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HYPERBOLIC METRIC ON THE STRIP AND THE SCHWARZ LEMMA FOR HQR MAPPINGS

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We give simple proofs of various versions of the Schwarz lemma for real valued harmonic functions and for holomorphic (more generally harmonic quasiregular, shortly HQR) mappings with the strip codomain. Along the way, we get a simple proof of a new version of the Schwarz lemma for real valued harmonic functions (without the assumption that 0 is mapped to 0 by the corresponding map). Using the Schwarz-Pick lemma related to distortion for harmonic functions and the elementary properties of the hyperbolic geometry of the strip we get optimal estimates for modulus of HQR mappings.

1. INTRODUCTION AND PRELIMINARIES

Motivated by the role of the Schwarz lemma in complex analysis and numerous fundamental results (see [**1, 32, 14, 25**] and references cited there and for some recent result which are in our research direction [**2, 13, 15, 20, 36**]), in 2016, cf. [**33**](a), the first author has posted the current research project "Schwarz lemma, the Carathéodory and Kobayashi Metrics and Applications in Complex Analysis"[*]. Various discussions regarding the subject can also be found in the Q&A section on ResearchGate under the question "What are the most recent versions of the Schwarz lemma ?",[**33**](b)[†]. In this project and in [**25**], cf. also [**13**] we developed the method related to holomorphic mappings with strip codomain (we refer to this

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[*]Motivated by S. G. Krantz paper [**14**].
[†]The subject has been presented at Belgrade analysis seminar [**27**].

method as the approach via the Schwarz-Pick lemma for holomorphic maps from the unit disc into a strip); see also [**28, 29**]. Note here that our use of terms the Schwarz lemma and the Schwarz-Pick lemma is refer to the corresponding versions for modulus and hyperbolic distances, respectively (we follow the terminology used in [**3**]).

In particular our work here is related to previous works M. Mateljević [**22**], M. Knežević and M. Mateljević [**19**], X. Chen and A. Fang [**6**], some recent results of D. Kalaj and M. Vuorinen [**13**] (shortly KV-results; see also D. Khavinson [**16**], G. Kresin and V. Maz'ya [**17**]) and S. Chen and D. Kalaj [**5**]. As we mentioned in [**25**], it seems that KV-results influenced further research by H. Chen [**4**], M. Marković [**20**], A. Khalfallah [**15**] and P. Melentijević [**31**].

One of the purpose of this paper, which is a relatively elementary contribution and continuation of these research, is to demonstrate our approach and make a common frame for previous works.

Throughout this paper by $\mathbb{U}$ we denote the unit disc $\{z \in \mathbb{C} : |z| < 1\}$. By the Riemann mapping theorem simply connected plane domains different from $\mathbb{C}$ (we call these domains hyperbolic) are conformally equivalent to $\mathbb{U}$. Accordingly if Ω is a hyperbolic domain then by $\rho_\Omega(z)|dz|$ we denote the hyperbolic metric of Ω. This metric induces a hyperbolic distance on Ω in the following way

$$d_\Omega(z_1, z_2) = \inf \int_\gamma \rho_\Omega(z)|dz|,$$

where the infimum is taken over all C^1 curves γ joining z_1 to z_2 in Ω.

It is well known that $\rho_{\mathbb{U}}(z)|dz| = \dfrac{2|dz|}{1-|z|^2}$ and immediately follows that for all $z_1, z_2 \in \mathbb{U}$ it holds

$$d_{\mathbb{U}}(z_1, z_2) = \ln \frac{1 + \sigma_{\mathbb{U}}(z_1, z_2)}{1 - \sigma_{\mathbb{U}}(z_1, z_2)} = 2 \operatorname{artanh} \sigma_{\mathbb{U}}(z_1, z_2),$$

where the pseudo-hyperbolic distance $\sigma_{\mathbb{U}}$ is given by $\sigma_{\mathbb{U}}(z_1, z_2) = \left|\dfrac{z_1 - z_2}{1 - z_1\overline{z_2}}\right|$.

If f is a conformal map from hyperbolic domain Ω onto $\mathbb{U}$ then the hyperbolic metric $\rho_\Omega(z)|dz|$ of Ω is defined by $\rho_\Omega(z)|dz| = \rho_{\mathbb{U}}(f(z))|f'(z)||dz|$. Hence, one can transfer the concept of the hyperbolic distance from $\mathbb{U}$ on hyperbolic domain Ω.

For more details related to hyperbolic domains, hyperbolic metric and distance, see, for example [**1, 3, 26**].

In this paper, except the disc $\mathbb{U}$, of other hyperbolic domains we will mainly use the strip $\mathbb{S} = \{z \in \mathbb{C} : -1 < \operatorname{Re} z < 1\}$.

Let D, G be domains in $\mathbb{C}$. By $\operatorname{Hol}(D, G)$ (respectively $\operatorname{Har}(D, G)$) we denote the set of the all holomorphic (respectively harmonic) mappings $f : D \to G$.

By d_e we denote Euclidean distance in $\mathbb{C}$ and for $z \in \mathbb{C}$ we define the functions e and R_e, by $e(z) = d_e(0, z) = |z|$ and $R_e(z) = \operatorname{Re} z$, respectively.

If $0 < r < 1$ then by U_r we denote Euclidean disc $\{z \in \mathbb{C} : |z| < r\}$ and by $\overline{U}_r$ we denote the corresponding closed disc.

For completeness we first give the classical Schwarz lemma which is a direct corollary of maximum modulus principle.

Theorem 1 (The classical Schwarz lemma - the Schwarz lemma for holomorphic maps from $\mathbb{U}$ into $\mathbb{U}$). *Let $f \in \mathrm{Hol}(\mathbb{U},\mathbb{U})$ and $f(0)=0$. Then*

$$|f(z)| \leqslant |z|, \quad \text{for all} \quad z \in \mathbb{U} \tag{1}$$

and

$$|f'(0)| \leqslant 1. \tag{2}$$

In (1) the equality holds for one $z \in \mathbb{U}-\{0\}$ and in (2) the equality holds if and only if $f(z)=\alpha z$, where $\alpha \in \mathbb{C}$ such that $|\alpha|=1$.

The following theorem is known as the Schwarz lemma for harmonic maps from $\mathbb{U}$ into itself.

Theorem 2 (The Schwarz lemma for harmonic maps from $\mathbb{U}$ into $\mathbb{U}$, [**11**], [**8**, p.77]). *Let $f \in \mathrm{Har}(\mathbb{U},\mathbb{U})$ and $f(0)=0$. Then*

$$|f(z)| \leqslant \frac{4}{\pi}\arctan|z|, \quad \text{for all} \quad z \in \mathbb{U}, \tag{3}$$

and this inequality is sharp for each point $z \in \mathbb{U}$.

In the literature this result is often attributed to E. Heinz [**11**]. Later, in 1977, H. W. Hethcote [**12**] improved the above result by removing the assumption $f(0)=0$ and showed the following:

Theorem 3 ([**12**, Theorem 1]). *Let $f \in \mathrm{Har}(\mathbb{U},\mathbb{U})$. Then*

$$\left| f(z) - \frac{1-|z|^2}{1+|z|^2} f(0) \right| \leqslant \frac{4}{\pi}\arctan|z|, \quad \text{for all} \quad z \in \mathbb{U}.$$

It seems that the researchers have overlooked H. W. Hethcote result and they have had some difficulties to handle the case $f(0) \neq 0$ in this context; see [**30**]. By our method, we get a simple proof of an optimal version of the Schwarz lemma for real valued harmonic functions (without the assumption that 0 is mapped to 0 by the corresponding map), see Theorem 6 which improves H. W. Hethcote result.

Using the concept of the hyperbolic metric and hyperbolic distance on hyperbolic domains one can derive the Schwarz-Pick lemma for simply connected domains as a corollary of the classical Schwarz lemma:

Theorem 4 (The Schwarz-Pick lemma for simply connected domains, [**3**, Theorem 6.4.]). *Let Ω_1 and Ω_2 be hyperbolic domains and $f \in \mathrm{Hol}(\Omega_1,\Omega_2)$. Then*

$$\rho_{\Omega_2}(f(z))|f'(z)| \leqslant \rho_{\Omega_1}(z), \quad \text{for all} \quad z \in \Omega_1 \tag{4}$$

and

$$d_{\Omega_2}(f(z_1), f(z_2)) \leqslant d_{\Omega_1}(z_1, z_2), \quad \textit{for all} \quad z_1, z_2 \in \Omega_1. \tag{5}$$

In (4) and (5) the equalities hold if and only if f is a conformal isomorphism from Ω_1 into Ω_2.

In this paper we will use only the special case of this result if the domain is the unit disc and the codomain is the strip.

In our consideration also important role plays the so-called subordination principle. In order to formulate the corresponding version of subordination principle it is convenient first to give the following definition.

Let $f, g \in \mathrm{Hol}(\mathbb{U}, \mathbb{C})$. Function f is said to be subordinate to g in $\mathbb{U}$ if there exists a function $w \in \mathrm{Hol}(\mathbb{U}, \mathbb{U})$ and $w(0) = 0$ such that

$$f(z) = g(w(z)), \quad \text{for all} \quad z \in \mathbb{U}. \tag{6}$$

If the function g is univalent in $\mathbb{U}$, then (6) is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}). \tag{7}$$

Proposition 1 (Subordination principle [**9**, p. 368-369])**.** *Let $f, g \in \mathrm{Hol}(\mathbb{U}, \mathbb{C})$. If g is univalent and if conditions (7) hold, then $f(\overline{U}_r) \subset g(\overline{U}_r)$ for all $0 < r < 1$ and $|f'(0)| \leqslant |g'(0)|$.*

In Example 1 we consider the conformal mapping ϕ from the unit disc $\mathbb{U}$ onto the strip $\mathbb{S}$. In Lemmas 1 and 2 we explicitly find the maximum and minimum of the function R_e and the maximum of the function e on the closed hyperbolic disc in $\mathbb{S}$ which is obtained as image of the closed hyperbolic disc with center 0 in $\mathbb{U}$ by mapping ϕ. Note that the proof of Lemma 1 is elementary and it is based on the properties of mapping ϕ. The proof of Lemma 2 is based on Proposition 3 which gives us an interesting relation between the hyperbolic and Euclidean distance on $\mathbb{S}$. Theorem 5 follows directly from the formula (15) of Lemma 1 and the subordination principle. Further development of this method yields Theorem 6 (without hypothesis that 0 is mapped to 0) which seems to be a new result (see also Example 3 and Lemma 3).

It seems here that it is right place to emphasize the following difference between holomorphic and harmonic maps.

If f is holomorphic mapping from $\mathbb{U}$ into itself such that $f(0) = b$, where $b \in \mathbb{U}$, then using the mapping $f^b = \varphi_b \circ f$, where φ_b is conformal automorphism of $\mathbb{U}$ (see Example 2 below), we reduce this situation to the case $b = 0$, since $f^b(0) = 0$. As far as we know the researchers have some difficulties to handle the case $f(0) = b$ if f is harmonic mapping from $\mathbb{U}$ into $(-1, 1)$, since in that case the mapping f^b is not harmonic in general. Our method overcome this difficulty.

To get optimal estimate for modulus of holomorphic (more generally HQR) mappings we use the elementary properties of the hyperbolic geometry of the strip, see Lemmas 2 and 4, and Theorems 7 and 8.

In order to establish Theorem 8 (the Schwarz lemma for HQR maps from $\mathbb{U}$ into $\mathbb{S}$) among other things we will use the following elementary considerations (see [**25**]):

(I) Suppose that $f \in \mathrm{Hol}(\mathbb{U},\mathbb{S})$. Then by Theorem 4 we have $\rho_{\mathbb{S}}(f(z))|f'(z)| \leqslant \rho_{\mathbb{U}}(z)$, for all $z \in \mathbb{U}$.

(II) If $f = u + iv$ is a complex valued harmonic and $F = U + iV$ is a holomorphic function on a domain D such that $\mathrm{Re}\, f = \mathrm{Re}\, F$ on D (in this setting we say that F is associated to f or to u), then $F' = U_x + iV_x = U_x - iU_y = u_x - iu_y$. Hence, if $\nabla u = (u_x, u_y) = u_x + iu_y$ then $F' = \overline{\nabla u}$ and $|F'| = |\overline{\nabla u}| = |\nabla u|$.

(III) Suppose that D is a simply connected plane domain and $f : D \to \mathbb{S}$ is a complex valued harmonic function. Then it is known from the standard course of complex analysis that there is a holomorphic function F on D such that $\mathrm{Re}\, f = \mathrm{Re}\, F$ on D, and it is clear that $F : D \to \mathbb{S}$.

(IV) The hyperbolic density $\rho_{\mathbb{S}}$ at point z depends only on $\mathrm{Re}\, z$.

By (I)-(IV) it is readable that we have

Proposition 2 ([**25**, Proposition 2.4], [**13, 4**]). *Let $u : \mathbb{U} \to (-1,1)$ be harmonic function and let F be holomorphic function which is associated to u. Then*

$$\rho_{\mathbb{S}}(u(z))|\nabla u(z)| = \rho_{\mathbb{S}}(F(z))|F'(z)| \leqslant \rho_{\mathbb{U}}(z) \quad \textit{for all} \quad z \in \mathbb{U}. \tag{8}$$

Note the above simple method described by the properties (I)-(IV) is basically based on the Schwarz-Pick lemma for holomorphic maps from $\mathbb{U}$ into $\mathbb{S}$ and it yields a proof of the above proposition to which we refer as the Schwarz-Pick lemma related to distortion for harmonic functions from $\mathbb{U}$ into $(-1,1)$. It is convenient that the properties (I)-(IV) together with Proposition 2 we shortly call the strip property of harmonic functions and refer to it as the *strip method*. By this proposition we control distortion of HQR mappings (see the proof of Lemma 4 below).

Note here that there is tightly connection between the subordination principle and the various versions of the Schwarz-Pick lemma for holomorphic maps from $\mathbb{U}$ into $\mathbb{S}$. Namely in the proof of Theorem 5 and Theorem 7 (the Schwarz lemma for holomorphic maps from $\mathbb{U}$ into $\mathbb{S}$) we have used a corollary of the subordination principle which can be stated in the form (see Definition 1 for notation):

(V) If $f \in \mathrm{Hol}(\mathbb{U},\mathbb{S})$ and $a \in \mathbb{U}$, then the image of the hyperbolic disc in $\mathbb{U}$ with hyperbolic center a and hyperbolic radius λ under f is in the hyperbolic disc in $\mathbb{S}$ with hyperbolic center at $f(a)$ and hyperbolic radius λ.

Note that (V) is the Schwarz-Pick lemma for holomorphic maps from $\mathbb{U}$ into $\mathbb{S}$. Instead of subordination principle in the proof of Theorem 8 (the Schwarz lemma for HQR maps from $\mathbb{U}$ into $\mathbb{S}$) we use Lemma 4, which can be consider as a generalization of (V):

(VI) If $f \in \mathrm{HQR}_K(\mathbb{U}, \mathbb{S})$ (see definition below) and $a \in \mathbb{U}$, then the image of the hyperbolic disc in $\mathbb{U}$ with hyperbolic center a and hyperbolic radius λ under f is in the hyperbolic disc in $\mathbb{S}$ with hyperbolic center $f(a)$ and hyperbolic radius $K\lambda$.

Note that proof of Lemma 4 is based on Proposition 2 (the Schwarz-Pick lemma related to distortion for harmonic functions from $\mathbb{U}$ into $(-1, 1)$).

2. SOME EXAMPLES AND SOME PROPERTIES OF THE STRIP

First of all we consider mappings related to extremal mappings.

Example 1. *Let φ be the mapping defined by $\varphi(z) = \tan\left(\frac{\pi}{4}z\right)$. It is easy to check that the mapping φ is holomorphic and injective on $\mathbb{S}$ and maps $\mathbb{S}$ onto $\mathbb{U}$. Therefore the inverse mapping of the φ maps $\mathbb{U}$ onto $\mathbb{S}$. Denote that inverse mapping by ϕ.*

Let's emphasize that throughout this text by φ and ϕ we always denote the mappings defined in Example 1.

Example 2. *For $a \in \mathbb{U}$, define $\varphi_a(z) = \dfrac{a+z}{1+\overline{a}z}$. It is well known that φ_a is a conformal automorphism of $\mathbb{U}$. Specially, for $a \in (-1, 1)$, the mapping φ_a has the following properties:*

i) it is increasing on $(-1, 1)$ and maps $(-1, 1)$ onto itself;

ii) for $r \in [0, 1)$ it holds $\varphi_a([-r, r]) = [\varphi_a(-r), \varphi_a(r)] = \left[\dfrac{a-r}{1-ar}, \dfrac{a+r}{1+ar}\right]$.

Example 3. *Let $b \in \mathbb{S}$ be arbitrary and let ϕ_b be conformal isomorphism from $\mathbb{U}$ onto $\mathbb{S}$ such that $\phi_b(0) = b$ and $\phi_b'(0) > 0$. It is straightforward to check that $\phi_b = \phi \circ \varphi_a$, where $a = \tan\dfrac{b\pi}{4}$ and φ_a is defined in Example 2. Specially, for $b \in (-1, 1)$, the mapping ϕ_b has the following properties:*

i) it is increasing on $(-1, 1)$ and maps $(-1, 1)$ onto itself;

ii) for $r \in [0, 1)$ it holds $\phi_b([-r, r]) = [m_b(r), M_b(r)]$, where $m_b(r) = \phi_b(-r) = \dfrac{4}{\pi}\arctan\dfrac{a-r}{1-ar}$ and $M_b(r) = \phi_b(r) = \dfrac{4}{\pi}\arctan\dfrac{a+r}{1+ar}$.

Since φ is a conformal isomorphism from $\mathbb{S}$ into $\mathbb{U}$ a simple computation gives

$$\rho_{\mathbb{S}}(z) = \rho_{\mathbb{U}}(\varphi(z))|\varphi'(z)| = \frac{\pi}{2}\frac{1}{\cos\left(\frac{\pi}{2}\operatorname{Re} z\right)}, \quad \text{for all} \quad z \in \mathbb{S}.$$

Hence since

$$0 < \cos\left(\frac{\pi}{2}\operatorname{Re} z\right) \leqslant 1, \quad \text{for all} \quad z \in \mathbb{S},$$

we have simple but useful corollary that

$$\rho_{\mathbb{S}}(z) \geqslant \frac{\pi}{2}, \quad \text{for all} \quad z \in \mathbb{S}. \tag{9}$$

The following proposition gives us an interesting relation between the hyperbolic distance $d_{\mathbb{S}}$ and the Euclidean distance d_e. It turns out that this relation is crucial for some of our investigation (see Lemma 2 and Theorems 7 and 8 below).

Proposition 3. *Let $z_1, z_2 \in \mathbb{S}$. Then*

$$d_{\mathbb{S}}(z_1, z_2) \geqslant \frac{\pi}{2} d_e(z_1, z_2). \tag{10}$$

If z_1, z_2 are pure imaginary numbers then in (10) the equality holds.

Proof. Let γ be a C^1 curve such that joining z_1 to z_2 in $\mathbb{S}$. Since $\rho_{\mathbb{S}}(z) \geqslant \frac{\pi}{2}$ for all $z \in \mathbb{S}$, it follows that

$$\int_\gamma \rho_{\mathbb{S}}(z)|dz| \geqslant \frac{\pi}{2} \int_\gamma |dz|. \tag{11}$$

In other words the hyperbolic length of the curve γ is great or equal to product of $\frac{\pi}{2}$ and Euclidean length of the curve γ. Since Euclidean length of the curve γ is great or equal to $d_e(z_1, z_2)$ according to the inequality (11), we have

$$\int_\gamma \rho_{\mathbb{S}}(z)|dz| \geqslant \frac{\pi}{2} d_e(z_1, z_2). \tag{12}$$

Take in (12) infimum over all C^1 curves γ joining z_1 to z_2 in $\mathbb{S}$ we obtain

$$d_{\mathbb{S}}(z_1, z_2) \geqslant \frac{\pi}{2} d_e(z_1, z_2).$$

On other hand, let $y_1, y_2 \in \mathbb{R}$ be arbitrary and let $\widehat{\gamma} : [0, 1] \to \mathbb{S}$ be defined by $\widehat{\gamma}(t) = iy_1 + i(y_2 - y_1)t$. An easy computation shows that

$$\int_{\widehat{\gamma}} \rho_{\mathbb{S}}(z)|dz| = \frac{\pi}{2}|y_2 - y_1| = \frac{\pi}{2} d_e(iy_1, iy_2). \tag{13}$$

□

3. EUCLIDEAN PROPERTIES OF HYPERBOLIC DISCS

Definition 1. *Let $\lambda > 0$ be arbitrary. By $D_\lambda(a)$ (respectively $S_\lambda(b)$) we denote the hyperbolic disc in $\mathbb{U}$ (respectively in $\mathbb{S}$) with hyperbolic center $a \in \mathbb{U}$ (respectively $b \in \mathbb{S}$) and hyperbolic radius λ. More precisely $D_\lambda(a) = \{z \in \mathbb{U} : d_{\mathbb{U}}(z, a) < \lambda\}$ and $S_\lambda(b) = \{z \in \mathbb{S} : d_{\mathbb{S}}(z, b) < \lambda\}$. Also, $\overline{D}_\lambda(a) = \{z \in \mathbb{U} : d_{\mathbb{U}}(z, a) \leqslant \lambda\}$ and $\overline{S}_\lambda(b) = \{z \in \mathbb{S} : d_{\mathbb{S}}(z, b) \leqslant \lambda\}$ are corresponding closed discs. Specially, if $a = 0$ (respectively $b = 0$) we omit a (respectively b) from the notations.*

Remark 1. *If f is a conformal isomorphism from $\mathbb{U}$ onto $\mathbb{S}$ such that $f(a) = b$ then $f(D_\lambda(a)) = S_\lambda(b)$ and $f(\overline{D}_\lambda(a)) = \overline{S}_\lambda(b)$.*

Let $r \in (0,1)$ be arbitrary and let

$$\lambda(r) = d_{\mathbb{U}}(r,0) = \ln\frac{1+r}{1-r} = 2\,\mathrm{artanh}\, r.$$

Since $d_{\mathbb{U}}(z,0) = \ln\dfrac{1+|z|}{1-|z|} = 2\,\mathrm{artanh}\,|z|$ for all $z \in \mathbb{U}$, we have

$$D_{\lambda(r)} = \{z \in \mathbb{C} : 2\,\mathrm{artanh}\,|z| < 2\,\mathrm{artanh}\, r\} = \{z \in \mathbb{C} : |z| < r\} = U_r,$$

and similarly

$$\overline{D}_{\lambda(r)} = \overline{U}_r.$$

The closed discs $\overline{D}_{\lambda(r)}$ and $\overline{S}_{\lambda(r)}$ are shown on the Figure 1 and the following lemma states that disc $\overline{S}_{\lambda(r)}$ is contained in a Euclidean rectangle.

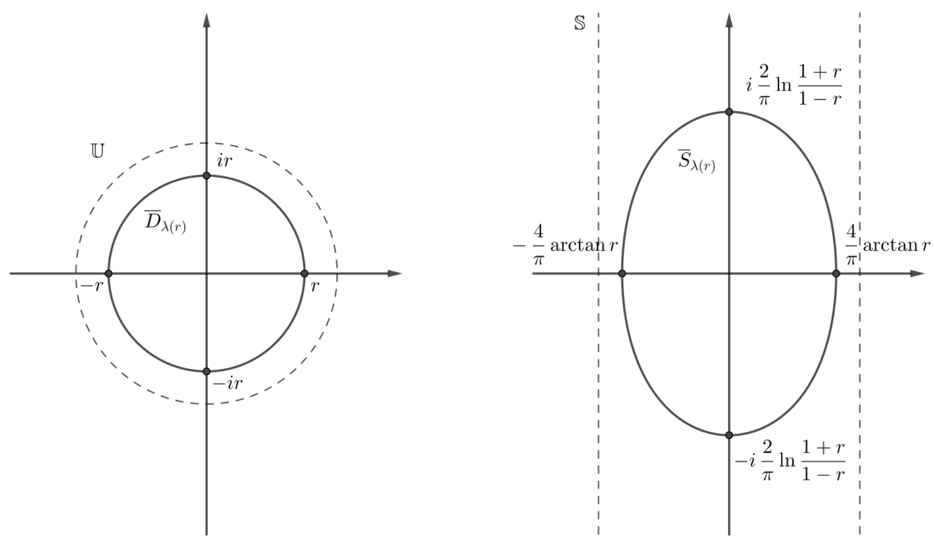

Figure 1: $\overline{D}_{\lambda(r)}$ and $\overline{S}_{\lambda(r)}$

Lemma 1. *Let $r \in (0,1)$ be arbitrary. Then*

$$\overline{S}_{\lambda(r)} \subset \left[-\frac{4}{\pi}\arctan r, \frac{4}{\pi}\arctan r\right] \times \left[-\frac{2}{\pi}\lambda(r), \frac{2}{\pi}\lambda(r)\right]. \tag{14}$$

In particular,

$$R_e(\overline{S}_{\lambda(r)}) = \left[-\frac{4}{\pi}\arctan r, \frac{4}{\pi}\arctan r\right]. \tag{15}$$

Proof. Since $\overline{S}_{\lambda(r)} = \phi(\overline{U}_r)$, where ϕ is defined in Example 1, it is sufficient to show that

$$\max\{|\operatorname{Re}\phi(z)| : z \in \overline{U}_r\} = \frac{4}{\pi}\arctan r \tag{16}$$

and

$$\max\{|\operatorname{Im}\phi(z)| : z \in \overline{U}_r\} = \frac{2}{\pi}\lambda(r). \tag{17}$$

Further, it is convenient to present the mapping ϕ as the composition $\phi_4\circ\phi_3\circ\phi_2\circ\phi_1$, where $\phi_1(z) = iz$, $\phi_2(z) = \dfrac{1+z}{1-z}$, $\phi_3(z) = \ln z^{\ddagger}$ and $\phi_4(z) = -i\dfrac{2}{\pi}z$. It is easy to check that $\phi_2\circ\phi_1$ maps ∂U_r onto l_r, where l_r is circle with center $c = \dfrac{1+r^2}{1-r^2}$ and radius $R = \dfrac{2r}{1-r^2}$. Also, for all $z \in l_r$ we have $\operatorname{Re}(\phi_4(\phi_3(z))) = \dfrac{2}{\pi}\arg z^{\S}$ and $\operatorname{Im}(\phi_4(\phi_3(z))) = -\dfrac{2}{\pi}\ln|z|$.

Set $\theta_0 = \max\{|\arg z| : z \in l_r\}$ and $L_0 = \max\{|\ln|z|| : z \in l_r\}$.

Let's look at Figure 2.

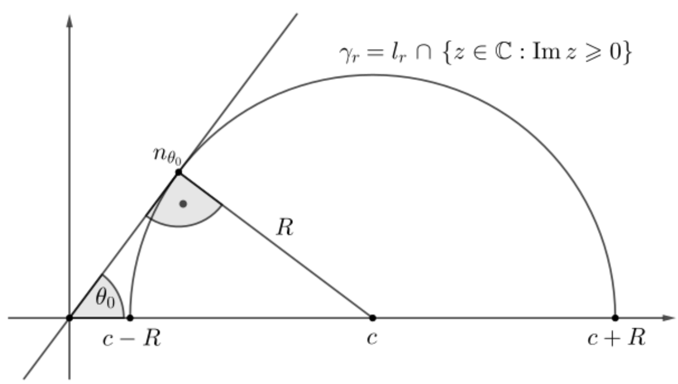

Figure 2: Angle θ_0.

It is clear that line $y = (\tan\theta_0)x$ is a tangent from the point 0 on the circle l_r and denote by n_{θ_0} the point of tangency. Also, note that l_r intersect the $x-$axis at the points $c - R = \dfrac{1-r}{1+r}$ and $c + R = \dfrac{1+r}{1-r}$ which are reciprocal numbers. Thus,

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‡Here ln is branch of logarithm defined on $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ and determined by $\ln 1 = 0$.

§Here arg is imaginary part of ln. It is evident that values of arg belong to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

the power of the point 0 with respect to the circle l_r is equal to 1 and therefore $|n_{\theta_0}| = 1$. Now, it is obviously that $\tan\theta_0 = R$ and therefore

$$\theta_0 = \arctan R = \arctan\frac{2r}{1-r^2} = 2\arctan r. \tag{18}$$

Further, since $\ln|n_{\theta_0}| = 0$ it is easy seen that $L_0 = \max\{-\ln(c-R), \ln(c+R)\}$. But, since $(c-R)(c+R) = 1$ it follows that $\ln(c+R) = -\ln(c-R)$ and therefore

$$L_0 = \ln(c+R) = -\ln(c-R) = \ln\frac{1+r}{1-r}. \tag{19}$$

From (18) and (19) we can get the equalities (16) and (17) (the details are left to the reader). □

Remark 2. *Lemma 1 can be easily deduced by properties of conformal maps of the unit disc onto a convex domain. Namely, since the strip $\mathbb{S}$ is Euclidean convex set, it follows that $\overline{S}_{\lambda(r)}$ is Euclidean convex set (see [**3**, Theorem 7.11]). Further, we have*

i) $\overline{S}_{\lambda(r)}$ *is symmetric with respect to the both coordinate axes;*

ii) for $r \in [0,1)$ *it holds* $\phi([-r,r]) = \left[-\frac{4}{\pi}\arctan r, \frac{4}{\pi}\arctan r\right]$ *and* $\phi([-ir,ir]) = \left[-i\frac{2}{\pi}\lambda(r), i\frac{2}{\pi}\lambda(r)\right]$.

Hence Lemma 1 immediately follows.

In order to appreciate the proof of the next lemma we give an example. For $s > 0$ set $R^s = [-1,1]\times[-s,s]$ and let ψ^s be conformal mapping of $\mathbb{U}$ onto R^s such that ψ^s maps $(-i,i)$ onto $(-is,is)$ with $\psi^s(0) = 0$. Also, for $r \in (0,1)$ set $\overline{E}_{r,s} = \psi^s(\overline{U}_r)$. We leave to the interested reader to check that for r for r close enough to 1 the function e on $\overline{E}_{r,s}$ does not attain maximum at $\psi^s(ir)$.

Lemma 2. *Let $\lambda > 0$ be arbitrary. Then*

$$\max\{d_e(z,0) : z \in \overline{S}_\lambda\} = \frac{2}{\pi}\lambda. \tag{20}$$

Proof. Let $z \in \overline{S}_\lambda$ be arbitrary. By Proposition 3 and since $z \in \overline{S}_\lambda$ we have

$$d_e(z,0) \leqslant \frac{2}{\pi}d_{\mathbb{S}}(z,0) \leqslant \frac{2}{\pi}\lambda. \tag{21}$$

It remains to show that there exists a $z_0 \in \overline{S}_\lambda$ such that $d_e(z_0,0) = \frac{2}{\pi}\lambda$. Let $z_0 = i\frac{2}{\pi}\lambda$ or $z_0 = -i\frac{2}{\pi}\lambda$. Then it is clear that $d_e(z_0,0) = \frac{2}{\pi}\lambda$ and by Proposition 3 we have $d_{\mathbb{S}}(z_0,0) = \lambda$, i.e. $z_0 \in \overline{S}_\lambda$.

Recall that the above proof is based on the hyperbolic geometry of the strip. The reader can try to get a direct analytic proof without appeal to the geometry. □

Lemma 3. *Let $r \in (0,1)$ and $b \in (-1,1)$ be arbitrary. Then*

$$R_e(\overline{S}_{\lambda(r)}(b)) = [m_b(r), M_b(r)],$$

where m_b and M_b are defined in Example 3.

Proof. Let's repeat that $\overline{D}_{\lambda(r)} = \overline{U}_r$ and $\overline{S}_{\lambda(r)}(b) = \phi_b(\overline{D}_{\lambda(r)}) = \phi_b(\overline{U}_r)$. Further, one can show that

$$\overline{S}_{\lambda(r)}(b) \text{ is symmetric with respect to the } x\text{-axis.} \tag{22}$$

Also, by [**3**, Theorem 7.11]

$$\overline{S}_{\lambda(r)}(b) \text{ is Euclidean convex.} \tag{23}$$

Now, from (22), (23) and parts i) and ii) in Example 3 the lemma follows. □

4. THE SCHWARZ LEMMA FOR HARMONIC FUNCTIONS FROM $\mathbb{U}$ INTO $(-1,1)$

In this section we first give a simple proof of the classical Schwarz lemma for harmonic functions and then use the same method to prove a new version (Theorem 6).

Theorem 5 ([**11**],[**8**, p. 77])**.** *Let $u : \mathbb{U} \to (-1,1)$ be harmonic function such that $u(0) = 0$. Then*

$$|u(z)| \leqslant \frac{4}{\pi} \arctan |z|, \quad \textit{for all} \quad z \in \mathbb{U}, \tag{24}$$

and this inequality is sharp for each point $z \in \mathbb{U}$.

Proof. Let $z \in \mathbb{U}$ be arbitrary and $r = |z|$. Since $\mathbb{U}$ is simply connected it is well known that there exists $f \in \mathrm{Hol}(\mathbb{U}, \mathbb{S})$ such that $u = \mathrm{Re}\, f$ and $f(0) = 0$. By subordination principle we have $f(\overline{U}_r) \subset \phi(\overline{U}_r)$, where ϕ is mapping defined in Example 1. Now, since $\overline{U}_r = \overline{D}_{\lambda(r)}$ and since $\overline{S}_{\lambda(r)} = \phi(\overline{D}_{\lambda(r)})$ by Lemma 1 we obtain $u(\overline{U}_r) \subset \left[-\frac{4}{\pi} \arctan r, \frac{4}{\pi} \arctan r\right]$ and the inequality (24) follows.

If $z = 0$ it is clear that in (24) the equality holds. In order to show that the inequality (24) is sharp also for $z \neq 0$, we define function $\widehat{u} : \mathbb{U} \to (-1,1)$ in the following way $\widehat{u}(\zeta) = (\mathrm{Re}\, \phi)(e^{-i \arg z} \zeta)$¶, where ϕ is mapping defined in Example 1. Note that function $\widehat{u}$ depend on the point z. It immediately follows that $\widehat{u}$ is harmonic function and $\widehat{u}(0) = 0$. A simple computation gives

$$|\widehat{u}(z)| = |(\mathrm{Re}\, \phi)(e^{-i \arg z} z)| = |(\mathrm{Re}\, \phi)(|z|)| = \frac{2}{\pi} \arctan \frac{2\, \mathrm{Re}\, |z|}{1 - |z|^2} = \frac{4}{\pi} \arctan |z|.$$

□

¶Here values of arg belong to the interval $[0, 2\pi)$.

We leave to the interested reader to elaborate proofs of Theorems 5 and 7 using the Schwarz-Pick lemma (as in Lemma 4) instead of the subordination principle.

Remark 3. *Using the rotation Theorem 2, stated in the introduction, follows easily from Theorem 5. For details see [**8**, p. 77] cf. also [**11**].*

Theorem 6. *Let $u : \mathbb{U} \to (-1, 1)$ be harmonic function such that $u(0) = b$ and let m_b and M_b be defined in Example 3. Then*

$$m_b(|z|) \leqslant u(z) \leqslant M_b(|z|), \quad \text{for all} \quad z \in \mathbb{U}, \tag{25}$$

and these inequalities are sharp for each point $z \in \mathbb{U}$.

Proof. Let $z \in \mathbb{U}$ be arbitrary and $r = |z|$. Since $\mathbb{U}$ is simply connected it is well known that there exists $f \in \operatorname{Hol}(\mathbb{U}, \mathbb{S})$ such that $u = \operatorname{Re} f$ and $f(0) = b$. By Theorem 4 we have

$$d_{\mathbb{S}}(f(z), b) = d_{\mathbb{S}}(f(z), f(0)) \leqslant d_{\mathbb{U}}(z, 0). \tag{26}$$

Since $d_{\mathbb{U}}(z, 0) = \lambda(r)$, from (26) it follows that $f(z) \in \overline{S}_{\lambda(r)}(b)$. Hence, by Lemma 3 we get $m_b(r) \leqslant u(z) \leqslant M_b(r)$.

Since $m_b(0) = M_b(0) = b$ it follows that $u(0) = m_b(0) = M_b(0)$.

For $z \neq 0$ we define functions $\widehat{u}, \widetilde{u} : \mathbb{U} \to (-1, 1)$ in the following way

$$\widehat{u}(\zeta) = (\operatorname{Re} \phi_b)(-e^{-i \arg z}\zeta)$$

and

$$\widetilde{u}(\zeta) = (\operatorname{Re} \phi_b)(e^{-i \arg z}\zeta).$$

Then $\widehat{u}(z) = m_b(|z|)$ and $\widetilde{u}(z) = M_b(|z|)$ i.e. inequalities (25) are sharp. □

5. THE SCHWARZ LEMMA FOR HOLOMORPHIC MAPS FROM $\mathbb{U}$ INTO $\mathbb{S}$

Theorem 2 is usually considered as harmonic version of Theorem 1. In analogy with Theorems 1 and 2 we prove the next results (Theorems 7 and 8). Whereby, the codomain $\mathbb{U}$ and the function arctan are replaced by the strip $\mathbb{S}$ and the function artanh, respectively. For $K = 1$ Theorem 8 is reduced to Theorem 7.

Theorem 7 (The Schwarz lemma for holomorphic maps from $\mathbb{U}$ into $\mathbb{S}$)**.**
Let $f \in \operatorname{Hol}(\mathbb{U}, \mathbb{S})$ and $f(0) = 0$. Then

$$|f(z)| \leqslant \frac{4}{\pi} \operatorname{artanh} |z|, \quad \text{for all} \quad z \in \mathbb{U}. \tag{27}$$

The inequality (27) is sharp for each point $z \in \mathbb{U}$. Also,

$$|f'(0)| \leqslant \frac{4}{\pi}. \tag{28}$$

In (28) the equality holds if and only if $f(z) = \phi(\alpha z)$, where $\alpha \in \mathbb{C}$ such that $|\alpha| = 1$, and ϕ is mapping defined in Example 1.

Proof. Let $z \in \mathbb{U}$ be arbitrary and $r = |z|$. By subordination principle we have $f(\overline{U}_r) \subset \phi(\overline{U}_r)$, where ϕ is mapping defined in Example 1. Since $\overline{U}_r = \overline{D}_{\lambda(r)}$ and since $\overline{S}_{\lambda(r)} = \phi(\overline{D}_{\lambda(r)})$ we have $f(\overline{U}_r) \subset \overline{S}_{\lambda(r)}$. Hence, by Lemma 2 we obtain

$$|f(z)| \leqslant \frac{2}{\pi}\lambda(|z|) = \frac{4}{\pi} \operatorname{artanh} |z|. \tag{29}$$

If $z = 0$ it is clear that in (27) the equality holds. In order to show that the inequality (27) is sharp also for $z \neq 0$, we define function $\widehat{f} : \mathbb{U} \to \mathbb{S}$ in the following way $\widehat{f}(\zeta) = \phi(ie^{-i \arg z}\zeta)$, where ϕ is defined in Example 1. Note that function $\widehat{f}$ depend on the point z. It immediately follows that $\widehat{f} \in \mathrm{Hol}(\mathbb{U}, \mathbb{S})$ and $\widehat{f}(0) = 0$. A simple computation gives

$$|\widehat{f}(z)| = |\phi(ie^{-i \arg z} z)| = |\phi(i|z|)| = \left| -i\frac{2}{\pi} \ln \frac{1 - |z|}{1 + |z|} \right| = \frac{4}{\pi} \operatorname{artanh} |z|.$$

Finally, by subordination principle we obtain $|f'(0)| \leqslant |\phi'(0)| = \dfrac{4}{\pi}$ and theorem follows. □

6. THE SCHWARZ LEMMA FOR HARMONIC K-QUASIREGULAR MAPS FROM $\mathbb{U}$ INTO $\mathbb{S}$

Quasiregular maps are a class of continuous maps between Euclidean spaces $\mathbb{R}^n$ of the same dimension or, more generally, between Riemannian manifolds of the same dimension, which share some of the basic properties with holomorphic functions of one complex variable.

Let D and G be domains in $\mathbb{C}$. A C^1 mapping $f : D \to G$ we call sense-preserving $K-$quasiregular mapping if

a) $|f_z(z)| > |f_{\overline{z}}(z)|$ for all $z \in D$;

b) there exists $K \geqslant 1$ such that $\dfrac{|f_z(z)| + |f_{\overline{z}}(z)|}{|f_z(z)| - |f_{\overline{z}}(z)|} \leqslant K$ for all $z \in D$.

Thus the linear map $(df)_z = f_z(z)dz + f_{\overline{z}}(z)d\overline{z}$ maps circles with center at z onto ellipses such that the ratio between the big axis and the small axis is uniformly bounded by K with respect to $z \in D$.

Injective $K-$quasiregular mappings are called $K-$quasiconformal mappings. Quasiconformal maps play a crucial role in Teichmüller theory and complex dynamics.

The class of all harmonic sense-preserving $K-$ quasiregular ($K-$ quasiconformal) mappings $f : D \to G$ we denote by $\mathrm{HQR}_K(D,G)$ (respectively $\mathrm{HQC}_K(D,G)$).

Example 4. *Let $K \geqslant 1$ and let $A_K : \mathbb{S} \to \mathbb{S}$ be defined by $A_K(x,y) = (x, Ky)$. It is clear that the mapping A_K is sense-preserving $K-$quasiregular. Let $\psi_K = A_K \circ \phi$, where ϕ is the mapping defined in Example 1. It is easy to check that $\psi_K \in \mathrm{HQR}_K(\mathbb{U},\mathbb{S})$.*

Lemma 4. *Let $K \geqslant 1$, $f \in \mathrm{HQR}_K(\mathbb{U},\mathbb{S})$. Then*

$$d_{\mathbb{S}}(f(z_1), f(z_2)) \leqslant K d_{\mathbb{U}}(z_1, z_2) \quad \textit{for all} \quad z_1, z_2 \in \mathbb{U}. \tag{30}$$

Remark 4. *Note that the assumption that the codomain is a strip has here a specific role and is crucial for application of the strip method and that in the lemma we consider quasiregular mappings. In particular for quasiconformal mappings some results of this type are known. If $f \in \mathrm{HQC}_K(\mathbb{U},\mathbb{U})$ the corresponding estimate is obtained in [**19**] using a different method. Further X. Chen and A. Fang [**6**] obtained results related to Euclidean harmonic quasiconformal mappings with convex ranges. More generally, if G is a planar domain and $f \in \mathrm{HQC}_K(\mathbb{U}, G)$ then there are the corresponding estimates via the quasi-hyperbolic metric on G, cf. [**23, 24**], which are not optimal in general.*

Note that we can not pass here from a codomain G to the strip codomain in this context. Namely if g is conformal mapping from G onto $\mathbb{S}$ then $g \circ f$ is not harmonic in general and we can not apply Proposition 2 on $g \circ f$. See also subsection 6.1. below for further details related to this remark.

Proof. Set $u = \operatorname{Re} f$ and $\nabla u = (u_x, u_y)$. Since f is $K-$quasiregular one can check that

$$|f_z(z)| + |f_{\overline{z}}(z)| \leqslant K|\nabla u(z)| \quad \text{for all} \quad z \in \mathbb{U}. \tag{31}$$

By Proposition 2 we have

$$\rho_{\mathbb{S}}(u(z))|\nabla u(z)| \leqslant \rho_{\mathbb{U}}(z) \quad \text{for all} \quad z \in \mathbb{U}. \tag{32}$$

On other hand we have

$$\begin{aligned} \rho_{\mathbb{S}}(f(z)) &= \frac{\pi}{2}\frac{1}{\cos\left(\dfrac{\pi}{2}\operatorname{Re} f(z)\right)} \\ &= \frac{\pi}{2}\frac{1}{\cos\left(\dfrac{\pi}{2}u(z)\right)} = \rho_{\mathbb{S}}(u(z)) \quad \text{for all} \quad z \in \mathbb{U}. \end{aligned} \tag{33}$$

Now, from (33), (31) and (32) it follows that

$$\rho_{\mathbb{S}}(f(z))\,(|f_z(z)|+|f_{\overline{z}}(z)|)\leqslant K\rho_{\mathbb{U}}(z)\quad\text{for all}\quad z\in\mathbb{U}. \tag{34}$$

It is well known in general that the estimate of the gradient by means of the corresponding densities yields the corresponding estimate between the distances, see for example [**25**]. The detailed verification of it is left to the reader. In particular, we get (30). □

Note that if codomain is $\mathbb{U}$ the result of this type is proved in [**19**] and [**18**].

Theorem 8 (The Schwarz lemma for HQR maps from $\mathbb{U}$ into $\mathbb{S}$)**.** *Let* $K\geqslant 1$, $f\in \mathrm{HQR}_K(\mathbb{U},\mathbb{S})$ *and* $f(0)=0$. *Then*

$$|f(z)|\leqslant \frac{4}{\pi}K\,\mathrm{artanh}\,|z|,\qquad \textit{for all}\quad z\in\mathbb{U}, \tag{35}$$

and this inequality is sharp for each point $z\in\mathbb{U}$.

Proof. Let $z\in\mathbb{U}$ be arbitrary. By the Lemma 4 we have

$$d_{\mathbb{S}}(f(z),0)\leqslant K d_{\mathbb{U}}(z,0). \tag{36}$$

Since $d_{\mathbb{U}}(z,0)=\lambda(|z|)$, from (36) it follows that $f(z)$ belongs to the closed hyperbolic disc with hyperbolic center 0 and hyperbolic radius $K\lambda(|z|)$, i.e.

$$f(z)\ \in\ \overline{S}_{K\lambda(|z|)}.$$

Hence, by Lemma 2 we get $|f(z)|\leqslant \dfrac{2}{\pi}K\lambda(|z|)=\dfrac{4}{\pi}K\,\mathrm{artanh}\,|z|$.

As in the proof of Theorem 7, one can show that the inequality (35) is sharp. In this case, instead of mapping ϕ defined in Example 1 the mapping ψ_K defined in Example 4 should be used. □

The following question suggested by the reviewer naturally arises:

Question 1. What happens in Theorem 8 when one replaces condition $f(0)=0$ with $f(0)=b$? The authors have been aware of this question, but they had impression that additional techniques is needed to get a satisfying version. Here we outline an explanation. A possibility is to try to use similar procedure as in proving Theorem 6 which generalizes Theorem 5. Note that in this case for arbitrary $z\in\mathbb{U}$ it is valid that $f(z)$ belongs to the closed hyperbolic disc with hyperbolic center b and hyperbolic radius $K\lambda(|z|)$ i.e. $f(z)\ \in\ \overline{S}_{K\lambda(|z|)}(b)$. But in this case we do not have analog version of Lemma 2. It also seems useful to observe the following. Let ω_b be conformal automorphism of $\mathbb{S}$ such that $\omega_b(0)=b$ and $g=\omega_b\circ f$. Then g is $K-$quasiregular mapping but we can not directly apply Theorem 8 on $g=\omega_b\circ f$ because g is not harmonic in general.

6.1. Further comments. In this section we extract some parts from the paper of X. Chen and A. Fang [**6**]. For a sense-preserving quasiconformal mapping f from D onto G, we write

$$||\partial f(z)|| = \frac{\rho_G(f(z))}{\rho_D(z)}|f_z(z)|.$$

In order to study the property of quasi-isometry of the class of Euclidean harmonic quasiconformal mappings of the unit disk onto itself, M. Knežević and M. Mateljević [**19**] studied its the generalized Schwarz-Pick inequality and showed:

Theorem A. If f is a Euclidean harmonic $K-$quasiconformal mapping of the unit disk $\mathbb{U}$ onto itself then the following inequality

$$\frac{K+1}{2K} \leqslant ||\partial f(z)|| \leqslant \frac{K+1}{2} \tag{37}$$

holds for every $z \in \mathbb{U}$.

The following question is natural:

Question 2. Does the generalized Schwarz-Pick inequality (37) still hold for Euclidean harmonic quasiconformal mappings with other ranges than the unit disk? Is the inequality (37) sharp for Euclidean harmonic quasiconformal mappings?

The main result of the paper [**6**] is to answer the above question affirmatively for Euclidean harmonic quasiconformal mappings with convex ranges. See [**6**, Theorem 2.1] stated here as:

Theorem B. Let G be a simply connected convex hyperbolic domain in $\mathbb{C}$. If f is a Euclidean harmonic $K-$quasiconformal mapping of the unit disk $\mathbb{U}$ onto G , then the inequality

$$\frac{K+1}{2K} \leqslant ||\partial f(z)|| \leqslant \frac{K+1}{2}$$

holds for every $z \in \mathbb{U}$. Moreover, both the upper and lower bounds are sharp.

The main technique of its proof is to combine the Ahlfors-Schwarz lemma and its opposite type given by M. Mateljević [**22**] with a property of domain constants given by R. Harmelin [**10**].

For convenience of the reader we state those results here:

Lemma A([**22**]). If $\rho > 0$ is a C^2 metric density on $\mathbb{U}$ for which the gaussian curvature satisfies $K_\rho \geqslant -1$ and if $\rho(z)$ tends to $+\infty$ when $|z|$ tends to $1-$, then $\rho_{\mathbb{U}} \leqslant \rho$.

Lemma B([**10**]). A hyperbolic domain G in $\mathbb{C}$ is convex if and only if its hyperbolic metric $\rho_G(z)|dz|$ satisfies the differential inequality

$$|D_z(\ln \circ \rho_G)(z)| \leqslant \frac{\rho_G(z)}{2}, \quad z \in G,$$

and if it holds then we also have

$$|D^2_{zz}(\ln \circ \rho_G)(z)| \leqslant \frac{\rho^2_G(z)}{4}, \quad z \in G.$$

Here we write $D_z f$ and $D^2_{zz} f$ instead of f_z and $(f_z)_z$, respectively.

In [**6**](see Examples 4.1 and 4.2) two examples are given which show that the generalized Schwarz-Pick inequality (37) no longer holds for two classes of Euclidean harmonic quasiconformal mappings with ranges only convex in the horizontal direction.

In order to study Schoen conjecture, T. Wan [**34**] first built the generalized Schwarz-Pick inequality for the class of hyperbolic harmonic quasiconformal mappings (for further results see G. W. Yao [**35**], and M. Knežević and M. Mateljević [**19**] and literature cited there).

The existence of harmonic extensions of quasiconformal maps was conjectured by Schoen, and was proved by V. Marković [**21**].

More precisely, V. Marković shows the following:

Theorem C. Every quasisymmetric homeomorphism h of the circle $\partial\mathbb{U}$ admits a harmonic quasiconformal extension H_h to the hyperbolic plane $\mathbb{U}$.

This proves the Schoen conjecture.

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