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ON MATHIEU-TYPE SERIES FOR THE UNIFIED GAUSSIAN HYPERGEOMETRIC FUNCTIONS

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The main purpose of this paper is to present closed integral form expressions for the Mathieu-type $\boldsymbol{a}$-series and for the associated alternating versions whose terms contain a generalized p–extended Gauss' hypergeometric function. Related bounding inequalities for the p–generalized Mathieu–type series are also obtained. Finally, a set of various (known or new) special cases and consequences of the results earned are presented.

1. INTRODUCTION AND MOTIVATION

Various extensions of Gauss' hypergeometric function and other special functions were investigated recently by several authors, consult for instance [**5**]-[**10**], [**15, 16**], [**19**]–[**22**]. The importance of these functions is that they inherit most of the properties of the original functions and provide new relations between different special functions. In particular, the generalized Gauss hypergeometric function [**29**, p. 350, Eq. (1.13)] (see also, [**14**, p. 631, Eq. (1)]) and generalized confluent hypergeometric function [**2**, p. 3695, Eq. (9)] are defined as follows:

$$F_p^{(\alpha,\beta;\kappa,\mu)}(a,b;c;z) = \sum_{n\geq 0}^{\infty} (a)_n \, \frac{\mathrm{B}_{p;\kappa,\mu}^{(\alpha,\beta)}(b+n,\, c-b)}{\mathrm{B}(b,\, c-b)} \, \frac{z^n}{n!}, \tag{1.1}$$

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$$(p, \kappa, \mu \geq 0\,;\, \min\{\Re(\alpha),\, \Re(\beta)\} > 0;\, \Re(c) > \Re(b) > 0,\, |z| < 1);$$

(1.2)
$$\Phi_p^{(\alpha,\beta;\kappa,\mu)}(b;c;z) = \sum_{n\geq 0}^{\infty} \frac{\mathrm{B}_{p;\kappa,\mu}^{(\alpha,\beta)}(b+n,\, c-b)}{\mathrm{B}(b,\, c-b)}\, \frac{z^n}{n!},$$

$$(p, \kappa, \mu \geq 0\,;\, \min\{\Re(\alpha),\, \Re(\beta)\} > 0;\, \Re(c) > \Re(b) > 0,\, |z| < \infty),$$

where $\mathrm{B}_{p;\kappa,\mu}^{(\alpha,\beta)}(x,y)$ is the generalized Beta function [**29**] (see also, [**14**]) defined by

$$\mathrm{B}_{p;\kappa,\mu}^{(\alpha,\beta)}(x,y) = \int_0^1 t^{x-1}\,(1-t)^{y-1}\,{}_1F_1\Big(\alpha;\beta;-\frac{p}{t^\kappa(1-t)^\mu}\Big)\mathrm{d}t\,,$$

$$(\Re(p),\, \kappa,\, \mu \geq 0;\, \min\{\Re(\alpha),\, \Re(\beta)\} > 0, \Re(x) > -\Re(\kappa\alpha),\, \Re(y) > -\Re(\mu\alpha)).$$

Here
$${}_1F_1(a,b;z) = \sum_{n\geq 0} \frac{(a)_n}{(b)_n}\, \frac{z^n}{n!},$$

stands for the Kummer's function or the confluent hypergeometric function, see [**1**, p. 509][1]. The cases of (1.1) when $\kappa = \mu$ correspond to the generalized hypergeometric function introduced by Parmar [**18**, p. 44]:

$$F_p^{(\alpha,\beta;\mu)}(a,b;c;z) = \sum_{n\geq 0}^{\infty} (a)_n\, \frac{\mathrm{B}_p^{(\alpha,\beta;\mu)}(b+n,\, c-b)}{\mathrm{B}(b,\, c-b)}\, \frac{z^n}{n!},$$

$$(p \geq 0,\, \mu \geq 0;\, \min\{\Re(\alpha),\, \Re(\beta)\} > 0;\, \Re(c) > \Re(b) > 0,\, |z| < 1),$$

which again, in case $\alpha = \beta$ and $\kappa = \mu$ in (1.1), reduces to the definition by Lee *et al.* [**12**]:

$$F_p^{(\mu)}(a,b;c;z) = \sum_{n\geq 0}^{\infty} (a)_n\, \frac{\mathrm{B}_p^{(\mu)}(b+n,\, c-b)}{\mathrm{B}(b,\, c-b)}\, \frac{z^n}{n!},$$

$$(p, \mu \geq 0\,;\, \Re(c) > \Re(b) > 0,\, |z| < 1).$$

Yet another case $\kappa = \mu = 1$ in (1.1) was studied by Özergin *et al.* [**17**]

$$F_p^{(\alpha,\beta)}(a,b;c;z) = \sum_{n\geq 0}^{\infty} (a)_n\, \frac{\mathrm{B}_p^{(\alpha,\beta)}(b+n,\, c-b)}{\mathrm{B}(b,\, c-b)}\, \frac{z^n}{n!},$$

$$(p \geq 0;\, \min\{\Re(\alpha),\, \Re(\beta)\} > 0;\, \Re(c) > \Re(b) > 0,\, |z| < 1).$$

In a recent article, Choi *et al.* [**9**] considered the general Mathieu–type series and their alternating variants whose terms contain the (p,q)–extended Gaussian hypergeometric function $F_{p,q}(z)$, and in turn, when $p = q$ the p–extended Gaussian hypergeometric function $F_p(z)$ and obtained the closed integral form expressions

[1] We point out that there is a wide class of elementary and special functions covered by Kummer's ${}_1F_1$, consult for instance [**1**, pp. 509-510, §13.6. Special cases].

for the considered series and bilateral bounding inequalities. For various other investigations involving other special functions, the interested reader may be referred to several recent papers on the subject (see, for instance [22–28] *and* the references cited therein). Here we are interested in generalizing the integral expressions for the Mathieu–type series and its alternating variants whose terms contain the generalized p–extended Gauss' hypergeometric function $F_p^{(\alpha,\beta;\kappa,\mu)}(z)$ which extends the results of the so–called p–extended Gaussian hypergeometric function $F_p(z)$ recently developed by Choi *et al.* [**9**]. These functions are built by intervention in the kernel replacing the occurring p–extended Beta function by its generalized p–variant $\mathrm{B}_{p;\kappa,\mu}^{(\alpha,\beta)}(x,y)$.

Now, extending the Mathieu–type series studied in [**9**] by introducing the $F_p^{(\alpha,\beta;\kappa,\mu)}(a,b;c;z)$ kernel instead of the $F_p(z)$ in the summands for $p,\ \kappa,\ \mu \geq 0$ and $\min\{\Re(\alpha),\ \Re(\beta)\} > 0$, we define the Mathieu–type **a**–series $\mathfrak{F}_{\lambda,\eta}$ and its alternating variant $\widetilde{\mathfrak{F}}_{\lambda,\eta}$ in the form of the series

$$\mathfrak{F}_{\lambda,\eta}(F_p^{(\alpha,\beta;\kappa,\mu)};\boldsymbol{a};r) := \sum_{n\geq 1} \frac{F_p^{(\alpha,\beta;\kappa,\mu)}\left(\lambda,\, b;\, c;\, -\frac{r^2}{a_n}\right)}{a_n^{\lambda}(a_n+r^2)^{\eta}} \tag{1.3}$$

where $\lambda, \eta, r > 0$; $\Re(c) > \Re(b) > 0$, and in the same range of parameters:

$$\widetilde{\mathfrak{F}}_{\lambda,\eta}(F_p^{(\alpha,\beta;\kappa,\mu)};\boldsymbol{a};r) := \sum_{n\geq 1} \frac{(-1)^{n-1}F_p^{(\alpha,\beta;\kappa,\mu)}\left(\lambda,\, b;\, c;\, -\frac{r^2}{a_n}\right)}{a_n^{\lambda}(a_n+r^2)^{\eta}}\,. \tag{1.4}$$

Here, and in what follows, the real sequence $\boldsymbol{a} = (a_n)_{n\geq 1}$ is the restriction of an increasing function $a\colon \mathbb{R}^+ \mapsto \mathbb{R}^+$ such that $a(x)|_{x\in\mathbb{N}} = \boldsymbol{a}$. The main purposes of this paper are to obtain integral representations and allied bounding inequalities for these functions in the widest possible range of the parameters involved.

2. INTEGRAL REPRESENTATIONS OF $\mathfrak{F}_{\lambda,\eta}(F_P^{(\alpha,\beta;\kappa,\mu)})$ AND $\widetilde{\mathfrak{F}}_{\lambda,\eta}(F_P^{(\alpha,\beta;\kappa,\mu)})$

In this section, we first give the closed integral form expressions for the series $\mathfrak{F}_{\lambda,\eta}(F_p^{(\alpha,\beta;\kappa,\mu)};\boldsymbol{a};r)$ and $\widetilde{\mathfrak{F}}_{\lambda,\eta}(F_p^{(\alpha,\beta;\kappa,\mu)};\boldsymbol{a};r)$ in the form of linear combinations of two principal integrals. Then we list certain special cases of our first main result.

Theorem 1. *Let $\lambda > 0$, $\eta > 0$, $r > 0$ and the real sequence $\boldsymbol{a} = (a_n)_{n\geq 1}$ monotone increases and tends to ∞. Then for $p \geq 0$, $\kappa \geq 0$, $\mu \geq 0$ and $\min\{\Re(\alpha),\ \Re(\beta)\} > 0$, we have*

$$\mathfrak{F}_{\lambda,\eta}(F_p^{(\alpha,\beta;\kappa,\mu)};\boldsymbol{a};r) = \lambda\ \mathscr{I}_p^{(\alpha,\beta;\kappa,\mu)}(\lambda+1,\eta,a_1) + \eta\ \mathscr{I}_p^{(\alpha,\beta;\kappa,\mu)}(\lambda,\eta+1,a_1) \tag{2.5}$$

$$\widetilde{\mathfrak{F}}_{\lambda,\eta}(F_p^{(\alpha,\beta;\kappa,\mu)};\boldsymbol{a};r) = \lambda\ \widetilde{\mathscr{I}}_p^{(\alpha,\beta;\kappa,\mu)}(\lambda+1,\eta,a_1) + \eta\ \widetilde{\mathscr{I}}_p^{(\alpha,\beta;\kappa,\mu)}(\lambda,\eta+1,a_1)\,, \tag{2.6}$$

where for all $\Re(c) > \Re(b) > 0$

$$\mathscr{I}_p^{(\alpha,\beta;\kappa,\mu)}(\lambda,\eta,a_1) = \int_{a_1}^\infty \frac{F_p^{(\alpha,\beta;\kappa,\mu)}\,(\lambda,\, b;\, c;\, -\frac{r^2}{x})\,[a^{-1}(x)]}{x^\lambda (x+r^2)^\eta}\,\mathrm{d}x \tag{2.7}$$

$$\widetilde{\mathscr{I}}_p^{(\alpha,\beta;\kappa,\mu)}(\lambda,\eta,a_1) = \int_{a_1}^\infty \frac{F_p^{(\alpha,\beta;\kappa,\mu)}\,(\lambda,\, b;\, c;\, -\frac{r^2}{x})\,\sin^2\left(\frac{\pi}{2}[a^{-1}(x)]\right)}{x^\lambda (x+r^2)^\eta}\,\mathrm{d}x \tag{2.8}$$

and a^{-1} *denotes the inverse of* a *while* $[a^{-1}]$ *stands for the integer part of* a^{-1}.

Proof. Consider the Laplace transform of the function $t^{\lambda-1}\Phi_p^{(\alpha,\beta;\kappa,\mu)}(b;c;z)$ [**2**, p. 3695, Eq. (9)] by using the definition (1.1). For all real ω it equals

$$F_p^{(\alpha,\beta;\kappa,\mu)}\left(\lambda, b; c; \frac{\omega}{z}\right) = \frac{z^\lambda}{\Gamma(\lambda)} \int_0^\infty \mathrm{e}^{-zt} t^{\lambda-1}\, \Phi_p^{(\alpha,\beta;\kappa,\mu)}(b;c;\omega t)\,\mathrm{d}t. \tag{2.9}$$

Taking $\xi = a_n + r^2$ in the gamma function formula

$$\Gamma(\eta)\,\xi^{-\eta} = \int_0^\infty \mathrm{e}^{-\xi t} t^{\eta-1}\,\mathrm{d}t, \qquad (\Re(\xi), \Re(\eta) > 0),$$

after rearrangement $\omega = -r^2$, $z = a_n$ in (2.9), the integral (2.7) becomes

$$\mathscr{I}_p^{(\alpha,\beta;\kappa,\mu)}(\lambda,\eta) = \int_0^\infty\int_0^\infty \frac{\mathrm{e}^{-r^2 s} t^{\lambda-1} s^{\eta-1}}{\Gamma(\lambda)\Gamma(\eta)} \sum_{n\ge 1} \mathrm{e}^{-a_n(t+s)}\, \Phi_p^{(\alpha,\beta;\kappa,\mu)}(b;c;-r^2 t)\,\mathrm{d}t\,\mathrm{d}s.$$

Using the Cahen formula [**4**] for summing up the resulting Dirichlet series by virtue of the technique developed in [**22, 24**], we conclude

$$\mathcal{D}_a(t+s) = \sum_{n\ge 1} \mathrm{e}^{-a_n(s+t)} = (s+t)\int_{a_1}^\infty e^{-(t+s)x}\,[a^{-1}(x)]\,\mathrm{d}x\,.$$

This gives

$$\begin{aligned} \mathscr{I}_p^{(\alpha,\beta;\kappa,\mu)}(\lambda,\eta) &= \frac{1}{\Gamma(\lambda)\Gamma(\eta)} \int_0^\infty\int_0^\infty\int_{a_1}^\infty \mathrm{e}^{-(r^2+x)s-tx}(t+s)t^{\lambda-1}s^{\eta-1}[a^{-1}(x)] \\ &\quad\times \Phi_p^{(\alpha,\beta;\kappa,\mu)}(b;c;-r^2 t)\,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}x =: \mathcal{I}_t + \mathcal{I}_s\,, \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} \mathcal{I}_t &= \int_0^\infty \left(\int_{a_1}^\infty \left(\int_0^\infty \frac{\Phi_p^{(\alpha,\beta;\kappa,\mu)}(b;c;-r^2 t)}{\Gamma(\eta)\Gamma(\lambda)\mathrm{e}^{xt}}\, t^\lambda\,\mathrm{d}t\right) \mathrm{e}^{-xs}[a^{-1}(x)]\,\mathrm{d}x\right) \mathrm{e}^{-r^2 s} s^{\eta-1}\,\mathrm{d}s \\ &= \frac{\lambda}{\Gamma(\eta)}\int_{a_1}^\infty \left(\int_0^\infty \mathrm{e}^{-(x+r^2)s} s^{\eta-1}\,\mathrm{d}s\right) \frac{[a^{-1}(x)]}{x^{\lambda+1}}\, F_p^{(\alpha,\beta;\kappa,\mu)}\left(\lambda+1,\, b;\, c;\, -\frac{r^2}{x}\right)\mathrm{d}x \\ &= \lambda \int_{a_1}^\infty F_p^{(\alpha,\beta;\kappa,\mu)}\left(\lambda+1,\, b; c; -\frac{r^2}{x}\right) \frac{[a^{-1}(x)]\,\mathrm{d}x}{x^{\lambda+1}(x+r^2)^\eta} = \lambda\, \mathscr{I}_p^{(\alpha,\beta;\kappa,\mu)}(\lambda+1,\eta)\,. \end{aligned} \tag{2.11}$$

In a similar way, we get

$$\mathcal{I}_s = \eta \int_{a_1}^{\infty} \frac{[a^{-1}(x)]}{(x+r^2)^{\eta+1}} \left(\int_0^{\infty} \frac{\mathrm{e}^{-xt} t^{\lambda-1}}{\Gamma(\lambda)} \Phi_p^{(\alpha,\beta;\kappa,\mu)}(b;c;-r^2 t)\, \mathrm{d}t \right) \mathrm{d}x$$

$$= \eta \int_{a_1}^{\infty} \frac{[a^{-1}(x)]}{x^{\lambda}(x+r^2)^{\eta+1}}\, F_p^{(\alpha,\beta;\kappa,\mu)} \left(\lambda,\, b;\, c;\, -\frac{r^2}{x}\right) \mathrm{d}x = \eta\, \mathscr{I}_p^{(\alpha,\beta;\kappa,\mu)}(\lambda, \eta+1). \tag{2.12}$$

Now, applying (2.11) and (2.12) to (2.10) we get the representation (2.5).

The derivation of (2.6) is similar to this proving procedure. As to the alternating Dirichlet series $\mathcal{D}_a(x)$ integral form, having in mind again the Cahen formula, we have [**24, 27**]

$$\widetilde{\mathcal{D}}_a(x) = \sum_{n\geq 1} (-1)^{n-1} \mathrm{e}^{-a_n(x)} = x \int_{a_1}^{\infty} \mathrm{e}^{-xt} \widetilde{A}(t)\, \mathrm{d}t,$$

and therefore

$$\widetilde{\mathcal{D}}_a(x) = x \int_{a_1}^{\infty} \mathrm{e}^{-xt} \sin^2\left(\frac{\pi}{2}[a^{-1}(x)]\right) \mathrm{d}t,$$

since the counting function turns out to be

$$\widetilde{A}(t) = \sum_{n:\; a_n \leq t} (-1)^{n-1} = \frac{1-(-1)^{[a^{-1}(t)]}}{2} = \sin^2\left(\frac{\pi}{2}[a^{-1}(t)]\right).$$

Hence, because

$$\widetilde{\mathcal{D}}_a(t+s) = (t+s) \int_{a_1}^{\infty} \mathrm{e}^{-(t+s)x} \sin^2\left(\frac{\pi}{2}[a^{-1}(t)]\right) \mathrm{d}x,$$

we conclude (2.6) by the obvious remaining steps. □

Now, in the case $\kappa = \mu$, Theorem 1 reduces to the following

Corollary 1.1. *Let $\lambda > 0$, $\eta > 0$, $r > 0$, and let the real sequence $\boldsymbol{a}$ monotone increases and tends to ∞. Then for $p \geq 0$, $\mu \geq 0$ and $\min\{\Re(\alpha), \Re(\beta)\} > 0$, we have*

$$\mathfrak{F}_{\lambda,\eta}(F_p^{(\alpha,\beta;\mu)}; \boldsymbol{a}; r) = \lambda\, \mathscr{I}_p^{(\alpha,\beta;\mu)}(\lambda+1, \eta, a_1) + \eta\, \mathscr{I}_p^{(\alpha,\beta;\mu)}(\lambda, \eta+1, a_1)$$
$$\widetilde{\mathfrak{F}}_{\lambda,\eta}(F_p^{(\alpha,\beta;\mu)}; \boldsymbol{a}; r) = \lambda\, \widetilde{\mathscr{I}}_p^{(\alpha,\beta;\mu)}(\lambda+1, \eta, a_1) + \eta\, \widetilde{\mathscr{I}}_p^{(\alpha,\beta;\mu)}(\lambda, \eta+1, a_1),$$

where

$$\mathscr{I}_p^{(\alpha,\beta;\mu)}(\lambda, \eta, a_1) = \int_{a_1}^{\infty} \frac{F_p^{(\alpha,\beta;\mu)}\left(\lambda,\, b;\, c;\, -\frac{r^2}{x}\right)[a^{-1}(x)]}{x^{\lambda}(x+r^2)^{\eta}}\, \mathrm{d}t$$
$$\widetilde{\mathscr{I}}_p^{(\alpha,\beta;\mu)}(\lambda, \eta, a_1) = \int_{a_1}^{\infty} \frac{F_p^{(\alpha,\beta;\mu)}\left(\lambda,\, b;\, c;\, -\frac{r^2}{x}\right) \sin^2\left(\frac{\pi}{2}[a^{-1}(x)]\right)}{x^{\lambda}(x+r^2)^{\eta}}\, \mathrm{d}t\,.$$

Again, in the case $\alpha = \beta$ and $\kappa = \mu$, Theorem 1 reduces to the following

Corollary 1.2. *Let $\lambda > 0$, $\eta > 0$, $r > 0$, and let the real sequence $\boldsymbol{a}$ monotone increases and tends to ∞. Then for $p \geq 0$, $\mu \geq 0$, we have*

$$\begin{aligned}\mathfrak{F}_{\lambda,\eta}(F_p^{(\mu)};\boldsymbol{a};r) &= \lambda\ \mathscr{I}_p^{(\mu)}(\lambda+1,\eta,a_1) + \eta\ \mathscr{I}_p^{(\mu)}(\lambda,\eta+1,a_1)\\ \widetilde{\mathfrak{F}}_{\lambda,\eta}(F_p^{(\mu)};\boldsymbol{a};r) &= \lambda\ \widetilde{\mathscr{I}}_p^{(\mu)}(\lambda+1,\eta,a_1) + \eta\ \widetilde{\mathscr{I}}_p^{(\mu)}(\lambda,\eta+1,a_1)\,,\end{aligned}$$

where

$$\begin{aligned}\mathscr{I}_p^{(\mu)}(\lambda,\eta,a_1) &= \int_{a_1}^{\infty} \frac{F_p^{(\mu)}\,(\lambda,\,b;\,c;\,-\frac{r^2}{x})\,[a^{-1}(x)]}{x^{\lambda}(x+r^2)^{\eta}}\,\mathrm{d}t\\ \widetilde{\mathscr{I}}_p^{(\mu)}(\lambda,\eta,a_1) &= \int_{a_1}^{\infty} \frac{F_p^{(\mu)}\,(\lambda,\,b;\,c;\,-\frac{r^2}{x})\,\sin^2\big(\frac{\pi}{2}[a^{-1}(x)]\big)}{x^{\lambda}(x+r^2)^{\eta}}\,\mathrm{d}t\,.\end{aligned}$$

Furthermore, in the case $\kappa = \mu = 1$, Theorem 1 becomes

Corollary 1.3. *Let $\lambda > 0$, $\eta > 0$, $r > 0$, and let the real sequence $\boldsymbol{a}$ monotone increases and tends to ∞. Then for $p \geq 0$ and $\min\{\Re(\alpha),\,\Re(\beta)\} > 0$, we have*

$$\begin{aligned}\mathfrak{F}_{\lambda,\eta}(F_p^{(\alpha,\beta)};\boldsymbol{a};r) &= \lambda\ \mathscr{I}_p^{(\alpha,\beta)}(\lambda+1,\eta,a_1) + \eta\ \mathscr{I}_p^{(\alpha,\beta)}(\lambda,\eta+1,a_1)\\ \widetilde{\mathfrak{F}}_{\lambda,\eta}(F_p^{(\alpha,\beta)};\boldsymbol{a};r) &= \lambda\ \widetilde{\mathscr{I}}_p^{(\alpha,\beta)}(\lambda+1,\eta,a_1) + \eta\ \widetilde{\mathscr{I}}_p^{(\alpha,\beta)}(\lambda,\eta+1,a_1)\,,\end{aligned}$$

where

$$\begin{aligned}\mathscr{I}_p^{(\alpha,\beta)}(\lambda,\eta,a_1) &= \int_{a_1}^{\infty} \frac{F_p^{(\alpha,\beta)}\,(\lambda,\,b;\,c;\,-\frac{r^2}{x})\,[a^{-1}(x)]}{x^{\lambda}(x+r^2)^{\eta}}\,\mathrm{d}t\\ \widetilde{\mathscr{I}}_p^{(\alpha,\beta)}(\lambda,\eta,a_1) &= \int_{a_1}^{\infty} \frac{F_p^{(\alpha,\beta)}\,(\lambda,\,b;\,c;\,-\frac{r^2}{x})\,\sin^2\big(\frac{\pi}{2}[a^{-1}(x)]\big)}{x^{\lambda}(x+r^2)^{\eta}}\,\mathrm{d}t\,.\end{aligned}$$

Remark 1. *The special case for $\alpha = \beta$ and $\kappa = \mu = 1$ reduces to the known result for the p–extended Gauss hypergeometric function F_p [**9**]. When $p = 0$ we have the claim of* Theorem 1 *for the Gaussian ${}_2F_1$ which is studied in [**22**].*

3. BOUNDING INEQUALITIES FOR THE P–GENERALIZED MATHIEU–TYPE SERIES

Very recently Luo *et al.* [**14**, Remark 2.6] have established an upper bounds for the generalized p–extended Beta function $\mathrm{B}_{p;\kappa,\mu}^{(\alpha,\beta)}(x,y)$. Namely, we have that for all real parameters $p,\kappa,\mu,\alpha,\beta > 0$, and $x,y > 0$, we have

$$\mathrm{B}_{p;\kappa,\mu}^{(\alpha,\beta)}(x,y) \leq \Omega_{\kappa,\mu}^{\alpha,\beta}(p)\,\mathrm{B}(x,y)\,, \tag{3.13}$$

where

$$\Omega_{\kappa,\mu}^{\alpha,\beta}(p) = {}_1F_1\Big(\alpha;\ \beta;\ -\frac{(\kappa+\mu)^{\kappa+\mu}}{\kappa^\kappa\,\mu^\mu}\,p\Big)\,.$$

We report here the following results [**14**, Corollary 2.7]

$$F_p^{(\alpha,\beta;\kappa,\mu)}(a,b;c;z) \le \Omega_{\kappa,\mu}^{\alpha,\beta}(p)\,{}_2F_1(a,b;c;z) \tag{3.14}$$

$$\Phi_p^{(\alpha,\beta;\kappa,\mu)}(b;c;z) \le \Omega_{\kappa,\mu}^{\alpha,\beta}(p)\,\Phi(b;c;z)\,, \tag{3.15}$$

where $p,\kappa,\mu,\alpha,\beta>0$, and $x,\,y>0$; $c>b>0$ and $|z|<1$. Next, we need also a certain Luke's upper bound exposed in [**13**] for the Gaussian hypergeometric function. Precisely, there holds [**13**, p. 52, Eq. (4.7)]

$$ {}_2F_1(a,b;c;-z) < 1 - \frac{2ab(c+1)}{c(a+1)(b+1)}\Big[1-\frac{2(c+1)}{2(c+1)+(a+1)(b+1)\,z}\Big]\,. \tag{3.16}$$

$$(b\in(0,1],\ c\ge a>0;\ z>0)\,.$$

For the sake of simplicity we introduce the shorthand notation

$$\mathscr{U}_a(\lambda,\eta) := \int_{a_1}^\infty \frac{[a^{-1}(x)]}{x^\lambda(x+r^2)^\eta}\,\mathrm{d}x.$$

In the sequel we consider a class of Mathieu–type series (1.3) and (1.4) in which the defining functions $a\colon \mathbb{R}_+\mapsto\mathbb{R}_+$ behave so, that $\mathscr{U}_a(\lambda,\eta,s)$ converges.

Theorem 2. *Let $\lambda\in(0,1]$, $\eta>0$ and let the real sequence $\boldsymbol{a}=(a_n)_{n\ge1}$ monotone increases and tends to ∞. Then for all $r\in(0,\sqrt{a_1}\,)$, $p,\kappa,\mu,\alpha,\beta>0$ and $c>b>0$, we have*

$$\begin{aligned}\mathfrak{F}_{\lambda,\eta}(F_p^{(\alpha,\beta;\kappa,\mu)};\boldsymbol{a};r) \le \lambda\,\Omega_{\kappa,\mu}^{\alpha,\beta}(p)&\Bigg\{\Big(1-\frac{2(\lambda+1)b(c+1)}{c(\lambda+2)(b+1)}\Big)\,\mathscr{U}_a(\lambda+1,\epsilon)\\ &+\frac{4(\lambda+1)b(c+1)^2\,\mathscr{U}_a(\lambda,\epsilon)}{c(\lambda+2)(b+1)\,[(\lambda+2)(b+1)r^2+2(c+1)a_1]}\Bigg\}\\ &+\eta\,\Omega_{\kappa,\mu}^{\alpha,\beta}(p)\Bigg\{\Big(1-\frac{2\lambda b(c+1)}{c(\lambda+1)(b+1)}\Big)\,\mathscr{U}_a(\lambda,\epsilon+1)\\ &+\frac{4\lambda b(c+1)^2\,\mathscr{U}_a(\lambda-1,\epsilon+1)}{c(\lambda+1)(b+1)\,[(\lambda+1)(b+1)r^2+2(c+1)a_1]}\Bigg\}.\end{aligned}$$

Moreover, for all $\lambda+\eta>1$; $r\in(0,\sqrt{a_1}\,)$, $p,\kappa,\mu,\alpha,\beta>0$ and $c>b>0$ we have

$$\begin{aligned}&\widetilde{\mathfrak{F}}_{\lambda,\eta}(F_p^{(\alpha,\beta;\kappa,\mu)};\boldsymbol{a};r)\\ &\quad\le \lambda\,\Omega_{\kappa,\mu}^{\alpha,\beta}(p)\Bigg\{\Big(1-\frac{2(\lambda+1)b(c+1)}{c(\lambda+2)(b+1)}\Big)\,\frac{a_1^{-\lambda-\eta}}{\lambda+\eta}\,{}_2F_1\Big(\eta,\lambda+\eta;\eta+1;-\frac{r^2}{a_1}\Big)\end{aligned}$$

$$
\begin{aligned}
&+ \frac{4(\lambda+1)b(c+1)^2}{c(\lambda+2)(b+1)} \frac{a_1^{1-\lambda-\eta}\, {}_2F_1\Big(\eta, \lambda+\eta-1; \eta+1; -\frac{r^2}{a_1}\Big)}{(\lambda+\eta-1)[(\lambda+2)(b+1)r^2+2(c+1)a_1]} \Bigg\} \\
&+ \eta\, \Omega_{\kappa,\mu}^{\alpha,\beta}(p) \Bigg\{ \Bigg(1 - \frac{2\lambda b(c+1)}{c(\lambda+1)(b+1)}\Bigg) \frac{a_1^{-\lambda-\eta}}{\lambda+\eta}\, {}_2F_1\Big(\eta+1, \lambda+\eta; \eta+2; -\frac{r^2}{a_1}\Big) \\
&+ \frac{4\lambda b(c+1)^2}{c(\lambda+1)(b+1)} \frac{a_1^{1-\lambda-\eta}\, {}_2F_1\Big(\eta+1, \lambda+\eta-1; \eta+2; -\frac{r^2}{a_1}\Big)}{(\lambda+\eta-1)[(\lambda+1)(b+1)r^2+2(c+1)a_1]} \Bigg\} .
\end{aligned} \tag{3.17}
$$

Proof. One starts with the relation (2.5)

$$
\mathfrak{F}_{\lambda,\eta}(F_p^{(\alpha,\beta;\kappa,\mu)}; \boldsymbol{a}; r) = \lambda\, \mathscr{I}_p(\lambda+1, \eta, a_1) + \eta\, \mathscr{I}_p(\lambda, \eta+1, a_1)\,,
$$

in which we bound from above the auxiliary integral $\mathscr{I}_p$ described in (2.7). To do this we quote that $F_p^{(\alpha,\beta;\kappa,\mu)}(a,b;c;z) > 0$ for all $a,b,c>0$ and all negative values of z, compare (3.16). Indeed, it is enough to consider the integral expression (2.7) by making use of the generalized p–extended Beta function $\mathrm{B}_{p;\kappa,\mu}^{(\alpha,\beta)}(x,y)$ where now the parameters involved become real in the definition (1.1)

$$
F_p^{(\alpha,\beta;\kappa,\mu)}(a,b;c;z) = \frac{1}{\mathrm{B}(b,c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-zt)^a}\, {}_1F_1\Big(\alpha;\beta; -\frac{p}{t^\kappa(1-t)^\mu}\Big)\, \mathrm{d}t\,.
$$

More precisely here and in what follows the parameters p, κ, $\mu \ge 0$; $\min\{\alpha,\beta\} > 0$; $c > b > 0$ and $-1 < z < 1$. Therefore, by virtue of (3.14) and (3.16) it follows

$$
\begin{aligned}
\mathscr{I}_p(\lambda,\eta,a_1) &= \int_{a_1}^\infty \frac{F_p^{(\alpha,\beta;\kappa,\mu)}\,(\lambda,\, b;\, c;\, -\frac{r^2}{x})[a^{-1}(x)]}{x^\lambda(x+r^2)^\eta}\, \mathrm{d}x \\
&\le \Omega_{\kappa,\mu}^{\alpha,\beta}(p) \int_{a_1}^\infty \frac{{}_2F_1\,(\lambda,\, b;\, c;\, -\frac{r^2}{x})[a^{-1}(x)]}{x^\lambda(x+r^2)^\eta}\, \mathrm{d}x \\
&\le \Omega_{\kappa,\mu}^{\alpha,\beta}(p) \Bigg\{ \Bigg(1 - \frac{2\lambda b(c+1)}{c(\lambda+1)(b+1)}\Bigg) \int_{a_1}^\infty \frac{[a^{-1}(x)]}{x^\lambda(x+r^2)^\eta}\, \mathrm{d}x \\
&\qquad + \frac{4\lambda b(c+1)^2}{c(\lambda+1)(b+1)} \int_{a_1}^\infty \frac{x^{1-\lambda}\,[a^{-1}(x)]\, \mathrm{d}x}{(x+r^2)^\eta\,[(\lambda+1)(b+1)r^2+2(c+1)x]} \Bigg\} \\
&\le \Omega_{\kappa,\mu}^{\alpha,\beta}(p) \Bigg\{ \Bigg(1 - \frac{2\lambda b(c+1)}{c(\lambda+1)(b+1)}\Bigg) \mathscr{U}_a(\lambda,\epsilon) \\
&\qquad + \frac{4\lambda b(c+1)^2\, \mathscr{U}_a(\lambda-1,\epsilon)}{c(\lambda+1)(b+1)\,[(\lambda+1)(b+1)r^2+2(c+1)a_1]} \Bigg\} .
\end{aligned}
$$

The rest is obvious. Next, we recall (2.6):

$$
\widetilde{\mathfrak{F}}_{\lambda,\eta}(F_p^{(\alpha,\beta;\kappa,\mu)}; \boldsymbol{a}; r) = \lambda\, \widetilde{\mathscr{I}_p}(\lambda+1, \eta, a_1) + \eta\, \widetilde{\mathscr{I}_p}(\lambda, \eta+1, a_1)\,.
$$

By the positivity of the integrand of (2.8) and in view of (3.14) we have

$$\widetilde{\mathscr{I}_p}(\lambda,\eta) \le \int_{a_1}^\infty \frac{F_p^{(\alpha,\beta;\kappa,\mu)}\left(\lambda,\, b;\, c;\, -\frac{r^2}{x}\right)}{x^\lambda(x+r^2)^\eta}\,\mathrm{d}x \le \Omega_{\kappa,\mu}^{\alpha,\beta}(p)\int_{a_1}^\infty \frac{{}_2F_1\left(\lambda,\, b;\, c;\, -\frac{r^2}{x}\right)}{x^\lambda(x+r^2)^\eta}\,\mathrm{d}x\,.$$

In turn, with the aid of (3.16) we conclude

$$\widetilde{\mathscr{I}_p}(\lambda,\eta) \le \Omega_{\kappa,\mu}^{\alpha,\beta}(p)\left\{\left(1-\frac{2\lambda b(c+1)}{c(\lambda+1)(b+1)}\right)\int_{a_1}^\infty \frac{\mathrm{d}x}{x^\lambda(x+r^2)^\eta}\right.$$
$$\left. + \frac{4\lambda b(c+1)^2}{c(\lambda+1)(b+1)}\int_{a_1}^\infty \frac{\mathrm{d}x}{x^{\lambda-1}(x+r^2)^\eta\,[(\lambda+1)(b+1)r^2+2(c+1)x]}\right\}.$$

Using [**10**, p. 313, Eq. **3.194** 1.] for $\lambda+\eta>1$ we have

$$\int_{a_1}^\infty \frac{\mathrm{d}x}{x^\lambda(x+r^2)^\eta} = \int_0^{\frac{1}{a_1}} \frac{t^{\lambda+\eta-2}}{(1+r^2t)^\eta}\,\mathrm{d}t = \frac{a_1^{1-\lambda-\eta}}{\lambda+\eta-1}\,{}_2F_1\Big(\eta,\lambda+\eta-1;\eta+1;-\frac{r^2}{a_1}\Big)\,,$$

which for $\lambda+\eta>2$ implies

$$\int_{a_1}^\infty \frac{\mathrm{d}x}{x^{\lambda-1}(x+r^2)^\eta\,[(\lambda+1)(b+1)r^2+2(c+1)x]}$$
$$\le \frac{a_1^{2-\lambda-\eta}\,{}_2F_1\Big(\eta,\lambda+\eta-2;\eta+1;-\frac{r^2}{a_1}\Big)}{(\lambda+\eta-2)[(\lambda+1)(b+1)r^2+2(c+1)a_1]}\,.$$

Collecting these formulae we get the upper bound

$$\widetilde{\mathscr{I}_p}(\lambda,\eta) \le \Omega_{\kappa,\mu}^{\alpha,\beta}(p)\left\{\left(1-\frac{2\lambda b(c+1)}{c(\lambda+1)(b+1)}\right)\frac{1}{a_1^{\lambda+\eta-1}\,(\lambda+\eta-1)}\right.$$
$$\times\,{}_2F_1\Big(\eta,\lambda+\eta-1;\eta+1;-\frac{r^2}{a_1}\Big)$$
$$\left. + \frac{4\lambda b(c+1)^2}{c(\lambda+1)(b+1)}\,\frac{a_1^{2-\lambda-\eta}\,{}_2F_1\Big(\eta,\lambda+\eta-2;\eta+1;-\frac{r^2}{a_1}\Big)}{(\lambda+\eta-2)[(\lambda+1)(b+1)r^2+2(c+1)a_1]}\right\}.$$

Now, obvious steps lead to the asserted upper bound (3.17). □

Remark 2. *Specifying the parameters in* (3.13), *we arrive at corollaries of* Theorem 2. *However, the upper bound expressions for related Mathieu–type series and its alternating variants* $\mathfrak{F}_{\lambda,\eta}(F_p^{(\alpha,\beta;\mu)};\boldsymbol{a};r)$, $\widetilde{\mathfrak{F}}_{\lambda,\eta}(F_p^{(\alpha,\beta;\mu)};\boldsymbol{a};r)$, $\mathfrak{F}_{\lambda,\eta}(F_p^{(\mu)};\boldsymbol{a};r)$, $\widetilde{\mathfrak{F}}_{\lambda,\eta}(F_p^{(\mu)};\boldsymbol{a};r)$ *and* $\mathfrak{F}_{\lambda,\eta}(F_p^{(\alpha,\beta)};\boldsymbol{a};r)$, $\widetilde{\mathfrak{F}}_{\lambda,\eta}(F_p^{(\alpha,\beta)};\boldsymbol{a};r)$ *we leave to the interested reader.*

4. DISCUSSION

Our research methodology is based on the following steps. We consider a Mathieu–type series with terms containing special functions (Gaussian hypergeometric function [**22**], Fox–Wright generalized hypergeometric ${}_p\Psi_q$ function [**26**], generalized hypergeometric function ${}_pF_q$ and Meijer G function [**27**], Fox's H function [**24**]) which *all* possess integral representations. The parameters and the constitutional coefficients families permit either series convergence and summation – integration interchange. *Mutatis mutandis*, by this procedure the 'inner' Dirichlet–series sum in the integrand becomes summable. Moreover, with the help of the Cahen–formula we next deduce a Laplace–integral expression; for instance, displays (2.7), (2.8) in Theorem 1 are illustrative examples. Finally, the resulting integrand's structure enables to construct contiguous relations by the related output integrals, see (2.5) and (2.6).

It is worth to mention the Mathieu–type series of more general structure like Mathieu $(\boldsymbol{a}, \boldsymbol{\lambda})$–series introduced by Pogány in [**23**] and the $\boldsymbol{a}$–series in [**28**] which integral expressions were obtained by similar derivation process.

An open problem can be posed concerning the existence of a generic (appropriately convergent) power series instead of $F_p^{(\alpha,\beta;\kappa,\mu)}$ precised by (1.1) in (1.3) and subsequently in (1.4) which use could lead to general formulae similar to (2.7), (2.8). By these efforts the re–formulated results in terms of generic series $S(x) = \sum_n g_n x^n$, say, would contain among others the case of $F_p^{(\alpha,\beta;\kappa,\mu)}$ considered in Theorem 1 as an obvious corollary. Unfortunately, the use of a generic power series needs at least the highly strong assumption by which $S(x)$ should have an suitable integral expression by which we could follow our consideration methodology exposed above.

However, these goals heavily overgrow the purposes and tasks of the recent article, we leave it for another address.

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