Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **14** (2020), 001–019.
https://doi.org/10.2298/AADM180826006M

ASYMPTOTIC PROPERTIES OF SOLUTIONS OF THIRD ORDER DIFFERENCE EQUATIONS

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We consider the difference equation of the form

$$\Delta(r_n\Delta(p_n\Delta x_n)) = a_n f(x_{\sigma(n)}) + b_n.$$

We present sufficient conditions under which, for a given solution y of the equation $\Delta(r_n\Delta(p_n\Delta y_n)) = 0$, there exists a solution x of the nonlinear equation with the asymptotic behavior

$$x_n = y_n + z_n,$$

where z is a sequence convergent to zero. Our approach allows us to control the degree of approximation, i.e., the rate of convergence of the sequence z. We examine two types of approximation: harmonic approximation when $z_n = \mathrm{o}(n^s)$, $s \le 0$, and geometric approximation when $z_n = \mathrm{o}(\mu^n)$, $\mu \in (0,1)$.

1. INTRODUCTION

Let $\mathbb{N}$, $\mathbb{R}$ denote the set of positive integers and the set of real numbers, respectively. In this paper we consider the third order nonlinear difference equation of the form

$$\text{(E)} \qquad \Delta(r_n\Delta(p_n\Delta x_n)) = a_n f(x_{\sigma(n)}) + b_n,$$

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2010 Mathematics Subject Classification. 39A10, 39A22
Keywords and Phrases. asymptotic behavior, bounded solution, convergent solution, harmonic approximation, geometric approximation

where $a_n, b_n \in \mathbb{R}$, $r_n, p_n > 0$, for any $n \in \mathbb{N}$, $f : \mathbb{R} \to \mathbb{R}$, and $\sigma : \mathbb{N} \to \mathbb{N}$, with $\sigma(n) \to \infty$. By a solution of (E) we mean a sequence $x : \mathbb{N} \to \mathbb{R}$ satisfying (E) for large n.

The purpose of this paper is to study the asymptotic behavior of solutions of equation (E), which means that we establish conditions under which, for a given solution y of the equation

$$\Delta(r_n \Delta(p_n \Delta y_n)) = 0, \tag{1}$$

there exists a solution x of (E) such that

$$x_n = y_n + \mathrm{o}(1). \tag{2}$$

If condition (2) is satisfied, then x is called a solution with *prescribed asymptotic behavior*, and y is called an *approximative solution* of (E). Using, in (2), a quickly converging to zero sequence we get a more precise approximation. Replacing (2) by $x_n = y_n + \mathrm{o}(n^s)$ for a given $s \in (-\infty, 0]$, we obtain a harmonic approximation. Next, replacing (2) by $x_n = y_n + \mathrm{o}(\mu^n)$ for a given $\mu \in (0, 1)$, we obtain a geometric approximation.

In our investigations the set of all solutions of equation (1) plays an important role. It is not difficult to see that a sequence y is a solution of (1) if and only if there exist $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$y_n = c_1 \sum_{j=1}^{n-1} \frac{1}{p_j} \sum_{i=1}^{j-1} \frac{1}{r_i} + c_2 \sum_{j=1}^{n-1} \frac{1}{p_j} + c_3.$$

We regard equation (E) as a generalization of the ordinary equation

$$\Delta^3 x_n = a_n f(x_{\sigma(n)}) + b_n.$$

Then equation (1) takes the form

$$\Delta^3 y_n = 0. \tag{3}$$

From the point of view of asymptotic behavior, it is worth mentioning that there exist positive sequences r and p such that any solution of (1) is convergent. It is very different from case (3) for which only constant solutions are convergent.

The considered problems appear in the study of asymptotic properties of solutions of both differential and difference equations. This study, in the case of second order differential equations, has been extensively conducted during the last five decades; see for example, the survey paper of Agarwal et al. [**1**] and the references therein. The considerations for equations of order higher than two have been given less attention in the literature, see [**15**], [**23**]. Philos et al. [**22**] proved sufficient and necessary conditions for the existence of solutions of the equation $x^{(m)}(t) = f(t, x)$ with prescribed asymptotic behavior via the solution of $y^{(m)} = 0$, which means that the solution x of nonlinear equations possesses the form

$$x(t) = c_0 + c_1 t + \ldots + c_k t^k + \mathrm{o}(1), \quad \text{as } t \to \infty$$

where $k \in \{1, \dots, m-1\}$.

Asymptotically polynomial solutions were also studied in the discrete case. This subject was begun by Popenda and Drozdowicz [**9**] and [**10**]. In [**9**], asymptotically linear solutions of difference equations of order two were studied. In [**10**], necessary and sufficient conditions were presented for the equation

$$\Delta^m x_n = a_n f(x_n)$$

to have a convergent solution (that, is asymptotically polynomial of degree zero). Zafer [**25**] obtained sufficient conditions for equation

$$\Delta^m x_n = f(n, x_{\sigma(n)}) + b_n$$

to have a solution of the form $x_n = an^{m-1} + \mathrm{o}(n^{m-1})$. In [**16**] and [**21**], the first author presented generalizations of these results. In these generalizations the "measure" of approximation was the space o(1). Further, in [**17**] and [**18**], were presented results, in which the "measure" of approximation is the space $\mathrm{o}(n^s)$, where $s \in (-\infty, 0]$. For some background on asymptotically polynomial solutions we refer to [**17**]. This paper is a continuation of these investigations.

Motivation of our study was also the fact, that in the investigation of the third order trinomial differential equations of the form

$$y'''(t) + a(t)y'(t) + b(t)F(y(\tau(t))) = 0 \tag{4}$$

a transformation of this equation to a binomial third order differential equation of type

$$\left(r(t)\left(p(t)x'(t)\right)'\right)' = a(t)f\left(y(\tau(t))\right), \tag{E'}$$

is often applied, see [**2**], [**4**], [**5**] or [**11**]. Such a method was also used to the discrete counterpart of equation (4). For example, in [**13**] asymptotic properties of nonoscillatory solutions of the equation

$$\Delta^3 x_n + p_n \Delta x_{n+1} + q_n x_{n-\tau} = 0 \tag{5}$$

were studied by transforming this equation to a binomial difference equation with quasi-differences of type (E). A criteria which ensures that all nonoscillatory solutions of (5) tend to zero was established. In [**12**], criteria for oscillation of bounded solutions and sufficient conditions for the existence of certain types of nonoscillatory solutions of nonlinear equation of type (5), are obtained. Here also the transformation to a binomial difference equation with quasidifferences was used. Moreover, third order difference equations with quasi differences appear in the study of neutral type equations, see [**6**] or [**24**]. In these papers, the substitution which transforms a second order difference equation of neutral type to a third order non-neutral equation with quasidifference, was applied. For some other asymptotic properties of solutions to third-order difference equations with quasi-differences, we refer to [**3**], [**7**], [**8**], [**14**].

The paper is organized as follows. In Section 2, we present some preliminary lemmas. Next, in Section 3, we present our theory of harmonic approximations. The main result of this section is presented in Theorem 1. Section 4 is devoted to geometric approximation. The main result of this section is Theorem 2. In the proofs of Theorems 1 and 2 we use the Schauder type fixed point lemma from [**18**]. We present two examples which prove that the assumptions of the main theorems are essential. Moreover, we construct three examples which show how the presented theorems can be used.

2. PRELIMINARIES

The space of all sequences $x : \mathbb{N} \to \mathbb{R}$ we denote by $\mathbb{R}^{\mathbb{N}}$. If x, y in $\mathbb{R}^{\mathbb{N}}$, then xy and $|x|$ denotes the sequences defined by $xy(n) = x_n y_n$ and $|x|(n) = |x_n|$, respectively. Moreover,

$$\|x\| = \sup\{|x_n| : n \in \mathbb{N}\}.$$

If $u \in \mathbb{R}^{\mathbb{N}}$ and $m < n$, then by convention,

$$\sum_{k=n}^{m} u_k = 0.$$

Lemma 1. [**20**, Lemma 5] *If U, W are positive sequences and*

$$\sum_{k=1}^{\infty} W_k \sum_{j=k}^{\infty} U_j < \infty,$$

then

$$\sum_{k=1}^{\infty} U_k \sum_{j=1}^{k} W_j < \infty \quad \textit{and} \quad \sum_{k=n}^{\infty} W_k \sum_{j=k}^{\infty} U_j \leq \sum_{k=n}^{\infty} U_k \sum_{j=1}^{k} W_j$$

for any $n \in \mathbb{N}$.

Lemma 2. *Assume $u : \mathbb{N} \to \mathbb{R}$, $s \in (-\infty, 0]$, $t_1 \in [s, \infty)$, $t_2 \in [s - t_1, \infty)$,*

$$\frac{1}{p_n} = \mathrm{O}(n^{t_1}), \quad \frac{1}{r_n} = \mathrm{O}(n^{t_2}), \quad \tau = 3 + t_1 + t_2 - s, \quad \textit{and}$$

$$\liminf_{k\to\infty} k \left(\frac{|u_k|}{|u_{k+1}|} - 1 \right) > \tau \quad \textit{or} \quad \liminf_{k\to\infty} k \log \frac{|u_k|}{|u_{k+1}|} > \tau.$$

Then

$$\sum_{k=1}^{\infty} \frac{1}{k^s p_k} \sum_{j=k}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} |u_i| < \infty.$$

Proof. Using [**19**, Lemma 6.3, Lemma 6.4] we get that

$$\sum_{k=1}^{\infty} k^{2+t_1+t_2-s}|u_k| < \infty.$$

By the assumptions, we can choose a positive constant L such that $p_k^{-1} \le Lk^{t_1}$ for any k and $r_j^{-1} \le Lj^{t_2}$ for any j. Then

$$\sum_{k=1}^{\infty} \frac{1}{k^s p_k} \sum_{j=k}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} |u_i| \le L^2 \sum_{k=1}^{\infty} k^{t_1-s} \sum_{j=k}^{\infty} j^{t_2} \sum_{i=j}^{\infty} |u_i|$$

$$= L^2 \sum_{k=1}^{\infty}\sum_{j=k}^{\infty} k^{t_1-s} j^{t_2} \sum_{i=j}^{\infty} |u_i| \le L^2 \sum_{k=1}^{\infty}\sum_{j=k}^{\infty} j^{t_1+t_2-s} \sum_{i=j}^{\infty} |u_i|$$

$$= L^2 \sum_{k=1}^{\infty}\sum_{j=k}^{\infty}\sum_{i=j}^{\infty} j^{t_1+t_2-s}|u_i| \le L^2 \sum_{k=1}^{\infty}\sum_{j=k}^{\infty}\sum_{i=j}^{\infty} i^{t_1+t_2-s}|u_i|.$$

Hence, by [**18**, Lemma 4.2] we get

$$\sum_{k=1}^{\infty} \frac{1}{k^s p_k} \sum_{j=k}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} |u_i| \le L^2 \sum_{k=1}^{\infty}\sum_{j=k}^{\infty}\sum_{i=j}^{\infty} \frac{i(i+1)}{2} i^{t_1+t_2-s}|u_i| \le L^2 \sum_{k=1}^{\infty} k^{2+t_1+t_2-s}|u_k| < \infty.$$

□

Lemma 3. [**18**, Lemma 4.7] *Assume* $y, \rho : \mathbb{N} \to \mathbb{R}$, *and* $\lim_{n\to\infty} \rho_n = 0$. *In the set* $X = \{x \in \mathbb{R}^{\mathbb{N}} : \ |x - y| \le |\rho|\}$ *we define a metric by the formula*

$$d(x, z) = \|x - z\|. \tag{6}$$

Then any continuous map $H : X \to X$ *has a fixed point.*

3. HARMONIC APPROXIMATION

In this section we present sufficient conditions under which a given solution y of the equation $\Delta(r_n\Delta(p_n\Delta y_n)) = 0$ is an approximative solution of (E). More precisely, we establish conditions for the existence of a solution x of equation (E) such that

$$y_n = x_n + \mathrm{o}(n^s),$$

where $s \in (-\infty, 0]$. Recall that a sequence $y : \mathbb{N} \to \mathbb{R}$ is a solution of the equation

$$\Delta(r_n\Delta(p_n\Delta y_n)) = 0 \tag{7}$$

if and only if there exist real constants c_1, c_2, c_3 such that

$$y_n = c_1 \sum_{j=1}^{n-1} \frac{1}{p_j} \sum_{i=1}^{j-1} \frac{1}{r_i} + c_2 \sum_{j=1}^{n-1} \frac{1}{p_j} + c_3 \tag{8}$$

for any n.

Theorem 1. *Assume $s \in (-\infty, 0]$ and*

$$\sum_{k=1}^{\infty} \frac{1}{k^s p_k} \sum_{j=k}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} (|a_i| + |b_i|) < \infty.$$

Moreover, assume that $q \in \mathbb{N}$, $\alpha \in (0, \infty)$, $y : \mathbb{N} \to \mathbb{R}$ a solution of (7) are such that f is continuous and bounded on U, where

$$U = \bigcup_{n=q}^{\infty} [y_n - \alpha, y_n + \alpha].$$

Then there exists a solution x of (E) such that $x_n = y_n + \mathrm{o}(n^s)$.

Proof. For $n \in \mathbb{N}$ and $x \in \mathbb{R}^{\mathbb{N}}$, let

$$x_n^* = a_n f(x_{\sigma(n)}) + b_n.$$

There exists a positive constant L, such that $|f(t)| \le L$ for any $t \in U$. Let

$$Y = \{x \in \mathbb{R}^{\mathbb{N}} : |x - y| \le \alpha\}, \quad \rho \in \mathbb{R}^{\mathbb{N}}, \quad \rho_n = \sum_{k=n}^{\infty} \frac{1}{p_k} \sum_{j=k}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} (L|a_i| + |b_i|).$$

Note that $\sigma(n) \ge q$ for large n. Hence, the sequence $(f(x_{\sigma(n)}))$ is bounded for any $x \in Y$. Therefore,

$$\sum_{k=1}^{\infty} \frac{1}{p_k} \sum_{j=k}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} |x_i^*| < \infty$$

for any $x \in Y$. Define a sequence z by

$$z_n = \sum_{k=n}^{\infty} \frac{1}{k^s p_k} \sum_{j=k}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} (L|a_i| + |b_i|).$$

By assumption, $z_n = \mathrm{o}(1)$. Since $s \le 0$, we have

$$\sum_{k=1}^{\infty} \frac{1}{p_k} \sum_{j=k}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} (L|a_i| + |b_i|) < \infty. \tag{9}$$

Then

$$n^{-s}|\rho_n| = n^{-s}\sum_{k=n}^{\infty}\frac{1}{p_k}\sum_{j=k}^{\infty}\frac{1}{r_j}\sum_{i=j}^{\infty}(L|a_i|+|b_i|) = \sum_{k=n}^{\infty}\frac{1}{n^s p_k}\sum_{j=k}^{\infty}\frac{1}{r_j}\sum_{i=j}^{\infty}(L|a_i|+|b_i|).$$

Since $s \le 0$, we get

$$n^{-s}|\rho_n| \le \sum_{k=n}^{\infty}\frac{1}{k^s p_k}\sum_{j=k}^{\infty}\frac{1}{r_j}\sum_{i=j}^{\infty}(L|a_i|+|b_i|) = z_n = \mathrm{o}(1).$$

Thus,

$$\rho_n = n^s\mathrm{o}(1) = \mathrm{o}(n^s).$$

Hence, there exists an index p such that

$$\rho_n \le \alpha \quad \text{and} \quad \sigma(n) \ge q \quad \text{for} \quad n \ge p.$$

Let

$$X = \{x \in \mathbb{R}^{\mathbb{N}} : |x-y| \le \rho \text{ and } x_n = y_n \text{ for } n < p\},$$

$$H : Y \to \mathbb{R}^{\mathbb{N}}, \qquad H(x)(n) = \begin{cases} y_n & \text{for } n < p, \\ y_n - \sum_{k=n}^{\infty}\frac{1}{p_k}\sum_{j=k}^{\infty}\frac{1}{r_j}\sum_{i=j}^{\infty}x_i^* & \text{for } n \ge p. \end{cases}$$

Note that $X \subset Y$. If $x \in X$, then for $n \ge p$ we have

$$|H(x)(n) - y_n| = \left|\sum_{k=n}^{\infty}\frac{1}{p_k}\sum_{j=k}^{\infty}\frac{1}{r_j}\sum_{i=j}^{\infty}x_i^*\right| \le \sum_{k=n}^{\infty}\frac{1}{p_k}\sum_{j=k}^{\infty}\frac{1}{r_j}\sum_{i=j}^{\infty}|x_i^*| \le \rho_n.$$

Therefore, $HX \subset X$. To prove the continuity of H, firstly we notice that because $s \le 0$, we have

$$\sum_{k=1}^{\infty}\frac{1}{p_k}\sum_{j=k}^{\infty}\frac{1}{r_j}\sum_{i=j}^{\infty}|a_i| < \infty,$$

which implies

$$\sum_{k=1}^{\infty}|a_k|\sum_{j=1}^{k}\frac{1}{r_j}\sum_{i=1}^{j}\frac{1}{p_i} < \infty \quad \text{and} \quad \sum_{k=n}^{\infty}\frac{1}{p_k}\sum_{j=k}^{\infty}\frac{1}{r_j}\sum_{i=j}^{\infty}|a_i| \le \sum_{k=n}^{\infty}|a_k|\sum_{j=1}^{k}\frac{1}{r_j}\sum_{i=1}^{j}\frac{1}{p_i}$$

for any $n \in \mathbb{N}$. Indeed, for $U_j = \frac{1}{r_j}\sum_{i=j}^{\infty}|a_i|$,

$$\sum_{k=1}^{\infty}\frac{1}{p_k}\sum_{j=k}^{\infty}U_j = \sum_{k=1}^{\infty}\frac{1}{p_k}\sum_{j=k}^{\infty}\frac{1}{r_j}\sum_{i=j}^{\infty}|a_i| < \infty.$$

By Lemma 1,

$$\sum_{k=1}^{\infty}U_k\sum_{j=1}^{k}\frac{1}{p_j} < \infty.$$

Hence, for $W_k = \frac{1}{r_k}\sum_{j=1}^{k}\frac{1}{p_j}$, we have

$$\sum_{k=1}^{\infty} U_k \sum_{j=1}^{k} \frac{1}{p_j} = \sum_{k=1}^{\infty} \frac{1}{r_k} \sum_{i=k}^{\infty} |a_i| \sum_{j=1}^{k} \frac{1}{p_j} = \sum_{k=1}^{\infty} W_k \sum_{i=k}^{\infty} |a_i|.$$

From Lemma 1 we get

$$\sum_{k=1}^{\infty} |a_k| \sum_{i=1}^{k} \frac{1}{r_i} \sum_{j=1}^{i} \frac{1}{p_j} = \sum_{k=1}^{\infty} |a_k| \sum_{i=1}^{k} W_i < \infty. \tag{10}$$

Moreover, by Lemma 1, for any $n \in \mathbb{N}$,

$$\sum_{k=n}^{\infty} \frac{1}{p_k} \sum_{j=k}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} |a_i| = \sum_{k=n}^{\infty} \frac{1}{p_k} \sum_{j=k}^{\infty} U_j \le \sum_{k=n}^{\infty} U_k \sum_{j=1}^{k} \frac{1}{p_j} = \sum_{k=n}^{\infty} W_k \sum_{j=k}^{\infty} |a_j| \le$$

$$\sum_{k=1}^{\infty} |a_k| \sum_{j=1}^{k} W_j = \sum_{k=n}^{\infty} |a_k| \sum_{j=1}^{k} \frac{1}{r_j} \sum_{i=1}^{j} \frac{1}{p_i}. \tag{11}$$

Now, we can prove that H is continuous. Let $x \in X$, and $\varepsilon > 0$. By (10),

$$\sum_{k=1}^{\infty} |a_k| \sum_{j=1}^{k} \frac{1}{r_j} \sum_{i=1}^{j} \frac{1}{p_i} < \infty.$$

Choose an index $m \ge p$ and a positive constant γ such that

$$L \sum_{k=m}^{\infty} |a_k| \sum_{j=1}^{k} \frac{1}{r_j} \sum_{i=1}^{j} \frac{1}{p_i} < \varepsilon \quad \text{and} \quad \gamma \sum_{k=1}^{m} |a_k| \sum_{j=1}^{k} \frac{1}{r_j} \sum_{i=1}^{j} \frac{1}{p_i} < \varepsilon. \tag{12}$$

Let

$$C = \bigcup_{n=1}^{m} [y_{\sigma(n)} - \alpha, y_{\sigma(n)} + \alpha].$$

Choose a positive δ such that if $t_1, t_2 \in C$ and $|t_1 - t_2| < \delta$, then

$$|f(t_1) - f(t_2)| < \gamma.$$

Choose $z \in X$ such that $\|x - z\| < \delta$. Then

$$\|Hx - Hz\| = \sup_{n \ge p} \left| \sum_{k=n}^{\infty} \frac{1}{p_k} \sum_{j=k}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} (x_i^* - z_i^*) \right| \le$$

$$\sum_{k=p}^{\infty} \frac{1}{p_k} \sum_{j=k}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} |x_i^* - z_i^*| \le \sum_{k=p}^{\infty} \frac{1}{p_k} \sum_{j=k}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} |a_i||f(x_{\sigma(i)}) - f(z_{\sigma(i)})|.$$

Using (11) and (12) we get

$$\|Hx - Hz\| \le \sum_{k=p}^{\infty} |a_k||f(x_{\sigma(k)}) - f(z_{\sigma(k)})| \sum_{j=1}^{k} \frac{1}{r_j} \sum_{i=1}^{j} \frac{1}{p_i}$$

$$\le \gamma \sum_{k=1}^{m} |a_k| \sum_{j=1}^{k} \frac{1}{r_j} \sum_{i=1}^{j} \frac{1}{p_i} + 2L \sum_{k=m}^{\infty} |a_k| \sum_{j=1}^{k} \frac{1}{r_j} \sum_{i=1}^{j} \frac{1}{p_i} < 3\varepsilon.$$

Hence, the map $H : X \to X$ is continuous with respect to the metric defined by (6). By Lemma 3, there exists a point $x \in X$ such that $x = Hx$. Then, for $n \ge p$, we have

$$x_n = y_n - \sum_{k=n}^{\infty} \frac{1}{p_k} \sum_{j=k}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} x_i^*.$$

Hence, for $n \ge p$ we get

$$\Delta(r_n\Delta(p_n\Delta x_n)) = \Delta(r_n\Delta(p_n\Delta y_n)) - \Delta\left(r_n\Delta\left(p_n\Delta\left(\sum_{k=n}^{\infty} \frac{1}{p_k} \sum_{j=k}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} x_i^*\right)\right)\right)$$

$$= 0 - (-x_n^*) = a_n f(x_{\sigma(n)}) + b_n.$$

Therefore, x is a solution of (E). Since $x \in X$ and $\rho_n = \mathrm{o}(n^s)$, we have

$$x_n = y_n + \mathrm{o}(n^s).$$

□

From Theorem 1 we obtain the following corollary.

Corollary 1. *Assume* $s \in (-\infty, 0]$, $t_1 \in [s, \infty)$,

$$\sum_{k=1}^{\infty} k^{2+t_1+t_2-s}(|a_k| + |b_k|) < \infty, \quad \frac{1}{p_n} = \mathrm{O}(n^{t_1}), \quad \frac{1}{r_n} = \mathrm{O}(n^{t_2}),$$

$q \in \mathbb{N}$, $\alpha \in (0, \infty)$, $y : \mathbb{N} \to \mathbb{R}$ *is a solution of* (7),

$$U = \bigcup_{n=q}^{\infty} [y_n - \alpha, y_n + \alpha],$$

and f *is continuous and bounded on* U. *Then there exists a solution* x *of* (E) *such that* $x_n = y_n + \mathrm{o}(n^s)$.

Corollary 2. *Assume* f *is continuous,* $s \in (-\infty, 0]$ *and*

$$\sum_{k=1}^{\infty} \frac{1}{k^s p_k} \sum_{j=k}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} (|a_i| + |b_i|) < \infty.$$

Then for any bounded solution y *of the equation* $\Delta(r_n\Delta(p_n\Delta y_n)) = 0$ *there exists a solution* x *of* (E) *such that* $x_n = y_n + \mathrm{o}(n^s)$.

Proof. Assume y is a bounded solution of the equation $\Delta(r_n\Delta(p_n\Delta y_n)) = 0$ and

$$U = \bigcup_{n=1}^{\infty}[y_n - 1, y_n + 1].$$

Since the sequence y is bounded, U is bounded. Hence f is continuous and bounded on U. Now, using Theorem 1, we obtain the result. □

Corollary 3. *Assume f is continuous, $s \in (-\infty, 0]$ and*

$$\sum_{k=1}^{\infty}\frac{1}{k^s p_k}\sum_{j=k}^{\infty}\frac{1}{r_j}\sum_{i=j}^{\infty}(|a_i| + |b_i|) < \infty.$$

Then for any real constant c there exists a solution x of (E) *such that $x_n = c + \mathrm{o}(n^s)$.*

Proof. Obviously any constant sequence $y_n = c$ is a bounded solution of the equation $\Delta(r_n\Delta(p_n\Delta y_n)) = 0$. Hence the assertion is a consequence of Corollary 2. □

Corollary 4. *Assume f is continuous and bounded, $s \in (-\infty, 0]$ and*

$$\sum_{k=1}^{\infty}\frac{1}{k^s p_k}\sum_{j=k}^{\infty}\frac{1}{r_j}\sum_{i=j}^{\infty}(|a_i| + |b_i|) < \infty.$$

Then for any solution y a of (7) *there exists a solution x of* (E) *such that*

$$x_n = y_n + \mathrm{o}(n^s).$$

Corollary 5. *Assume $s \in (-\infty, 0]$, $t_1 \in [s, \infty)$, $t_2 \in [s - t_1, \infty)$,*

$$\frac{1}{p_n} = \mathrm{O}(n^{t_1}), \quad \frac{1}{r_n} = \mathrm{O}(n^{t_2}), \quad \liminf_{k\to\infty} k\left(\frac{|a_k| + |b_k|}{|a_{k+1}| + |b_{k+1}|} - 1\right) > 3 + t_1 + t_2 - s,$$

and f is continuous. Moreover, assume that $y : \mathbb{N} \to \mathbb{R}$ is a solution of (7) *such that the sequence y is bounded or one of the following conditions holds*

(*a*) $y_n \to \infty$ *and* $\limsup_{\tau\to\infty}|f(\tau)| < \infty$,

(*b*) $y_n \to -\infty$ *and* $\limsup_{\tau\to-\infty}|f(\tau)| < \infty$.

Then there exists a solution x of (E) *such that $x_n = y_n + \mathrm{o}(n^s)$.*

Proof. If y is bounded, then by Lemma 2 and Corollary 2 we get the result. Assume condition (a) is satisfied. Then f is continuous and bounded on $[0, \infty)$. By Lemma 2 and Corollary 1, there exists a solution x of (E) such that $x_n = y_n + \mathrm{o}(n^s)$. The proof for case (b) is similar. □

Now we present an example that proves the assumption that the function f is bounded on some "neighborhood" of y which solves $\Delta\left(r_n\Delta\left(p_n\Delta y_n\right)\right)=0$ in Theorem 1, is essential.

Example 1. Assume

$$r_n=\frac{1}{n},\quad p_n=n,\quad a_n=\frac{1}{n^4\sqrt{n}},\quad b_n=0,\quad \sigma(n)=n-2,\quad f(x)=x^2.$$

Then equation (E) takes the form

$$\Delta\left(\frac{1}{n}\Delta\left(n\Delta x_n\right)\right)=\frac{\left(x_{n-2}\right)^2}{\sqrt{n}\,n^4}. \tag{13}$$

Let $c_1=1$, $c_2=c_3=0$. Define a sequence y by

$$y_n=c_1\sum_{j=1}^{n-1}\frac{1}{p_j}\sum_{i=1}^{j-1}\frac{1}{r_i}+c_2\sum_{j=1}^{n-1}\frac{1}{p_j}+c_3=\sum_{j=1}^{n-1}\frac{1}{j}\sum_{i=1}^{j-1}i=\frac{(n-1)(n-2)}{4}.$$

Notice that f is continuous and unbounded on

$$\bigcup_{n=q}^{\infty}\left[y_n-\alpha,y_n+\alpha\right]$$

for any $q\in\mathbb{N}$ and any $\alpha>0$. Let $s=0$. Then

$$\begin{aligned}
&\sum_{k=1}^{\infty}\frac{1}{k^s p_k}\sum_{j=k}^{\infty}\frac{1}{r_j}\sum_{i=j}^{\infty}(|a_i|+|b_i|)\\
&=\sum_{k=1}^{\infty}\frac{1}{k}\sum_{j=k}^{\infty}j\sum_{i=j}^{\infty}\frac{1}{\sqrt{i}\,i^4}=\sum_{k=1}^{\infty}\frac{1}{k}\sum_{j=k}^{\infty}\sum_{i=j}^{\infty}\frac{j}{\sqrt{i}\,i^4}\le\sum_{k=1}^{\infty}\frac{1}{k}\sum_{j=k}^{\infty}\sum_{i=j}^{\infty}\frac{i}{\sqrt{i}\,i^4}\\
&=\sum_{k=1}^{\infty}\frac{1}{k}\sum_{j=k}^{\infty}\sum_{i=j}^{\infty}\frac{1}{\sqrt{i}\,i^3}\le\sum_{k=1}^{\infty}\frac{1}{k}\sum_{j=k}^{\infty}\frac{j}{\sqrt{j}\,j^3}=\sum_{k=1}^{\infty}\sum_{j=k}^{\infty}\frac{1}{k}\frac{1}{\sqrt{j}\,j^2}\le\sum_{k=1}^{\infty}\sum_{j=k}^{\infty}\frac{1}{\sqrt{j}\,j^2}\\
&\le\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}\,k}<\infty.
\end{aligned}$$

Assume x is a solution of (13) such that

$$x_n=y_n+z_n,\quad z_n=\mathrm{o}(n^s)=\mathrm{o}(1). \tag{14}$$

Notice that $x_n\to\infty$. Since

$$\Delta\left(\frac{1}{n}\Delta(n\Delta y_n)\right)=0,$$

we have

$$\Delta\left(\frac{1}{n}\Delta(n\Delta z_n)\right)=\Delta\left(\frac{1}{n}\Delta(n\Delta x_n)\right)=\frac{(x_{n-2})^2}{\sqrt{n}\,n^4}>0$$

for large n. Therefore, the sequence $n^{-1}\Delta(n\Delta z_n)$ is eventually increasing, and there exists the limit

$$\lambda=\lim_{n\to\infty}\frac{1}{n}\Delta(n\Delta z_n)>-\infty.$$

If $\lambda=\infty$, then for large n we get

$$\Delta(n\Delta z_n)>n.$$

Summing up the above inequality from n_0 to $n-1$ gives

$$n\Delta z_n-n_0\Delta z_{n_0}>n-n_0.$$

Hence,

$$\liminf_{n\to\infty}\Delta z_n\geq 1.$$

On the other hand, $\Delta z_n=\Delta\mathrm{o}(1)=\mathrm{o}(1)$. Therefore, we get $\lambda<\infty$. Thus, the series

$$\sum_{n=1}^{\infty}\Delta\left(\frac{1}{n}\Delta(n\Delta z_n)\right)$$

is convergent. On the other hand,

$$\Delta\left(\frac{1}{n}\Delta(n\Delta z_n)\right)=\Delta\left(\frac{1}{n}\Delta(n\Delta x_n)\right)=\frac{(x_{n-2})^2}{\sqrt{n}\,n^4}$$

for large n. Taking into account that $y_n\to\infty$ and $z_n\to 0$, from (14) we get $x_n>y_n/2$ for large n. Hence, there exists an index n_0 such that

$$\sum_{n=n_0}^{\infty}\Delta\left(\frac{1}{n}\Delta(n\Delta z_n)\right)>\sum_{n=n_0}^{\infty}\frac{((n-3)(n-4))^2}{16\sqrt{n}\,n^4}=\infty$$

which gives a contradiction. □

In the next example, we show that the assumption

$$\sum_{k=1}^{\infty}\frac{1}{k^s p_k}\sum_{j=k}^{\infty}\frac{1}{r_j}\sum_{i=j}^{\infty}(|a_i|+|b_i|)<\infty$$

in Theorem 1 is essential.

Example 2. Assume

$$r_n=n,\quad p_n=n^4,\quad a_n=\frac{1}{n},\quad b_n=0,\quad f(x)=\sqrt{|x|}+1,\quad \sigma:\mathbb{N}\to\mathbb{N}.$$

Then equation (E) takes the form

$$\Delta\left(n\Delta\left(n^4\Delta x_n\right)\right)=\frac{\sqrt{|x_{\sigma(n)}|+1}}{n}. \tag{15}$$

Let $c_1, c_2, c_3 \in \mathbb{R}$. Define a sequence y by

$$y_n = c_1\sum_{j=1}^{n-1}\frac{1}{p_j}\sum_{i=1}^{j-1}\frac{1}{r_i}+c_2\sum_{j=1}^{n-1}\frac{1}{p_j}+c_3.$$

It is easy to see that the sequence y is bounded. Define a sequence R by

$$R_n=\sum_{k=1}^{n-1}\frac{1}{r_k}.$$

Thus, $\Delta y_n = p_n^{-1}(c_1R_n+c_2)$. It is clear, that the sequence $c_1R_n+c_2$ has constant sign for large n. Since $p_n>0$, we get that y is eventually monotonic, and hence convergent. Hence, f is continuous and bounded on

$$\bigcup_{n=q}^{\infty}[y_n-\alpha, y_n+\alpha]$$

for any $q\in\mathbb{N}$ and any positive α. Let $s=-4$. Then,

$$\sum_{k=1}^{\infty}\frac{1}{k^s p_k}\sum_{j=k}^{\infty}\frac{1}{r_j}\sum_{i=j}^{\infty}(|a_i|+|b_i|)=\sum_{k=1}^{\infty}\sum_{j=k}^{\infty}\frac{1}{j}\sum_{i=j}^{\infty}\frac{1}{i}=\infty.$$

Assume x is a solution of (15) such that

$$x_n=y_n+z_n,\quad z_n=\mathrm{o}(n^s)=\mathrm{o}(n^{-4}).$$

Since $\Delta(n\Delta(n^4\Delta y_n))=0$, we have for large n

$$\Delta(n\Delta(n^4\Delta z_n))=\Delta(n\Delta(n^4\Delta x_n))=\frac{\sqrt{|x_{\sigma(n)}|+1}}{n}>\frac{1}{n}>0. \tag{16}$$

Therefore, the sequence $n\Delta(n^4\Delta z_n)$ is eventually increasing and there exists the limit

$$\lambda=\lim_{n\to\infty}n\Delta(n^4\Delta z_n)>-\infty.$$

If $\lambda<\infty$, then the series

$$\sum_{n=1}^{\infty}\Delta(n\Delta(n^4\Delta z_n))$$

is convergent which is contrary to (16). Hence, $\lambda=\infty$ and we get

$$\Delta(n^4\Delta z_n)>\frac{1}{n} \tag{17}$$

for large n. On the other hand,

$$n^4\Delta z_n = n^4\Delta(\mathrm{o}(n^{-4})) = n^4\mathrm{o}(n^{-4}) = \mathrm{o}(1).$$

Hence, the series

$$\sum_{n=1}^{\infty} \Delta(n^4\Delta z_n)$$

is convergent. In view of (17) this is impossible. □

The following two examples illustrate Theorem 1.

Example 3. Assume $s \in (-2/3, 0]$, $r_n = 1$, $p_n = n$, $b_n = 0$, $\sigma(n) = n$,

$$a_n = \frac{2\sqrt[3]{n^2-1}}{\sqrt[3]{n}(n+1)(n+2)(n+3)}, \quad f(x) = \frac{1}{\sqrt[3]{x}} \quad \text{for} \quad x \neq 0.$$

Then equation (E) takes the form

$$\Delta^2\left(n\Delta x_n\right) = \frac{2\sqrt[3]{n^2-1}}{\sqrt[3]{n}(n+1)(n+2)(n+3)}\,\frac{1}{\sqrt[3]{x_n}} \tag{18}$$

for $x_n \neq 0$. Let $c_1 = c_2 = c_3 = 1$. Then by (8) the sequence $y_n = n$ is a solution of (7). We have

$$\sum_{k=1}^{\infty}\frac{1}{k^s p_k}\sum_{j=k}^{\infty}\frac{1}{r_j}\sum_{i=j}^{\infty}(|a_i| + |b_i|) = \sum_{k=1}^{\infty}\frac{1}{k^s k}\sum_{j=k}^{\infty}\sum_{i=j}^{\infty}\frac{2\sqrt[3]{i^2-1}}{\sqrt[3]{i}(i+1)(i+2)(i+3)}$$

$$\leq 2\sum_{k=1}^{\infty}\frac{1}{k^s k}\sum_{j=k}^{\infty}\sum_{i=j}^{\infty}\frac{1}{\sqrt[3]{i^8}} \leq 2\sum_{k=1}^{\infty}\frac{1}{k^s k\sqrt[3]{k^2}} < \infty.$$

Let $q = 2$, $\alpha = 1$, and

$$U = \bigcup_{n=q}^{\infty}[y_n - \alpha, y_n + \alpha] = [1, \infty).$$

Since f is bounded and continuous on U, by Theorem 1, there exists a solution x of (18) such that $x_n = n + \mathrm{o}(n^s)$. Indeed the sequence $x_n = n - \frac{1}{n}$ is a solutions of (18) with this property.

Example 4. Assume

$$r_n = 2^n,\; p_n = 2^{-n},\; a_n = \frac{-3\cdot 2^{n-4}}{(1+2^n)^2},\; b_n = 0,\; \sigma(n) = n,\; f(x) = x^2.$$

Then equation (E) takes the form

$$\Delta\left(2^n\Delta\left(\frac{1}{2^n}\Delta x_n\right)\right) = \frac{-3\cdot 2^{n-4}}{(1+2^n)^2}\,x_n^2. \tag{19}$$

Let $c_1 = c_2 = 0, c_3 = 1$. Then by (8) the sequence $y_n = 1$ is a solution of (7). Let $s \in (-\infty, 0]$. We have

$$\begin{aligned}
&\sum_{k=1}^{\infty} \frac{1}{k^s p_k} \sum_{j=k}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} (|a_i| + |b_i|) \\
&= 3 \sum_{k=1}^{\infty} \frac{1}{k^s} 2^k \sum_{j=k}^{\infty} \frac{1}{2^j} \sum_{i=j}^{\infty} \frac{2^{i-4}}{(1+2^i)^2} \le 3 \sum_{k=1}^{\infty} \frac{1}{k^s} 2^k \sum_{j=k}^{\infty} \frac{1}{2^j} \sum_{i=j}^{\infty} \frac{1}{2^{i+4}} \\
&= \frac{3}{8} \sum_{k=1}^{\infty} \frac{1}{k^s} 2^k \sum_{j=k}^{\infty} \frac{1}{4^j} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k^s} \frac{1}{2^k} < \infty.
\end{aligned}$$

Hence, by Corollary 2, there exists a solution x of (19) such that $x_n = 1 + \mathrm{o}(n^s)$. Indeed the sequence $x_n = 1 + \frac{1}{2^n}$ is a solutions of (19) with this property.

4. GEOMETRIC APPROXIMATION

In this section, we obtain results analogous to those in Section 3. We replace harmonic approximation by geometric approximation which means with measure of approximation $\mathrm{o}(\mu^n)$, $\mu \in (0, 1)$.

We will use the following condition:

(G) $t \in \mathbb{R}$, $p_n^{-1} = \mathrm{O}(n^t)$, $r_n^{-1} = \mathrm{O}(n^t)$, and

$$\limsup_{n\to\infty} \sqrt[n]{|a_n| + |b_n|} < \mu < 1.$$

Lemma 4. *Assume $u \in \mathbb{R}^{\mathbb{N}}$, $t \in \mathbb{R}$, $p_n^{-1} = \mathrm{O}(n^t)$, $r_n^{-1} = \mathrm{O}(n^t)$, and*

$$\lambda = \limsup_{n\to\infty} \sqrt[n]{|u_n|} < \mu < 1.$$

Then

$$\sum_{k=n}^{\infty} \frac{1}{p_k} \sum_{j=k}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} u_i = \mathrm{o}(\mu^n).$$

Proof. Note that if $\beta \in (0, 1)$, then

$$\sum_{i=j}^{\infty} \beta^i = \frac{\beta^j}{1-\beta}. \tag{20}$$

Moreover, if $0 < \alpha < \beta$, then

$$\sqrt[k]{k^t \alpha^k} = \left(\sqrt[k]{k}\right)^t \alpha < \beta \;\Rightarrow\; k^t \alpha^k < \beta^k \tag{21}$$

for large k. Choose $\beta_1, \beta_2, \beta_3$ such that $\lambda < \beta_1 < \beta_2 < \beta_3 < \mu$, and let

$$z_n = \sum_{k=n}^{\infty} \frac{1}{p_k} \sum_{j=k}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} u_i.$$

Choose positive constants L_1, L_2 such that $p_n^{-1} \leq L_1 n^t$ and $r_n^{-1} \leq L_2 n^t$ for any n. Then, using (20) and reasoning as in (21), we get

$$|z_n| \leq L_1 L_2 \sum_{k=n}^{\infty} k^t \sum_{j=k}^{\infty} j^t \sum_{i=j}^{\infty} |u_i| < L_1 L_2 \sum_{k=n}^{\infty} k^t \sum_{j=k}^{\infty} j^t \sum_{i=j}^{\infty} \beta_1^i$$

$$= \frac{L_1 L_2}{1-\beta_1} \sum_{k=n}^{\infty} k^t \sum_{j=k}^{\infty} j^t \beta_1^j < \frac{L_1 L_2}{1-\beta_1} \sum_{k=n}^{\infty} k^t \sum_{j=k}^{\infty} \beta_2^j$$

$$= \frac{L_1 L_2}{(1-\beta_1)(1-\beta_2)} \sum_{k=n}^{\infty} k^t \beta_2^k < \frac{L_1 L_2 \beta_3^n}{(1-\beta_1)(1-\beta_2)(1-\beta_3)}$$

for large n. Hence $z_n = \mathrm{O}(\beta_3^n) = \mathrm{o}(\mu^n)$. □

Theorem 2. *Assume* (G) *holds,* $q \in \mathbb{N}$, $\alpha \in (0, \infty)$, $y : \mathbb{N} \to \mathbb{R}$ *is a solution of* (7),

$$U = \bigcup_{n=q}^{\infty} [y_n - \alpha, y_n + \alpha],$$

and f *is continuous and bounded on* U. *Then there exists a solution* x *of* (E) *such that* $x_n = y_n + \mathrm{o}(\mu^n)$.

Proof. Repeat the proof of Theorem 1 using Lemma 4. □

Corollary 6. *Assume* (G) *holds and* f *is continuous. Then for any bounded solution* y *of the equation* $\Delta(r_n \Delta(p_n \Delta y_n)) = 0$ *there exists a solution* x *of* (E) *such that*

$$x_n = y_n + \mathrm{o}(\mu^n).$$

Corollary 7. *Assume* (G) *holds and* f *is continuous and bounded. Then for any solution* y *of the equation* $\Delta(r_n \Delta(p_n \Delta y_n)) = 0$ *there exists a solution* x *of* (E) *such that*

$$x_n = y_n + \mathrm{o}(\mu^n).$$

The following example illustrates Theorem 2.

Example 5. Assume $f(x) = x^4$ for $x \in \mathbb{R}$,

$$r_n = \tfrac{1}{n},\ a_n = 4^{-n},\ \sigma(n) = n-3,\ \text{for } n \in \mathbb{N}, \quad t = 1, \quad \mu \in (4^{-1}, 1),$$

$$p_n = \begin{cases} 1 & \text{for } n = 1, \\ n^2(n^2-1) & \text{for } n \geq 2, \end{cases}$$

$$b_n = \begin{cases} 0 & \text{for } n < 4, \\ \tfrac{3}{64}4^{-n}\left(-9n^3 + 21n^2 + 28n - 4 - \tfrac{1}{12}\left(\tfrac{n-5}{n-3} + 4^{-n+4}\right)^4\right) & \text{for } n \geq 4. \end{cases}$$

Then equation (E) takes the form

$$\Delta\left(\frac{1}{n}\Delta\left(p_n \Delta x_n\right)\right) = \frac{(x_{n-3})^4}{4^n} + b_n. \tag{22}$$

Notice that

$$\limsup_{n\to\infty} \sqrt[n]{|a_n| + |b_n|} = \tfrac{1}{4} < \mu.$$

Let $c_1, c_2, c_3 \in \mathbb{R}$. Define a sequence y by

$$y_n = c_1 \sum_{j=1}^{n-1} \frac{1}{p_j} \sum_{i=1}^{j-1} \frac{1}{r_i} + c_2 \sum_{j=1}^{n-1} \frac{1}{p_j} + c_3.$$

Then, for $n \geq 2$, we have

$$y_n = \frac{c_1}{4}\left(1 - \frac{2}{n}\right) + c_2 \sum_{j=2}^{n-1} \frac{1}{j^2(j^2-1)} + c_2 + c_3.$$

Hence, the sequence y is convergent and, by Corollary 6, there exists a solution x of (22) such that $x_n = y_n + \mathrm{o}(\mu^{-n})$. If $c_1 = 1$, $c_2 = c_3 = 0$, then one of such solutions is

$$x_n = \tfrac{1}{2}\left(\tfrac{1}{2} - \tfrac{1}{n}\right) + 4^{-n}.$$

□

Acknowledgments. The authors wish to express their thanks to referees for insightful remarks improving quality of the paper. The second author was supported by the Ministry of Science and Higher Education of Poland (04/43/DSPB/0105).

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(Received 26.08.2018)
(Revised 16.11.2019)

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