negative decreasing function. If  $p \geq 1$ , then holds the inequality*

$$(22) \quad \int_a^b (x-a)^{p-1} f^p(x)\Delta x \leq \frac{1}{p} \left(\int_a^b f(x)\Delta x\right)^p.$$

**Remark 1.** *Applying Hölder's inequality (18) with exponents  $1/p$  and  $(p-1)/p$  to the right-hand side of (22), we obtain the following inequality*

$$\int_a^b (x-a)^{p-1} f^p(x)\Delta x \leq \frac{(b-a)^{p-1}}{p} \int_a^b f^p(x)\Delta x.$$

Our next intention is to rewrite inequality (22) in a form which will be more suitable in our further discussion. First of all, it should be noticed here that if a non-negative function  $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$  is decreasing, then the function  $(x - a)^{\gamma-1}f$ , where  $\gamma \leq 1$ , is also decreasing on  $[a, b]_{\mathbb{T}}$ . Therefore, considering relation (22) with  $(x - a)^{\gamma-1}f$  instead of  $f$ , we obtain the following result.

**Corollary 2.** *Let  $\mathbb{T}$  be a time scale with  $a, b \in \mathbb{T}$ , and let  $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$  be a non-negative decreasing function. If  $p \geq 1$  and  $\gamma \leq 1$ , then holds the inequality*

$$(23) \quad \int_a^b (x - a)^{p\gamma-1} f^p(x) \Delta x \leq \frac{1}{p} \left( \int_a^b (x - a)^{\gamma-1} f(x) \Delta x \right)^p.$$

**Remark 2.** *Let  $r$  and  $s$  be positive real numbers such that  $r \leq s$ . Considering relation (23) with the function  $f^r$  instead of  $f$ , and with parameters  $p = s/r \geq 1$ ,  $\gamma = 1/p \leq 1$ , we obtain the inequality*

$$(24) \quad \left( \int_a^b f^s(x) \Delta x \right)^{\frac{r}{s}} \leq \frac{r}{s} \int_a^b (x - a)^{\frac{r}{s}-1} f^r(x) \Delta x.$$

The above inequality will be an important relation that will be used later in the proof of our main higher integrability theorem. On the other hand, it is important in its own right, since it is the time scale version of an inequality due to Hardy, Littlewood and Pólya (for more details, see [11]).

Our next intention is to give a time scale extension and refinement of the famous Hardy inequality (5). In order to summarize our further discussion, we first define an operator  $\mathcal{G}$  by

$$(25) \quad \mathcal{G}(x) := \frac{1}{x - a} \int_a^x g(t) \Delta t, \quad \text{for all } x \in [a, T]_{\mathbb{T}},$$

where  $g : [a, T]_{\mathbb{T}} \rightarrow \mathbb{R}$  is a non-negative function. Next, we give several simple facts about operator  $\mathcal{G}$  following from its definition.

**Lemma 1.** *If  $g : [a, T]_{\mathbb{T}} \rightarrow \mathbb{R}$  is a non-negative decreasing function, then  $\mathcal{G}(x) \geq g(x)$  for  $x \in [a, T]_{\mathbb{T}}$ .*

*Proof.* Since  $g$  is decreasing, it follows that

$$\mathcal{G}(x) = \frac{1}{x - a} \int_a^x g(t) \Delta t \geq \frac{1}{x - a} \int_a^x g(x) \Delta t = g(x), \quad x \in [a, T]_{\mathbb{T}},$$

which completes the proof.  $\square$

It should be also noticed here that  $\mathcal{G}$  inherits the decreasing nature of the function  $g$ .

**Lemma 2.** *If  $g : [a, T]_{\mathbb{T}} \rightarrow \mathbb{R}$  is a non-negative decreasing function, then so is  $\mathcal{G}$ .*



*Proof.* By utilizing the quotient rule in (14), it follows that

$$\mathcal{G}^\Delta(x) = \frac{g(x)(x-a) - \int_a^x g(t)\Delta t}{(x-a)(\sigma(x)-a)}, \quad x \in [a, T]_{\mathbb{T}}.$$

Hence, by virtue of Lemma 1, we have  $(\sigma(x)-a)\mathcal{G}^\Delta(x) = g(x) - \mathcal{G}(x) \leq 0$ , for  $x \in [a, T]_{\mathbb{T}}$ , which proves our assertion.  $\square$

Further, in the proof of our refinement of the Hardy inequality (5) we will use the following elementary inequality

$$(26) \quad (u+v)^p \geq u^p + pu^{p-1}v, \quad \text{where } p > 1 \text{ or } p < 0.$$

Recall that this relation is a variant of the well-known Bernoulli inequality and it is valid for all  $u \geq 0$  and  $u+v \geq 0$ , if  $p > 1$ , or for  $u > 0$  and  $u+v > 0$ , if  $p < 0$ . The equality in (26) holds if and only if  $v = 0$ .

Finally, in order to establish the new refinement of Hardy's inequality on a time scale, we will need yet another assumption. Namely, we will assume that the forward jump operator is uniformly bounded from above by a linear function. More precisely, we suppose that there exists a real number  $m \geq 1$  such that

$$(27) \quad \sigma(t) - a \leq m(t-a), \quad \text{for } t > a.$$

It should be noticed here that the condition  $\sigma(t) \leq \lambda t$  may be removed if the graininess function  $\mu(t)$  on the time scale  $\mathbb{T}$  satisfies relation  $\mu(t) = O(t)$ . Indeed, if  $\mu(t) = O(t)$ , then there exists  $\lambda > 1$  such that  $0 < \mu(t)/t \leq \lambda - 1$ , for all  $t \in \mathbb{T}$ . Hence,  $1 \leq (t + \mu(t))/t \leq \lambda$  and therefore,  $1 \leq \sigma(t)/t \leq \lambda$ , for all  $t \in \mathbb{T}$ . Note also that if  $\mathbb{T} = \mathbb{R}$ , then  $\sigma(t) = t$ , while for  $\mathbb{T} = \mathbb{N}$ , we have  $\sigma(t) = t + 1$ .

Now, we are ready to state and prove a refinement of Hardy's inequality on time scales.

**Theorem 2.** *Let  $\mathbb{T}$  be a time scale with  $a, b \in \mathbb{T}$ , and let  $g$  be a non-negative decreasing function on  $[a, b]_{\mathbb{T}}$ . Further, assume that (27) holds. If  $\alpha \leq 1$  and  $\beta > 1$ , then holds the inequality*

$$(28) \quad \int_a^b (x-a)^{\alpha-1} (\mathcal{G}^\sigma(x))^\beta \Delta x + \left(\frac{\beta}{\beta-\alpha}\right) (b-a)^\alpha \mathcal{G}^\beta(b) \\ \leq \left(\frac{\beta}{\beta-\alpha}\right)^\beta \int_a^b (x-a)^{\alpha-1} g^\beta(x) \Delta x,$$

where  $\mathcal{G}$  is defined as in (25).

*Proof.* Taking into account the chain rule (15) and utilizing the fact that  $\mathcal{G}$  is

decreasing on  $[a, b]_{\mathbb{T}}$  by Lemma 2, we obtain the inequality

$$\begin{aligned} (\mathcal{G}^\beta(x))^\Delta &= \beta \int_0^1 [h\mathcal{G}^\sigma(x) + (1-h)\mathcal{G}(x)]^{\beta-1} dh \mathcal{G}^\Delta(x) \\ &\leq \beta \int_0^1 [h\mathcal{G}^\sigma(x) + (1-h)\mathcal{G}^\sigma(x)]^{\beta-1} dh \mathcal{G}^\Delta(x) \\ &= \beta \mathcal{G}^\Delta(x) (\mathcal{G}^\sigma(x))^{\beta-1}. \end{aligned}$$

On the other hand, since  $(x-a)\mathcal{G}(x) = \int_a^x g(t) \Delta t$ , the product rule (14) implies the equality  $(x-a)\mathcal{G}^\Delta(x) + \mathcal{G}^\sigma(x) = g(x)$ , and then, we have

$$(29) \quad (x-a)^\alpha \mathcal{G}^\Delta(x) = (x-a)^{\alpha-1} [g(x) - \mathcal{G}^\sigma(x)].$$

Further, applying integration by parts formula (17) with  $u(x) = \int_a^x (t-a)^{\alpha-1} \Delta t$  and  $v(x) = \mathcal{G}^\beta(x)$ , it follows that

$$\begin{aligned} &\int_a^b (x-a)^{\alpha-1} (\mathcal{G}^\sigma(x))^\beta \Delta x \\ &= u(b)\mathcal{G}^\beta(b) - \lim_{x \rightarrow a^+} u(x)\mathcal{G}^\beta(x) - \int_a^b u(x) (\mathcal{G}^\beta(x))^\Delta \Delta x \\ &\geq \frac{(b-a)^\alpha \mathcal{G}^\beta(b)}{\alpha} - \frac{\beta}{\alpha} \int_a^b (x-a)^\alpha \mathcal{G}^\Delta(x) (\mathcal{G}^\sigma(x))^{\beta-1} \Delta x - \lim_{x \rightarrow a^+} u(x)\mathcal{G}^\beta(x) \\ &= \frac{(b-a)^\alpha \mathcal{G}^\beta(b)}{\alpha} - \frac{\beta}{\alpha} \int_a^b (x-a)^{\alpha-1} [g(x) - \mathcal{G}^\sigma(x)] (\mathcal{G}^\sigma(x))^{\beta-1} \Delta x \\ &= \frac{(b-a)^\alpha \mathcal{G}^\beta(b)}{\alpha} - \frac{\beta}{\alpha} \int_a^b (x-a)^{\alpha-1} g(x) (\mathcal{G}^\sigma(x))^{\beta-1} \Delta x \\ &\quad + \frac{\beta}{\alpha} \int_a^b (x-a)^{\alpha-1} (\mathcal{G}^\sigma(x))^\beta \Delta x - \lim_{x \rightarrow a^+} u(x)\mathcal{G}^\beta(x). \end{aligned}$$

Now, our intention is to show that  $\lim_{x \rightarrow a^+} u(x)\mathcal{G}^\beta(x) = 0$ . Taking into account

definitions of functions  $u(x)$  and  $\mathcal{G}^\beta(x)$ , we have

$$\begin{aligned}
 u(x)\mathcal{G}^\beta(x) &= \int_a^x (t-a)^{\alpha-1} \Delta t \left( \frac{1}{x-a} \int_a^x g(t) \Delta t \right)^\beta \\
 &= \left( \frac{1}{x-a} \right)^\beta \int_a^x (t-a)^{\alpha-1} \Delta t \left( \int_a^x g(t) \Delta t \right)^\beta \\
 &\leq g^\beta(a) \left( \frac{1}{x-a} \right)^\beta \int_a^x (t-a)^{\alpha-1} \Delta t \left( \int_a^x \Delta t \right)^\beta \\
 &= g^\beta(a) \int_a^x (t-a)^{\alpha-1} \Delta t \\
 &\leq g^\beta(a) \int_a^x \frac{1}{(\sigma(t)-a)^{1-\alpha}} \left( \frac{\sigma(t)-a}{t-a} \right)^{1-\alpha} \Delta t \\
 (30) \quad &\leq m^{1-\alpha} g^\beta(a) \int_a^x \frac{1}{(\sigma(t)-a)^{1-\alpha}} \Delta t.
 \end{aligned}$$

Since  $\alpha \leq 1$ , yet another application of the chain rule (15) yields the estimate

$$\begin{aligned}
 ((t-a)^\alpha)^\Delta &= \alpha \int_0^1 [h(\sigma(t)-a) + (1-h)(t-a)]^{\alpha-1} dh \\
 &\geq \alpha \int_0^1 [h(\sigma(t)-a) + (1-h)(\sigma(t)-a)]^{\alpha-1} dh \\
 &= \alpha (\sigma(t)-a)^{\alpha-1},
 \end{aligned}$$

which implies the inequality

$$\int_a^x (\sigma(t)-a)^{\alpha-1} \Delta t \leq \int_a^x \frac{((t-a)^\alpha)^\Delta}{\alpha} \Delta t = \frac{(x-a)^\alpha}{\alpha}.$$

Thus, utilizing the above inequality and (30), we obtain the estimate

$$u(x)\mathcal{G}^\beta(x) \leq m^{1-\alpha} g^\beta(a) \frac{(x-a)^\alpha}{\alpha},$$

and consequently,  $\lim_{x \rightarrow a^+} u(x)\mathcal{G}^\beta(x) = 0$ . Clearly, from the above discussion, we obtain the inequality

$$\begin{aligned}
 &\left( \frac{\beta-\alpha}{\alpha} \right) \int_a^b (x-a)^{\alpha-1} (\mathcal{G}^\sigma(x))^\beta \Delta x + \frac{(b-a)^\alpha \mathcal{G}^\beta(b)}{\alpha} \\
 &\leq \frac{\beta}{\alpha} \int_a^b (x-a)^{\alpha-1} g(x) (\mathcal{G}^\sigma(x))^{\beta-1} \Delta x.
 \end{aligned}$$

Now, applying Hölder's inequality with exponents  $1/\beta$  and  $(\beta-1)/\beta$  to the right-

hand side of the last inequality yields the relation

$$\begin{aligned} & \left( \frac{\beta - \alpha}{\alpha} \right) \int_a^b (x - a)^{\alpha-1} (\mathcal{G}^\sigma(x))^\beta \Delta x + \frac{(b-a)^\alpha \mathcal{G}^\beta(b)}{\alpha} \\ & \leq \frac{\beta}{\alpha} \left\{ \int_a^b (x - a)^{\alpha-1} g^\beta(x) \Delta x \right\}^{\frac{1}{\beta}} \left\{ \int_a^b (x - a)^{\alpha-1} (\mathcal{G}^\sigma(x))^\beta \Delta x \right\}^{\frac{\beta-1}{\beta}}, \end{aligned}$$

which can be rewritten in the following form

$$\begin{aligned} & \left( \frac{\beta}{\beta - \alpha} \right)^\beta \int_a^b (x - a)^{\alpha-1} g^\beta(x) \Delta x \\ & \geq \left[ \left\{ \int_a^b (x - a)^{\alpha-1} (\mathcal{G}^\sigma(x))^\beta \Delta x \right\}^{\frac{1}{\beta}} + \frac{\frac{(b-a)^\alpha}{(\beta-\alpha)} \mathcal{G}^\beta(b)}{\left\{ \int_a^b (x - a)^{\alpha-1} (\mathcal{G}^\sigma(x))^\beta \Delta x \right\}^{\frac{\beta-1}{\beta}}} \right]^\beta. \end{aligned}$$

Finally, applying Bernoulli's inequality (26), with

$$u = \left\{ \int_a^b (x - a)^{\alpha-1} (\mathcal{G}^\sigma(x))^\beta \Delta x \right\}^{\frac{1}{\beta}}, \text{ and } v = \frac{\frac{(b-a)^\alpha}{(\beta-\alpha)} \mathcal{G}^\beta(b)}{\left\{ \int_a^b (x - a)^{\alpha-1} (\mathcal{G}^\sigma(x))^\beta \Delta x \right\}^{\frac{\beta-1}{\beta}}},$$

to the right-hand side of the last relation, we obtain

$$\begin{aligned} & \left( \frac{\beta}{\beta - \alpha} \right)^\beta \int_a^b (x - a)^{\alpha-1} g^\beta(x) \Delta x \\ & \geq \int_a^b (x - a)^{\alpha-1} (\mathcal{G}^\sigma(x))^\beta \Delta x + \frac{\beta}{\beta - \alpha} (b - a)^\alpha \mathcal{G}^\beta(b), \end{aligned}$$

which represents the desired inequality (28). The proof is now complete.  $\square$

The established inequality (28) is both refinement and the time scale extension of the Hardy inequality (5) for a class of decreasing functions. To see this, let  $\alpha = q/p$  and  $\beta = q$ , where  $p \geq q > 1$ . In this setting, Theorem 2 reduces to the following form.

**Corollary 3.** *Let  $\mathbb{T}$  be a time scale with  $a, b \in \mathbb{T}$ , let  $g$  be a non-negative decreasing function on  $[a, b]_{\mathbb{T}}$ , and suppose that (27) holds. If  $p \geq q > 1$ , then holds the inequality*

$$\begin{aligned} (31) \quad & \int_a^b (x - a)^{\frac{q}{p}-1} (\mathcal{G}^\sigma(x))^q \Delta x + \frac{p}{p-1} (b - a)^{\frac{q}{p}} \mathcal{G}^q(b) \\ & \leq \left( \frac{p}{p-1} \right)^q \int_a^b (x - a)^{\frac{q}{p}-1} g^q(x) \Delta x, \end{aligned}$$

where  $\mathcal{G}$  is defined as in (25).

Clearly, if  $\mathbb{T} = \mathbb{R}$ ,  $a = 0$  and  $p = q$ , then  $\sigma(x) = x$  and  $\mathcal{G}(x) = (1/x) \int_0^x g(t) dt$ , so the inequality (31) provides refinement of the classical Hardy inequality (5) for a class of decreasing functions. In the next few remarks, we will compare our Theorem 2 with some results known from the literature.

**Remark 3.** If  $\alpha = 1$ , Theorem 2 provides the following refinement of the Hardy-type inequality

$$\int_a^b (\mathcal{G}^\sigma(x))^\beta \Delta x + \frac{\beta(b-a)}{\beta-1} \mathcal{G}^\beta(b) \leq \left(\frac{\beta}{\beta-1}\right)^\beta \int_a^b g^\beta(x) \Delta x,$$

which can be regarded as a time scale version of the inequality established by Shum [19, Theorem 2.1] and the inequality due to Yang and Hwang [20, Lemma 2].

**Remark 4.** If  $\mathbb{T} = \mathbb{R}$ ,  $a = 0$  and  $\gamma = \beta - \alpha + 1 > 1$ , the inequality (28) reduces to

$$\int_0^b x^{-\gamma} \mathcal{M}^\beta(x) dx + \frac{\beta b^{1-\gamma}}{\gamma-1} \mathcal{M}^\beta(b) \leq \left(\frac{\beta}{\gamma-1}\right)^\beta \int_0^b x^{\beta-\gamma} g^\beta(x) dx,$$

where  $\mathcal{M}(x) := \int_0^x g(t) dt$ . It should be noticed here that this inequality has been established by Shum [19, Theorem 2.1].

**Remark 5.** If  $\mathbb{T} = \mathbb{R}$ , our inequality (28) can be regarded as a refinement of the inequality

$$\int_a^b (x-a)^{\alpha-1} \mathcal{G}^\beta(x) dx \leq \left(\frac{\beta}{\beta-\alpha}\right)^\beta \int_a^b (x-a)^{\alpha-1} g^\beta(x) dx,$$

established by Popoli [16, Theorem 2.1], for  $\alpha \leq 1$  and  $\beta > 1$ . Clearly,  $\mathcal{G}$  is defined here by  $\mathcal{G}(x) := (1/(x-a)) \int_a^x g(t) dt$ .

In order to finish our discussion referring to Theorem 2, we give its discrete version. In a discrete case,  $\mathbb{T} = \mathbb{N}$  and  $\sigma(n) = n+1$ , so the condition (27) is satisfied for  $m = 2$ .

**Corollary 4.** Let  $g(n)$  be a non-negative decreasing sequence on  $[a, b]_{\mathbb{N}}$ ,  $a, b \in \mathbb{N}$ . If  $\alpha \leq 1$  and  $\beta > 1$ , then holds the inequality

$$\sum_{n=a}^{b-1} (n-a)^{\alpha-1} \mathcal{G}^\beta(n+1) + \frac{\beta(b-a)^\alpha}{\beta-\alpha} \mathcal{G}^\beta(b) \leq \left(\frac{\beta}{\beta-\alpha}\right)^\beta \sum_{n=a}^{b-1} (n-a)^{\alpha-1} g^\beta(n),$$

where  $\mathcal{G}(n) := (1/(n-a)) \sum_{k=a}^{n-1} g(k)$ .

In the sequel, we first state and prove several lemmas, interesting in their own right, which will be utilized in establishing the higher integrability theorem.

**Lemma 3.** Let  $\mathbb{T}$  be a time scale with  $a, b \in \mathbb{T}$ , and let  $\varphi, \psi$  be non-negative functions defined on  $[a, b]_{\mathbb{T}}$ . Then holds the relation

$$\int_a^b \varphi(t) \left( \int_t^b \psi(x) \Delta x \right) \Delta t = \int_a^b \psi(t) \left( \int_a^{\sigma(t)} \varphi(x) \Delta x \right) \Delta t.$$

*Proof.* Define  $\Psi(t) = \int_t^b \psi(x)\Delta x$ . Applying the integration by parts formula (17) to the term  $\int_a^b \varphi(t)\Psi(t)\Delta t$  with  $u(t) = \Psi(t)$  and  $v^\Delta(t) = \varphi(t)$ , we get

$$\int_a^b \varphi(t) \left( \int_t^b \psi(x)\Delta x \right) \Delta t = \int_a^b \varphi(t)\Psi(t)\Delta t = \Psi(t)v(t)|_a^b - \int_a^b \Psi^\Delta(t)v^\sigma(t)\Delta t,$$

where  $v(t) = \int_a^t \varphi(x)\Delta x$ . Now, since  $v(a) = 0$  and  $\Psi(b) = 0$ , it follows that

$$\begin{aligned} \int_a^b \varphi(t) \left( \int_t^b \psi(x)\Delta x \right) \Delta t &= \int_a^b [-\Psi^\Delta(t)] v^\sigma(t)\Delta t = \int_a^b \psi(t)v^\sigma(t)\Delta t \\ &= \int_a^b \psi(t) \left( \int_a^{\sigma(t)} \varphi(x)\Delta x \right) \Delta t, \end{aligned}$$

which completes the proof.  $\square$

**Lemma 4.** Let  $\mathbb{T}$  be a time scale with  $a, b \in \mathbb{T}$ , and let  $w$  be a non-negative increasing function on  $[a, b]_{\mathbb{T}}$ . Further, suppose that there exists  $m \geq 1$  such that

$$(32) \quad w^\sigma(x) \leq mw(x), \quad \text{for all } x \in [a, b]_{\mathbb{T}}.$$

If  $\lambda < 0$ , then holds the inequality

$$(33) \quad \begin{aligned} &\int_a^b w^\lambda(x) w^\Delta(x) \mathcal{H}^\sigma(x) \Delta x \\ &\geq \frac{1}{\lambda m} \left[ w^\lambda(b) \int_a^b \varphi(x) \Delta x - \int_a^b w^\lambda(x) \varphi(x) \Delta x \right], \end{aligned}$$

where  $\mathcal{H}(x) := (1/w(x)) \int_a^x \varphi(t) \Delta t$ .

*Proof.* Taking into account definition of  $\mathcal{H}$ , the assumption (32) and Lemma 3, we obtain

$$\begin{aligned} \int_a^b w^\lambda(x) w^\Delta(x) \mathcal{H}^\sigma(x) \Delta x &= \int_a^b w^\Delta(x) w^{\lambda-1}(x) \left( \frac{w}{w^\sigma} \int_a^{\sigma(x)} \varphi(t) \Delta t \right) \Delta x \\ &\geq \frac{1}{m} \int_a^b w^\Delta(x) w^{\lambda-1}(x) \left( \int_a^{\sigma(x)} \varphi(t) \Delta t \right) \Delta x \\ &= \frac{1}{m} \int_a^b \varphi(x) \left( \int_x^b w^\Delta(t) w^{\lambda-1}(t) \Delta t \right) \Delta x. \end{aligned}$$

Further, since  $\lambda < 0$  and  $w^\Delta(x) > 0$ , by applying the chain rule (15), we have

$$\begin{aligned} (w^\lambda(t))^\Delta &= \lambda \int_0^1 [hw^\sigma(t) + (1-h)w(t)]^{\lambda-1} dh w^\Delta(t) \\ &\geq \lambda \int_0^1 [hw(t) + (1-h)w(t)]^{\lambda-1} dh w^\Delta(t) \\ &= \lambda w^\Delta(t) w^{\lambda-1}(t), \end{aligned}$$

which implies the estimate  $w^\Delta(t) w^{\lambda-1}(t) \geq \frac{1}{\lambda} (w^\lambda(t))^\Delta$ . Hence, we obtain

$$\begin{aligned} \int_a^b w^\lambda(x) w^\Delta(x) \mathcal{H}^\sigma(x) \Delta x &\geq \frac{1}{\lambda m} \int_a^b \varphi(x) \left( \int_x^b (w^\lambda(t))^\Delta \Delta t \right) \Delta x \\ &= \frac{1}{\lambda m} \left[ w^\lambda(b) \int_a^b \varphi(x) \Delta x - \int_a^b w^\lambda(x) \varphi(x) \Delta x \right], \end{aligned}$$

which proves our assertion. The proof is complete.  $\square$

We will also employ the following Lemma which has been proved in [16, Lemma 2.2].

**Lemma 5.** *Let  $C > 1$ ,  $q > p > 0$ , and let  $L$  be defined by*

$$(34) \quad L(p, q, x, C) = 1 - C^q (1-x) \left( \frac{q}{q-px} \right)^{\frac{q}{p}},$$

where  $x \in [0, 1]$ . Then, there exists a unique solution  $x_q$  of the equation  $L(p, q, x, C) = 0$ . In addition,  $L(p, q, x, C) > 0$  if and only if  $x \in (x_q, 1]$ .

In order to establish higher integrability theorems, from now on we assume that a non-negative function  $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$  satisfies the reverse Hölder's inequality. In this context, this means that there exist constants  $0 < p < q$  and  $K > 1$  such that the inequality

$$(35) \quad \left[ \frac{1}{\sigma(x) - a} \int_a^{\sigma(x)} f^q(t) \Delta t \right]^{\frac{1}{q}} \leq K \left[ \frac{1}{\sigma(x) - a} \int_a^{\sigma(x)} f^p(t) \Delta t \right]^{\frac{1}{p}},$$

holds for every  $x \in [a, b]_{\mathbb{T}}$ .

Before we state and prove our main theorem, we first need the following auxiliary result.

**Theorem 3.** *Let  $0 < p < q$ ,  $K > 1$ , and let  $\mathbb{T}$  be a time scale with  $a, b \in \mathbb{T}$ . Further, let  $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$  be a non-negative decreasing function satisfying (35). If*

$$(36) \quad \mathcal{L}(p, q, \alpha, K) = 1 - K^q m (1 - \alpha) \left( \frac{q}{q - p\alpha} \right)^{\frac{q}{p}},$$

and the condition (27) holds, then the inequality

$$(37) \quad \int_a^b (x-a)^{\alpha-1} f^q(x) \Delta x \leq \frac{(b-a)^{\alpha-1}}{\mathcal{L}(p, q, \alpha, K)} \int_a^b f^q(x) \Delta x$$

holds for all  $\alpha \in (\alpha_q, 1]$ , where  $\alpha_q$  is the unique root of the equation (34) with  $C = Km^{1/q}$ .

*Proof.* Without loss of generality, we can suppose that  $\alpha \in (\alpha_q, 1)$  since for  $\alpha = 1$  the inequality (37) holds trivially. Now, since  $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$  satisfies the reverse Hölder's inequality (35), it follows that

$$F_q^\sigma(x) \leq K^q [F_p^\sigma(x)]^{\frac{q}{p}},$$

where  $F_p(x) = (1/(x-a)) \int_a^x f^p(t) \Delta t$  and  $F_q(x) = (1/(x-a)) \int_a^x f^q(t) \Delta t$ . By integrating, it follows that

$$(38) \quad \int_a^b (x-a)^{\alpha-1} F_q^\sigma(x) \Delta x \leq K^q \int_a^b (x-a)^{\alpha-1} (F_p^\sigma(x))^{\frac{q}{p}} \Delta x.$$

Further, it should be noticed that if  $w(x) = x-a$ , then the condition (32) reduces to (27). Therefore, applying Lemma 4 with  $\lambda = \alpha - 1 < 0$ ,  $w(x) = x-a$  and  $\varphi = f^q(x)$  to the left-hand side of (38), it follows that

$$\begin{aligned} & \int_a^b (x-a)^{\alpha-1} F_q^\sigma(x) \Delta x \\ & \geq \frac{1}{(\alpha-1)m} \left[ (b-a)^{\alpha-1} \int_a^b f^q(x) \Delta x - \int_a^b (x-a)^{\alpha-1} f^q(x) \Delta x \right]. \end{aligned}$$

On the other hand, applying Theorem 2 with  $\beta = q/p > 1$  and  $g = f^p$  to the right-hand side of (38), we have

$$\begin{aligned} & \int_a^b (x-a)^{\alpha-1} (F_p^\sigma(x))^{\frac{q}{p}} \Delta x \leq \left( \frac{q}{q-p\alpha} \right)^{\frac{q}{p}} \int_a^b (x-a)^{\alpha-1} f^q(x) \Delta x \\ & - \left( \frac{q}{q-p\alpha} \right) (b-a)^{\alpha-\frac{q}{p}} \left( \int_a^b f^p(x) \Delta x \right)^{\frac{q}{p}} \\ & \leq \left( \frac{q}{q-p\alpha} \right)^{\frac{q}{p}} \int_a^b (x-a)^{\alpha-1} f^q(x) \Delta x. \end{aligned}$$

Now, taking into account (38) and the previous two estimates, we obtain the inequality

$$\begin{aligned} & \frac{1}{(\alpha-1)m} \left[ (b-a)^{\alpha-1} \int_a^b f^q(x) \Delta x - \int_a^b (x-a)^{\alpha-1} f^q(x) \Delta x \right] \\ & \leq K^q \left( \frac{q}{q-p\alpha} \right)^{\frac{q}{p}} \int_a^b (x-a)^{\alpha-1} f^q(x) \Delta x, \end{aligned}$$



which can be rewritten as

$$\begin{aligned} & (b-a)^{\alpha-1} \int_a^b f^q(x) \Delta x \\ & \geq \left[ 1 - K^q (1-\alpha) m \left( \frac{q}{q-p\alpha} \right)^{\frac{q}{p}} \right] \int_a^b (x-a)^{\alpha-1} f^q(x) \Delta x \\ & = \mathcal{L}(p, q, \alpha, K) \int_a^b (x-a)^{\alpha-1} f^q(x) \Delta x. \end{aligned}$$

Finally, by Lemma 5 there exists a unique  $\alpha_q \in (0, 1)$  such that  $\mathcal{L}(p, q, \alpha, K) > 0$  for  $\alpha \in (\alpha_q, 1]$ . This provides the inequality (37) and the proof is complete.  $\square$

**Remark 6.** If  $p = 1$  and  $\alpha = \frac{q}{r}$ , provided that  $r \geq q$ , the inequality (37) reduces to

$$\int_a^b (x-a)^{\frac{q}{r}-1} f^q(x) \Delta x \leq \frac{(b-a)^{\frac{q}{r}-1}}{\mathcal{L}(1, q, q/r, K)} \int_a^b f^q(x) \Delta x,$$

where  $\mathcal{L}(1, q, q/r, K)$  is defined by (36). In particular, if  $\mathbb{T} = \mathbb{R}$  this inequality provides relation established by D'Apuzzo and Sbordonè [7, Lemma 3.2].

Finally, we are able to state and prove the higher integrability theorem for decreasing functions on time scales.

**Theorem 4.** Let  $0 < p < q$ ,  $K > 1$ , and let  $\mathbb{T}$  be a time scale with  $a, b \in \mathbb{T}$ . Further, suppose that  $f \in L_{\Delta}^q[a, b]_{\mathbb{T}}$  is a non-negative decreasing function satisfying (35). If the condition (27) holds, then  $f \in L_{\Delta}^s[a, b]_{\mathbb{T}}$  for  $q \leq s < q_0$ , where  $q_0$  is the unique solution of the equation

$$(39) \quad \left( \frac{x}{x-q} \right)^{\frac{1}{q}} = Km^{\frac{1}{q}} \left( \frac{x}{x-p} \right)^{\frac{1}{p}},$$

and the following inequality holds

$$(40) \quad \left( \frac{1}{b-a} \int_a^b f^s(x) \Delta x \right)^{\frac{q}{s}} \leq \frac{q}{s\mathcal{L}(p, q, \frac{q}{s}, K)} \left( \frac{1}{b-a} \int_a^b f^q(x) \Delta x \right).$$

*Proof.* Utilizing Theorem 3 with  $\alpha = q/s$  and  $\alpha_q = q/q_0$ , we obtain the inequality

$$\int_a^b (x-a)^{\frac{q}{s}-1} f^q(x) \Delta x \leq \frac{(b-a)^{\frac{q}{s}-1}}{\mathcal{L}(p, q, \frac{q}{s}, K)} \int_a^b f^q(x) \Delta x,$$

where  $q \leq s < q_0$ . In addition, applying the inequality (24) with  $r = q$  to the left-hand side of the previous inequality, we have

$$\left( \int_a^b f^s(x) \Delta x \right)^{\frac{q}{s}} \leq \frac{q(b-a)^{\frac{q}{s}-1}}{s\mathcal{L}(p, q, \frac{q}{s}, K)} \int_a^b f^q(x) \Delta x, \quad \text{for } q \leq s < q_0,$$

and thus,

$$\left( \frac{1}{b-a} \int_a^b f^s(x) \Delta x \right)^{\frac{q}{s}} \leq \frac{q}{s\mathcal{L}(p, q, \frac{q}{s}, K)} \left( \frac{1}{b-a} \int_a^b f^q(x) \Delta x \right),$$

which provides the inequality (40). Clearly,  $q_0$  is the unique solution of the equation

$$\mathcal{L}\left(p, q, \frac{q}{x}, K\right) = 0,$$

which reduces to (39). The proof is now complete.  $\square$

**Remark 7.** Since  $f$  is decreasing function, we have  $f^\sigma(x) \leq f(x)$ , so relation (40) implies that the inequality

$$(41) \quad \left( \frac{1}{b-a} \int_a^b f^s(\sigma(x)) \Delta x \right)^{\frac{q}{s}} \leq \frac{q}{s\mathcal{L}(p, q, \frac{q}{s}, K)} \left( \frac{1}{b-a} \int_a^b f^q(x) \Delta x \right),$$

holds for  $q \leq s < q_0$ , where  $q_0$  is the unique solution of the equation (39) and  $\mathcal{L}(p, q, q/s, K)$  is defined as in (36).

**Remark 8.** As a special case, when  $\mathbb{T} = \mathbb{R}$ , then  $\sigma(x) = x$  and  $m = 1$  in (27). Hence, in this setting the inequality (41) reduces to inequality (10) established by Popoli (for more details, see [16]). In addition, if  $\mathbb{T} = \mathbb{R}$  and  $p = 1$ , we obtain the inequality (7) due to D'Apuzzo and Sbordonc (for more details, see [7]).

We conclude this paper with a discrete version of the inequality (41) from the previous remark. Namely, if  $\mathbb{T} = \mathbb{N}$ , then  $\sigma(n) = n + 1$ , so the condition (27) is satisfied for  $m = 2$ . In this case, the constant  $\mathcal{L}$  defined by (36) reduces to

$$\bar{\mathcal{L}}\left(p, q, \frac{q}{s}, K\right) \equiv 1 - 2K^q \left(1 - \frac{q}{s}\right) \left(\frac{s}{s-p}\right)^{\frac{q}{p}},$$

so we have the following result.

**Corollary 5.** Let  $0 < p < q$ ,  $K > 1$ , and let  $f(n) \in l^q[a, b]_{\mathbb{N}}$ ,  $a, b \in \mathbb{N}$ , be a non-negative decreasing sequence such that

$$\left( \frac{1}{n+1-a} \sum_{k=a}^n f^q(k) \right)^{\frac{1}{q}} \leq K \left( \frac{1}{n+1-a} \sum_{k=a}^n f^p(k) \right)^{\frac{1}{p}},$$

holds for  $n \in [a, b]_{\mathbb{N}}$ . Then,  $f \in l^s[a, b]_{\mathbb{N}}$  for  $q \leq s < q_0$ , and

$$\left( \frac{1}{b-a} \sum_{n=a}^{b-1} f^s(n+1) \right)^{\frac{q}{s}} \leq \frac{q}{s\bar{\mathcal{L}}(p, q, \frac{q}{s}, K)} \left( \frac{1}{b-a} \sum_{n=a}^{b-1} f^q(n) \right),$$

where  $q_0$  is the unique solution of the equation

$$\left( \frac{x}{x-q} \right)^{\frac{1}{q}} = 2^{\frac{1}{q}} K \left( \frac{x}{x-p} \right)^{\frac{1}{p}}.$$

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(Received 08.12.2018)

(Revised 08.07.2019)

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