Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **14** (2020), 169–182.
https://doi.org/10.2298/AADM180123002K

THE APPLICATIONS OF CAUCHY-SCHWARTZ INEQUALITY FOR HILBERT MODULES TO ELEMENTARY OPERATORS AND I.P.T.I. TRANSFORMERS

Dragoljub J. Kečkić

We apply the inequality $|\langle x,y\rangle| \le ||x||\ \langle y,y\rangle^{1/2}$ to give an easy and elementary proof of many operator inequalities for elementary operators and inner type product integral transformers obtained during last two decades, which also generalizes many of them.

1. INTRODUCTION

Let A be a Banach algebra, and let $a_j,\ b_j \in A$. Elementary operators, introduced by Lummer and Rosenblum in [**12**] are mappings from A to A of the form

$$x \mapsto \sum_{j=1}^{n} a_j x b_j. \tag{1}$$

Finite sum may be replaced by infinite sum provided some convergence condition.

A similar mapping, called inner product type integral transformer (i.p.t.i. transformers in further), considered in [**6**], is defined by

$$X \mapsto \int_\Omega \mathcal{A}_t X \mathcal{B}_t \mathrm{d}\mu(t), \tag{2}$$

2010 Mathematics Subject Classification. Primary: 47A63, Secondary: 47B47, 46L08
Keywords and Phrases. Cauchy Schwartz inequality, unitarily invariant norm, elementary operator, inner product type transformers.

where (Ω, μ) is a measure space, and $t \mapsto \mathcal{A}_t$, $\mathcal{B}_t$ are fields of operators in $B(H)$.

During last two decades, there were obtained a number of inequalities involving elementary operators on $B(H)$ as well as i.p.t.i. type transformers. The aim of this paper is to give an easy and elementary proof of those proved in [**7, 8, 6, 9, 4, 21, 10**] and [**11**] using the Cauchy Schwartz inequality for Hilbert C^*-modules – the inequality stated in the abstract, which also generalizes all of them.

2. PRELIMINARIES

Throughout this paper A will always denote a semifinite von Neumann algebra, and τ will denote a semifinite trace on A. By $L^p(A;\tau)$ we will denote the non-commutative L^p space, $L^p(A;\tau) = \{a \in A \mid \|a\|_p = \tau(|a|^p)^{1/p} < +\infty\}$.

It is well known that $L^1(A;\tau)^* \cong A$, $L^p(A;\tau)^* \cong L^q(A;\tau)$, $1/p + 1/q = 1$. Both dualities are realized by

$$L^p(A;\tau) \ni a \mapsto \tau(ab) \in \mathbf{C}, \qquad b \in L^q(A;\tau) \text{ or } b \in A.$$

For more details on von Neumann algebras the reader is referred to [**13**], and for details on $L^p(A,\tau)$ to [**15**].

Let M be a *right* Hilbert W^*-module over A. (Since M is right we assume that A-valued inner product is A-linear in second variable, and adjoint A-linear in the first.) We assume, also, that there is a faithful left action of A on M, that is, an embedding (and hence an isometry) of A into $B^a(M)$ the algebra of all adjointable bounded A-linear operators on M. Hence, for x, $y \in M$ and a, $b \in A$ we have

$$\langle x, y\rangle\, a = \langle x, ya\rangle\,, \quad \langle xa, y\rangle = a^* \langle x, y\rangle\,, \quad \langle x, ay\rangle = \langle a^*x, y\rangle\,.$$

For more details on Hilbert modules, the reader is referred to [**14**] or [**16**].

We quote the basic property of A-valued inner product, a variant od Cauchy-Schwartz inequality.

Proposition 1. *Let M be a Hilbert C^*-module over A. For any x, $y \in M$ we have*

$$|\langle x, y\rangle|^2 \le \|x\|^2 \langle y, y\rangle\,, \qquad |\langle x, y\rangle| \le \|x\| \langle y, y\rangle^{1/2}\,, \tag{3}$$

in the ordering of A.

The proof can be found in [**14**, page 3] or [**16**, page 3]. Notice: 1° the left inequality implies the right one, since $t \mapsto t^{1/2}$ is operator increasing function; 2° Both inequalities hold for A-valued *semi-inner* product, i.e. even if $\langle\cdot,\cdot\rangle$ may be degenerate.

Finally, we need a counterpart of Tomita modular conjugation.

Definition 1. Let M be a Hilbert W^*-module over a semifinite von Neumann algebra A, and let there is a left action of A on M.

A (possibly unbounded) mapping J, defined on some A submodule $M_0 \subseteq M$ with values in M, we call modular conjugation if it satisfies: (i) $J(axb) = b^* J(x) a^*$; (ii) $\tau(\langle J(y), J(x)\rangle) = \tau(\langle x, y\rangle)$ whenever $\langle x, x\rangle$, $\langle y, y\rangle$, $\langle J(x), J(x)\rangle$, $\langle J(y), J(y)\rangle \in L^1(A, \tau)$.

In what follows, we shall use simpler notation $\overline{x}$ instead of $J(x)$. Thus, the determining equalities become

$$\overline{axb} = b^* \overline{x} a^*, \qquad \tau(\langle \overline{y}, \overline{x}\rangle) = \tau(\langle x, y\rangle). \tag{4}$$

We shall call the module M together with left action of A and the modular conjugation J *conjugated W^*-module.*

Definition 2. Let M be a conjugated W^*-module over A. We say that $x \in M_0$ is *normal*, if (i) $\langle x, x\rangle x = x \langle x, x\rangle$, (ii) $\langle x, x\rangle = \langle \overline{x}, \overline{x}\rangle$.

Remark 1. It might be a nontrivial question, whether J can be defined on an arbitrary Hilbert W^*-module in a way similar to the construction of Tomita's modular conjugation (see [**18**]). However for our purpose, the preceding definition is enough.

Examples of conjugated modules are following.

Example 1. Let A be a semifinite von Neumann algebra, and let $M = A^n$. For $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in M$, $a \in A$, define right multiplication, left action of A, the A-valued inner product and modular conjugation by

$$xa = (x_1 a, \dots, x_n a), \qquad ax = (ax_1, \dots, ax_n); \tag{5}$$

$$\langle x, y\rangle = x_1^* y_1 + \cdots + x_n^* y_n, \qquad \overline{x} = (x_1^*, \dots, x_n^*). \tag{6}$$

All required properties are easily verified. The element $x = (x_1, \dots, x_n)$ is normal whenever all x_j are normal and mutually commute.

We have

$$\langle x, ay\rangle = \sum_{j=1}^{n} x_j^* a y_j,$$

which is the term of the form (1).

There are two important modules with infinite number of summands.

Example 2. Let A be a semifinite von Neumann algebra. We consider the standard Hilbert module $l^2(A)$ over A and its dual module $l^2(A)'$ defined by

$$l^2(A) = \Big\{(x_1, \dots, x_n, \dots) \;\Big|\; \sum_{k=1}^{+\infty} a_k^* a_k \text{ converges in norm of } A\Big\}.$$

$$l^2(A)' = \Big\{(x_1, \dots, x_n, \dots) \;\Big|\; \Big\|\sum_{k=1}^{n} a_k^* a_k\Big\| \le M < +\infty\Big\}.$$

(It is clear that $x \in l^2(A)'$ if and only if the series $\sum x_k^* x_k$ weakly converges.)

The basic operation on these modules are given by (5) and (6) with infinite number of entries.

The main difference between $l^2(A)$ ($l^2(A)'$ respectively) and A^n is the fact that $\overline{x} = (x_1^*, \dots, x_n^*, \dots)$ is defined only on the subset of $l^2(A)$ consisting of those $x \in M$ for which $\sum x_k x_k^*$ converges in the norm of A.

The element $x = (x_1, \dots, x_n, \dots) \in M_0$ is normal whenever all x_j are normal and mutually commute.

Remark 2. The notation $l^2(A)'$ comes from the fact that $l^2(A)'$ is isomorphic to the module of all adjointable bounded A-linear functionals $\Lambda : M \to A$.

For more details on $l^2(A)$ or $l^2(A)'$ see [**16**, §1.4 and §2.5].

Example 3. Let A be a semifinite von Neumann algebra and let (Ω, μ) be a measure space. Consider the space $L^2(\Omega, A)$ consisting of all weakly-$*$ measurable functions such that $\int_\Omega x^* x \mathrm{d}\mu < +\infty$ weak-$*$ converges. The weak-$*$ measurability is reduced to the measurability of functions $\varphi(x(t))$ for all normal states φ, since the latter generate the predual of A.

Basic operations are given by

$$x(t) \cdot a = x(t)a, \quad a \cdot x(t) = ax(t), \quad \langle x, y \rangle = \int_\Omega x(t)^* y(t) \mathrm{d}\mu(t), \quad \overline{x}(t) = x(t)^*.$$

All required properties are easily verified. The mapping $x \mapsto \overline{x}$ is again defined on a proper subset of $L^2(\Omega, A)$. The element x is normal if $x(t)$ is normal for almost all t, and $x(t)x(s) = x(s)x(t)$ for almost all (s, t).

Again, for $a \in A$ we have

$$\langle x, ay \rangle = \int_\Omega x(t)^* a y(t) \mathrm{d}\mu(t),$$

which is the term of the form (2).

Thus, norm estimates of elementary operators (1), or i.t.p.i. transformers (2) are estimates of the term $\langle x, ay \rangle$.

In section 4 we need two more examples.

Example 4. Let M_1 and M_2 be conjugated W^*-modules over a semifinite von Neumann algebra A. Consider the *interior* tensor product of Hilbert modules M_1 and M_2 constructed as follows. The linear span of $x_1 \otimes x_2$, $x_1 \in M_1$, $x_2 \in M_2$ subject to the relations

$$a(x_1 \otimes x_2) = ax_1 \otimes x_2, \qquad x_1 a \otimes x_2 = x_1 \otimes ax_2, \qquad (x_1 \otimes x_2)a = x_1 \otimes x_2 a,$$

and usual bi-linearity of $x_1 \otimes x_2$, can be equipped by an A-valued semi-inner product

$$\langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle = \langle x_2, \langle x_1, y_1 \rangle y_2 \rangle . \tag{7}$$

The completion of the quotient of this linear span by the kernel of (7) is denoted by $M_1 \otimes M_2$ and called *interior tensor product* of M_1 and M_2. For more details on tensor products, see [**14**, Chapter 4].

If $M_1 = M_2 = M$, $M \otimes M$ can be endowed with a modular conjugation by

$$\overline{x_1 \otimes x_2} = \overline{x_2} \otimes \overline{x_1}.$$

All properties are easily verified. Also, x normal implies $x \otimes x$ is normal and $\langle x \otimes x, x \otimes x\rangle = \langle x, x\rangle^2$.

Example 5. Let M_n, $n \in \mathbf{N}$ be conjugated modules. Their infinite direct sum $\bigoplus_{n=1}^{+\infty} M_n$ is the module consisting of those sequences (x_n), $x_n \in M_n$ such that $\sum_{n=1}^{+\infty} \langle x_n, x_n\rangle$ weakly converges, with the A-valued inner product

$$\langle (x_n), (y_n)\rangle = \sum_{n=1}^{+\infty} \langle x_n, y_n\rangle .$$

The modular conjugation can be given by $\overline{(x_n)} = (\overline{x_n})$. In particular, we need the *full Fock module*

$$F = \bigoplus_{n=0}^{+\infty} M^{\otimes n},$$

where $M^{\otimes 0} = A$, $M^{\otimes 1} = M$, $M^{\otimes 2} = M \otimes M$, $M^{\otimes 3} = M \otimes M \otimes M$, etc.

For $x \in M$, $\|x\| < 1$ the element $\sum_{n=0}^{+\infty} x^{\otimes n} \in F$ (where $x^{\otimes 0} := 1$) is well defined. It is normal whenever x is normal. Also, for normal x, we have

$$\left\langle \sum_{n=0}^{+\infty} x^{\otimes n}, \sum_{n=0}^{+\infty} x^{\otimes n} \right\rangle = \sum_{n=0}^{+\infty} \left\langle x^{\otimes n}, x^{\otimes n}\right\rangle = \sum_{n=0}^{+\infty} \langle x, x\rangle^n = (1 - \langle x, x\rangle)^{-1}. \tag{8}$$

We shall deal with *unitarily invariant norms* on the algebra $B(H)$ of all bounded Hilbert space operators. For more details, the reader is referred to [**22**, Chapter III]. We use the following facts. For any unitarily invariant norm $|||\cdot|||$, we have $|||A||| = |||A^*||| = |||\,|A|\,||| = |||UAV||| = |||A|||$ for all unitaries U and V, as well as $\|A\| \le |||A||| \le \|A\|_1$. The latter allows the following well known interpolation Lemma, which we state with a proof.

Lemma 2. *Let T and S be linear mappings defined on the space $\mathcal{C}_\infty$ of all compact operators on Hilbert space H. If*

$$\|Tx\| \le \|Sx\| \text{ for all } x \in \mathcal{C}_\infty, \qquad \|Tx\|_1 \le \|Sx\|_1 \text{ for all } x \in \mathcal{C}_1$$

then

$$|||Tx||| \le |||Sx|||$$

for all unitarily invariant norms.

Proof. The norms $\|\cdot\|$ and $\|\cdot\|_1$ are dual to each other, in the sense

$$\|x\| = \sup_{\|y\|_1=1} |\operatorname{tr}(xy)|, \qquad \|x\|_1 = \sup_{\|y\|=1} |\operatorname{tr}(xy)|.$$

Hence, $\|T^*x\| \le \|S^*x\|$, $\|T^*x\|_1 \le \|S^*x\|_1$.

Consider the Ky Fan norm $\|\cdot\|_{(k)}$. Its dual norm is $\|\cdot\|_{(k)}^{\sharp} = \max\{\|\cdot\|, (1/k)\|\cdot\|_1\}$. Thus, by duality, $\|Tx\|_{(k)} \le \|Sx\|_{(k)}$ and the result follows by Ky Fan dominance property, [**22**, §3.4]. □

3. CAUCHY-SCHWARTZ INEQUALITIES

Cauchy-Schwartz inequality for $\|\cdot\|$ follows from (3), for $\|\cdot\|_1$ by duality and for other norms by interpolation.

Theorem 3. *Let A be a semifinite von Neumann algebra, let M be a conjugated W^*-module over A and let $a \in A$. Then:*

$$\|\langle x, ay\rangle\| \le \|x\|\|y\|\|a\|, \qquad \|\langle x, ay\rangle\|_1 \le \|\langle \overline{x}, \overline{x}\rangle^{1/2} a \langle \overline{y}, \overline{y}\rangle^{1/2}\|_1; \tag{9}$$

$$\|\langle x, ay\rangle\|_2 \le \|x\|\|a\langle \overline{y}, \overline{y}\rangle^{1/2}\|_2, \quad \text{and} \quad \|\langle x, ay\rangle\|_2 \le \|y\|\|\langle \overline{x}, \overline{x}\rangle^{1/2} a\|_2. \tag{10}$$

In particular, if $A = B(H)$, $\tau = \operatorname{tr}$ and x, y are normal, then

$$|||\langle x, ay\rangle||| \le \left|\left|\left|\langle x, x\rangle^{1/2} a \langle y, y\rangle^{1/2}\right|\right|\right| \tag{11}$$

for all unitarily invariant norms $|||\cdot|||$.

Proof. By (3), we have $\|\langle x, ay\rangle\| \le \|x\|\|ay\| \le \|x\|\|y\|\|a\|$, which proves the first inequality in (9).

For the proof of the second, note that by (4), for all $a \in L^1(A;\tau)$ we have $\tau(b\langle x, ay\rangle) = \tau(\langle xb^*, ay\rangle) = \tau(\langle \overline{y}a^*, b\overline{x}\rangle) = \tau(a\langle \overline{y}, b\overline{x}\rangle)$. Hence for $\langle \overline{x}, \overline{x}\rangle$, $\langle \overline{y}, \overline{y}\rangle \le 1$

$$\|\langle x, ay\rangle\|_1 = \sup_{\|b\|=1} |\tau(b\langle x, ay\rangle)| = \sup_{\|b\|=1} |\tau(a\langle \overline{y}, b\overline{x}\rangle)| \le \sup_{\|b\|=1} \|a\|_1 \|\langle \overline{y}, b\overline{x}\rangle\| \le \|a\|_1,$$

In the general case, let $\varepsilon > 0$ be arbitrary, and let $x_1 = (\langle \overline{x}, \overline{x}\rangle + \varepsilon)^{-1/2}x$ and $y_1 = (\langle \overline{y}, \overline{y}\rangle + \varepsilon)^{-1/2}y$. Then $\overline{x_1} = \overline{x}(\langle \overline{x}, \overline{x}\rangle + \varepsilon)^{-1/2}$ and $\overline{y_1} = \overline{y}(\langle \overline{y}, \overline{y}\rangle + \varepsilon)^{-1/2}$ (by (4)). Thus

$$\langle \overline{x_1}, \overline{x_1}\rangle = (\langle x, x\rangle + \varepsilon)^{-1/2} \langle x, x\rangle (\langle x, x\rangle + \varepsilon)^{-1/2} \le 1,$$

by continuous functional calculus. Hence

$$\begin{aligned} \|\langle x, ay\rangle\|_1 &= \left\|\left\langle (\langle \overline{x}, \overline{x}\rangle + \varepsilon)^{1/2}x_1, a(\langle \overline{y}, \overline{y}\rangle + \varepsilon)^{1/2}y_1 \right\rangle\right\|_1 = \\ &= \left\|\left\langle x_1, (\langle \overline{x}, \overline{x}\rangle + \varepsilon)^{1/2}a(\langle \overline{y}, \overline{y}\rangle + \varepsilon)^{1/2}y_1 \right\rangle\right\|_1 \le \\ &= \left\|(\langle \overline{x}, \overline{x}\rangle + \varepsilon)^{1/2}a(\langle \overline{y}, \overline{y}\rangle + \varepsilon)^{1/2}\right\|_1, \end{aligned} \tag{12}$$

and let $\varepsilon \to 0$. (Note $\|(\langle\overline{x},\overline{x}\rangle+\varepsilon)^{1/2}-\langle\overline{x},\overline{x}\rangle^{1/2}\| \le \varepsilon^{1/2}$.)

To prove (10), by (3) we have

$$|\langle x, ay\rangle|^2 \le \|x\|^2 \langle ay, ay\rangle = \|x\|^2 \langle y, a^*ay\rangle . \tag{13}$$

Apply $\|\cdot\|_1$ to the previous inequality. By (9) we obtain

$$\|\langle x, ay\rangle\|_2^2 \le \|x\|^2 \|\langle y, a^*ay\rangle\|_1 \le \|x\|^2 \|\langle\overline{y},\overline{y}\rangle^{\frac{1}{2}} a^*a \langle\overline{y},\overline{y}\rangle^{\frac{1}{2}}\|_1 = \|x\|^2 \|a\langle\overline{y},\overline{y}\rangle^{\frac{1}{2}}\|_2^2.$$

This proves the first inequality in (10). The second follows by duality

$$\|\langle x, ay\rangle\|_2 = \|\langle y, a^*x\rangle\|_2 \le \|y\|\|a^*\langle\overline{x},\overline{x}\rangle^{1/2}\|_2 = \|y\|\|\langle\overline{x},\overline{x}\rangle^{1/2} a\|_2.$$

Finally, if $A = B(H)$, $\tau = \mathrm{tr}$ and x, y normal. Then (11) holds for $\|\cdot\|_1$ by (9). For the operator norm, it follows by normality. Namely then $x\langle x, x\rangle = \langle x, x\rangle x$ and we can repeat argument from (12). Now, the general result follows from Lemma 2 □

Corollary 4. *If $A = B(H)$ and $M = l^2(A)'$ (Example 2), then (11) is [**7**, Theorem 2.2] (the first formula from the abstract). If $M = L^2(\Omega, A)$, $A = B(H)$, (11) is [**6**, Theorem 3.2] (the second formula from the abstract).*

Remark 3. The inequality (13) for $M = B(H)^n$ is proved in [**21**] using complicated identities and it plays an important role in that paper.

Using three line theorem (which is a standard procedure), we can interpolate results of Theorem 3 to $L^p(A,\tau)$ spaces.

Theorem 5. *Let A be a semifinite von Neumann algebra, and let M be a conjugated W^*-module over A. For all p, q, $r > 1$ such that $1/q + 1/r = 2/p$, we have*

$$\|\langle x, ay\rangle\|_p \le \left\|\left\langle \langle x, x\rangle^{q-1}\overline{x}, \overline{x}\right\rangle^{1/2q} a \left\langle \langle y, y\rangle^{r-1}\overline{y}, \overline{y}\right\rangle^{1/2r}\right\|_p . \tag{14}$$

Proof. Let u, $v \in M_0$ and let $b \in A$. For $0 \le \operatorname{Re}\lambda, \operatorname{Re}\mu \le 1$ consider the function

$$f(\lambda,\mu) = \left\langle (\langle\overline{u},\overline{u}\rangle+\varepsilon)^{-\frac{\lambda}{2}} u(\langle u,u\rangle+\varepsilon)^{\frac{\lambda-1}{2}}, b(\langle\overline{v},\overline{v}\rangle+\varepsilon)^{-\frac{\mu}{2}} v(\langle v,v\rangle+\varepsilon)^{\frac{\mu-1}{2}}\right\rangle .$$

This is an analytic function (obviously).

On the boundaries of the strips, we estimate. For $\operatorname{Re}\lambda = \operatorname{Re}\mu = 0$

$$f(it, is) = \left\langle (\langle\overline{u},\overline{u}\rangle+\varepsilon)^{-\frac{it}{2}} u(\langle u,u\rangle+\varepsilon)^{-\frac{1}{2}+\frac{it}{2}}, b(\langle\overline{v},\overline{v}\rangle+\varepsilon)^{-\frac{is}{2}} v(\langle v,v\rangle+\varepsilon)^{-\frac{1}{2}+\frac{is}{2}}\right\rangle .$$

Since $(\langle\overline{u},\overline{u}\rangle+\varepsilon)^{-it/2}$, $(\langle u,u\rangle+\varepsilon)^{it/2}$, $(\langle\overline{v},\overline{v}\rangle+\varepsilon)^{-is/2}$ and $(\langle v,v\rangle+\varepsilon)^{is/2}$ are unitary operators, and since the norm of $u(\langle u,u\rangle+\varepsilon)^{-1/2}$, $v(\langle v,v\rangle+\varepsilon)^{-1/2}$ does not exceed 1, by (9) we have

$$\|f(it, is)\| \le \|b\|. \tag{15}$$

For $\operatorname{Re}\lambda=\operatorname{Re}\mu=1$

$$f(1+it,1+is)=\left\langle(\langle\overline{u},\overline{u}\rangle+\varepsilon)^{-\frac{1}{2}-\frac{it}{2}}u(\langle u,u\rangle+\varepsilon)^{\frac{it}{2}},b(\langle\overline{v},\overline{v}\rangle+\varepsilon)^{-\frac{1}{2}-\frac{is}{2}}v(\langle v,v\rangle+\varepsilon)^{\frac{is}{2}}\right\rangle.$$

By a similar argument, by (9) we obtain

$$\|f(1+it,1+is)\|_1\le\|b\|_1. \tag{16}$$

For $\operatorname{Re}\lambda=0$, $\operatorname{Re}\mu=1$, by (10) we have

$$f(it,1+is)=\left\langle(\langle\overline{u},\overline{u}\rangle+\varepsilon)^{-\frac{it}{2}}u(\langle u,u\rangle+\varepsilon)^{-\frac{1}{2}+\frac{it}{2}},b(\langle\overline{v},\overline{v}\rangle+\varepsilon)^{-\frac{1}{2}-\frac{is}{2}}v(\langle v,v\rangle+\varepsilon)^{\frac{is}{2}}\right\rangle,$$

and hence

$$\|f(it,1+is)\|_2\le\|b\|_2. \tag{17}$$

Similarly

$$\|f(1+it,is)\|_2\le\|b\|_2. \tag{18}$$

Let us interpolate between (15) and (17). Let $r>1$. Then $1/r=\theta\cdot1+(1-\theta)\cdot0$ for $\theta=1/r$. Then $(\theta\cdot(1/2)+(1-\theta)\cdot0)^{-1}=2r$ and hence, by three line theorem (see [**23**] and [**15**]) we obtain

$$\|f(0+it,1/r)\|_{2r}\le\|b\|_{2r} \tag{19}$$

Similarly, interpolating between (18) and (16) we get

$$\|f(1+it,1/r)\|_{\frac{2r}{r+1}}\le\|b\|_{\frac{2r}{r+1}} \tag{20}$$

since $1/r=\theta\cdot1+(1-\theta)\cdot0$ for same $\theta=1/r$ and $(\theta\cdot1+(1-\theta)\cdot(1/2))^{-1}=2r/(r+1)$

Finally, interpolate between (19) and (20). Then $1/q=\theta\cdot1+(1-\theta)\cdot0$ for $\theta=1/q$ and therefore $(\theta\cdot\frac{r+1}{2r}+(1-\theta)\frac{1}{2r})^{-1}=\frac{1}{2qr}\cdot(r+1+q-1)=p$. Thus

$$\|f(1/q,1/r)\|_p\le\|b\|_p.$$

i.e.

$$\left\|\left\langle(\langle\overline{u},\overline{u}\rangle+\varepsilon)^{-\frac{1}{2q}}u(\langle u,u\rangle+\varepsilon)^{\frac{1-q}{2q}},b(\langle\overline{v},\overline{v}\rangle+\varepsilon)^{-\frac{1}{2r}}v(\langle v,v\rangle+\varepsilon)^{\frac{1-q}{2q}}\right\rangle\right\|_p\le\|b\|_p.$$

After substitutions

$$u=x\langle x,x\rangle^{(q-1)/2},\quad v=\langle y,y\rangle^{(r-1)/2},\quad b=(\langle\overline{u},\overline{u}\rangle+\varepsilon)^{1/2q}a(\langle\overline{v},\overline{v}\rangle+\varepsilon)^{1/2r},$$

we obtain

$$\left\|\left\langle x\langle x,x\rangle^{(q-1)/2}(\langle x,x\rangle^q+\varepsilon)^{(1-q)/2q},ay\langle y,y\rangle^{(q-1)/2}(\langle y,y\rangle^q+\varepsilon)^{(1-q)/2q}\right\rangle\right\|_p\le$$
$$\le\left\|\left(\left\langle\langle x,x\rangle^{q-1}\overline{x},\overline{x}\right\rangle+\varepsilon\right)^{1/2q}a\left(\left\langle\langle y,y\rangle^{r-1}\overline{y},\overline{y}\right\rangle+\varepsilon\right)^{1/2r}\right\|_p$$

which after $\varepsilon\to0$ yields (14), using the argument similar to that in [**16**, Lemma 1.3.9]. □

Remark 4. In a special case $A = B(H)$, $\tau = \mathrm{tr}$, $M = l^2(A)'$, $r = q = p$ formula (14) becomes [**8**, Theorem 2.1] (the main result).

Also, for $A = B(H)$, $\tau = \mathrm{tr}$, $M = L^2(\Omega, A)$ formula (14) becomes [**6**, Theorem 3.3] (the first displayed formula from the abstract), there proved with an additional assumption that Ω is σ-finite.

In the next two section we derive some inequalities that regularly arise from Cauchy-Schwartz inequality.

4. INEQUALITIES OF THE TYPE $|1 - \langle x, y\rangle| \geq (1 - \|x\|^2)^{1/2}(1 - \|y\|^2)^{1/2}$

The basic inequality can be proved as

$$|(1 - \langle x, y\rangle)^{-1}| \leq \sum_{n=0}^{+\infty} |\langle x, y\rangle|^n \leq \sum_{n=0}^{+\infty} \|x\|^n\|y\|^n \leq$$
$$\left(\sum_{n=0}^{+\infty} \|x\|^{2n}\right)^{1/2} \left(\sum_{n=0}^{+\infty} \|y\|^{2n}\right)^{1/2} = (1 - \|x\|^2)^{-1/2}(1 - \|y\|^2)^{-1/2}.$$

Following this method we prove:

Theorem 6. *Let M be a conjugated W^*-module over $A = B(H)$, let $x, y \in M_0$ be normal, and let $\langle x, x\rangle$, $\langle y, y\rangle \leq 1$. Then*

$$\left|\!\left|\!\left|(1 - \langle x, x\rangle)^{1/2} a (1 - \langle y, y\rangle)^{1/2}\right|\!\right|\!\right| \leq |\!|\!| a - \langle x, ay\rangle |\!|\!|$$

in any unitarily invariant norm.

Proof. We use examples 4 and 5.

Denote $Ta = \langle x, ay\rangle$. We have $T^2a = \langle x, \langle x, ay\rangle\, y\rangle = \langle x \otimes x, a(y \otimes y)\rangle$ and by induction $T^k a = \left\langle x^{\otimes k}, ay^{\otimes k}\right\rangle$. Suppose $\|x\|$, $\|y\| \leq \delta < 1$. Then $\|x^{\otimes k}\|$, $\|y^{\otimes k}\| \leq \delta^k$ and hence $|\!|\!|T^k|\!|\!| \leq \delta^{2k}$. Then

$$(I - T)^{-1} = \sum_{n=0}^{+\infty} T^k. \tag{21}$$

Put $b = (I - T)^{-1}a$. Then

$$|\!|\!| b |\!|\!| = \left|\!\left|\!\left|\sum_{k=0}^{+\infty} T^k a\right|\!\right|\!\right| = \left|\!\left|\!\left|\sum_{k=0}^{+\infty} \left\langle x^{\otimes k}, ay^{\otimes k}\right\rangle\right|\!\right|\!\right| = \left|\!\left|\!\left|\left\langle \sum_{k=0}^{+\infty} x^{\otimes k}, a\sum_{k=0}^{+\infty} y^{\otimes k}\right\rangle\right|\!\right|\!\right| \leq \tag{22}$$
$$\leq \left|\!\left|\!\left|\left\langle \sum_{k=0}^{+\infty} x^{\otimes k}, \sum_{k=0}^{+\infty} x^{\otimes k}\right\rangle^{1/2} a \left\langle \sum_{k=0}^{+\infty} y^{\otimes k}, \sum_{k=0}^{+\infty} y^{\otimes k}\right\rangle^{1/2}\right|\!\right|\!\right|.$$

by (11) and normality of x and y. Invoking (8), inequality (22) becomes

$$\left|\!\left|\!\left|(I-T)^{-1}a\right|\!\right|\!\right| \le \left|\!\left|\!\left|(1-\langle x,x\rangle)^{-1/2}a(1-\langle y,y\rangle)^{-1/2}\right|\!\right|\!\right|. \tag{23}$$

Finally, note that the mappings $I-T$ and $a\mapsto(1-\langle x,x\rangle)^{-1/2}a(1-\langle y,y\rangle)^{-1/2}$ commute (by normality of x and y) and put $(1-\langle x,x\rangle)^{-1/2}(a-Ta)(1-\langle y,y\rangle)^{-1/2}$ in place of a, to obtain the conclusion.

If $\|x\|$, $\|y\|=1$ then put δx instead of x and let $\delta\to 1-$. □

Remark 5. If $M=L^2(\Omega,A)$ this is [**6**, Theorem 4.1] (the last formula from the abstract). If $M=B(H)\times B(H)$, $x=(I,A)$, $y=(I,B)$ then it is [**7**, Theorem 2.3] (the last formula from the abstract).

Remark 6. Instead of $t\mapsto 1-t$ we may consider any other function f such that $1/f$ is well defined on some $[0,c)$ and has Taylor expansion with positive coefficients, say c_n. Then distribute $\sqrt{c_n}$ on both arguments in inner product in (22) and after few steps we get

$$\left|\!\left|\!\left|(f(x^*x))^{1/2}a(f(y^*y))^{1/2}\right|\!\right|\!\right| \le |\!|\!|f(T)|\!|\!|.$$

For instance, for $t\mapsto(1-t)^\alpha$, $\alpha>0$ we have $(1-t)^{-\alpha}=\sum c_nt^n$, where $c_n=\Gamma(n+\alpha)/(\Gamma(\alpha)n!)>0$ and we get

$$\left|\!\left|\!\left|(1-\langle x,x\rangle)^{\alpha/2}a(1-\langle y,y\rangle)^{\alpha/2}\right|\!\right|\!\right| \le |\!|\!|(I-T)^\alpha a|\!|\!| \tag{24}$$

in any unitarily invariant norm. For $M=A=B(H)$, (24) reduces to

$$\left|\!\left|\!\left|(1-x^*x)^{\alpha/2}a(1-y^*y)^{\alpha/2}\right|\!\right|\!\right| \le \left|\!\left|\!\left|\sum_{n=0}^{+\infty}(-1)^n\binom{a}{n}x^{*n}ay^n\right|\!\right|\!\right|,$$

which is the main result of [**11**]. Varying f, we may obtain many similar inequalities.

Finally, if normality condition on x and y is dropped, we can use (14) to obtain some inequalities in $L^p(A;\tau)$ spaces.

Theorem 7. *Let M be a conjugated W^*-module over a semifinite von Neumann algebra A, let x, $y\in M_0$, $\|x\|$, $\|y\|<1$ and let*

$$\Delta_z=\left\langle\sum_{n=0}^{+\infty}z^{\otimes n},\sum_{n=0}^{+\infty}z^{\otimes n}\right\rangle^{-1/2}, \quad \text{for } z\in\{x,y,\overline{x},\overline{y}\}. \tag{25}$$

Then

$$\|\Delta_x^{1-1/q}a\Delta_y^{1-1/r}\|_p \le \|\Delta_{\overline{x}}^{-1/q}(a-\langle x,ay\rangle)\Delta_{\overline{y}}^{-1/r}\|_p,$$

for all p, q, $r>1$ such that $1/q+1/r=2/p$.

Proof. Let $b = (I - T)a$. We have $a = (I - T)^{-1}b$ and hence

$$\|\Delta_x^{1-1/q} a \Delta_y^{1-1/r}\|_p = \left\|\Delta_x^{1-1/q} \sum_{n=0}^{+\infty} \left\langle x^{\otimes n}, by^{\otimes n}\right\rangle \Delta_y^{1-1/r}\right\|_p = $$
$$= \left\| \sum_{n=0}^{+\infty} \left\langle x^{\otimes n} \Delta_x^{1-1/q}, by^{\otimes n} \Delta_y^{1-1/r}\right\rangle \right\|_p \le \|ubv\|_p,$$

by (14), where

$$u = \sum_{n=0}^{+\infty} \left\langle \sum_{n=0}^{+\infty} \left\langle x^{\otimes n} \Delta_x^{1-1/q}, x^{\otimes n} \Delta_x^{1-1/q}\right\rangle^{q-1} \Delta_x^{1-1/q} \overline{x}^{\otimes n}, \Delta_x^{1-1/q} \overline{x}^{\otimes n}\right\rangle^{1/2q}.$$

After a straightforward calculation, we obtain $u = \Delta_{\overline{x}}^{-1/q}$ and similarly $v = \Delta_{\overline{y}}^{-1/r}$ and the conclusion follows. □

Remark 7. When $A = B(H)$, $\tau = \mathrm{tr}$, this is the main result of [**10**], from which we adapted the proof for our purpose. However, the application of Fock module technique significantly simplified the proof.

Also, in [**10**], the assumptions are relaxed to $r(T_{x,x})$, $r(T_{y,y}) \le 1$, where r stands for the spectral radius and $T_{x,y}(a) = \langle x, ay\rangle$. This easily implies $r(T_{x,y}) \le 1$. First, it is easy to see that $\|T_{z,z}\| = \|z\|^2$. Indeed, by (9) we have $\|T_{z,z}\| \le \|z\|^2$. On the other hand, choosing $a = 1$ we obtain $\|T_{z,z}\| \ge \|T_{z,z}(1)\| = \|\langle z, z\rangle\| = \|z\|^2$. Again, by (9), we have $\|T_{x,y}\| \le \|x\|\|y\| = \sqrt{\|T_{x,x}\|\|T_{y,y}\|}$. Apply this to $x^{\otimes n}$ and $y^{\otimes n}$ instead of x and y and we get $\|T_{x,y}^n\| \le \sqrt{\|T_{x,x}^n\|\|T_{y,y}^n\|}$ from which we easily conclude $r(T_{x,y})^2 \le r(T_{x,x})r(T_{y,y})$ by virtue of spectral radius formula. (In a similar way, we can conclude $r(T_{x,x}) = \|x\|$ for normal x.)

Thus, if both $r(T_{x,x})$, $r(T_{y,y}) < 1$, the series in (25) converge. If some of $r(T_{x,x})$, $r(T_{y,y}) = 1$ then define $\Delta_x = \lim_{\delta\to 0} \Delta_{\delta x} = \inf_{0<\delta<1} \Delta_{\delta x}$, etc, and the result follows, provided that series that defines $\Delta_{\overline{x}}$ and $\Delta_{\overline{y}}$ are weakly convergent.

5. GRÜSS TYPE INEQUALITIES

For classical Grüss inequality, see [**19**, §2.13]. We give a generalization to Hilbert modules following very simple approach from [**20**] in the case of Hilbert spaces.

Theorem 8. *Let M be a conjugated W^*-module over $B(H)$, and let $e \in M$ be such that $\langle e, e\rangle = 1$. Then the mapping $\Phi : M \times M \to B(H)$, $\Phi(x, y) = \langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle$. is a semi-inner product.*

If, moreover, e is central (i.e. $ae = ea$ for all a) and x, $y \in M_0$ are normal with respect to Φ and some conjugation then

$$\text{(26)}\quad \|\langle x, ay\rangle - \langle x, e\rangle \langle e, ay\rangle\| \le \left\|(\langle x, x\rangle - |\langle x, e\rangle|^2)^{1/2} a (\langle y, y\rangle - |\langle y, e\rangle|^2)^{1/2}\right\|$$

in any unitarily invariant norm.

Finally, if x, y *belongs to balls with diameters* $[me, Me]$ *and* $[pe, Pe]$ *(*m, M, p, $P \in \mathbf{R}$*), respectively, then*

$$\|\!|\langle x, ay\rangle - \langle x, e\rangle \langle e, ay\rangle|\!\| \leq \frac{1}{4} \|\!|a|\!\| \, |M - m||P - p|. \tag{27}$$

(Here, x *belongs to the ball with diameter* $[y, z]$ *iff* $\|x - \frac{y+z}{2}\| \leq \|\frac{z-y}{2}\|$*.)*

Proof. The mapping Φ is obviously linear in y and conjugate linear in x. Moreover, by inequality (3)

$$\langle x, e\rangle \langle e, x\rangle = |\langle e, x\rangle|^2 \leq \|e\|^2 \langle x, x\rangle = \langle x, x\rangle ,$$

i.e. $\Phi(x, x) \geq 0$. Hence Φ is an A-valued (semi)inner product.

If e is central it is easy to derive $\Phi(x, ay) = \Phi(a^* x, y)$. Hence, if x, y are normal, then, by (11) we obtain

$$\|\!|\Phi(x, ay)|\!\| \leq \left|\!\left|\!\left|\Phi(x, x)^{1/2} a \Phi(y, y)^{1/2}\right|\!\right|\!\right| \tag{28}$$

in any unitarily invariant norm. Writing down the expression for Φ we obtain (26).

Finally, for the last conclusion, note that $\Phi(x, x) = \Phi(x - ec, x - ec)$ for any $c \in \mathbf{C}$ (direct verification), and hence $\Phi(x, x) \leq \langle x - ec, x - ec\rangle$, which implies $\|\Phi(x, x)^{1/2}\| \leq \|x - ec\|$. Choosing $c = (M + m)/2$, we obtain $\|\Phi(x, x)^{1/2}\| \leq (M - m)/2$. Similarly, $\|\Phi(y, y)^{1/2}\| \leq (P - p)/2$. Thus (28) implies (27). $\square$

Remark 8. Choose $M = L^2(\Omega, \mu)$, $\mu(\Omega) = 1$ and choose e to be the function identically equal to 1. Then

$$\Phi(x, ay) = \int_\Omega x(t)^* ay(t) \mathrm{d}\mu(t) - \int_\Omega x(t)^* \mathrm{d}\mu(t) \int_\Omega ay(t) \mathrm{d}\mu(t),$$

and from (26) and (27) we obtain main results of [**4**].

Remark 9. Applying other inequalities from section 3, we can derive other results from [**4**]. Also, applying inequality $|\langle x, ay\rangle|^2 \leq \|x\|^2 \langle ay, ay\rangle$ to the mapping Φ instead of $\langle \cdot, \cdot \rangle$ we obtain the key result of [**9**], there proved by complicated identities.

6. CONCLUDING REMARKS

Both, elementary operators and i.p.t.i. transformers on $B(H)$ are special case of

$$\langle x, Ty\rangle \tag{29}$$

where x, y are vectors from some Hilbert W^*-module M over $B(H)$ and $T : M \to M$ is given by left action of $B(H)$.

Although there are many results independent of the representation (29), a lot of inequalities related to elementary operators and i.p.t.i. transformers can be reduced to elementary properties of the $B(H)$-valued inner product.

Acknowledgments. The author was supported in part by the Ministry of education and science, Republic of Serbia, Grant #174034.

REFERENCES

1. Matharu, Jagjit Singh and Moslehian, Mohammad Sal: *Gruss inequality for some types of positive linear maps*, J Operator Theory, Journal of Operator Theory (73), (1), 2015, pp. 265–278
2. Moslehian, Mohammad Sal and Rajić, Rajna: *A Gruss inequality for n-positive linear maps*, Linear Algebra Appl., Linear Algebra and its Applications, (433), 2010, pp. 1555–1560
3. Balasubramanian, Sriram: *On the Grüss inequality for unital 2-positive linear maps*,Oper. Matrices, Operators and Matrices, (10:3), 2016, pp. 643–649
4. Jocić, Danko R. and Krtinić, Đorđe and Moslehian, Mohammad Sal: *Landau and Grüss type inequalities for inner product type integral transformers in norm ideals*, Math. Inequal. Appl., Mathematical Inequalities & Applications, (16:1), 2013, pp. 109–125
5. Ghazanfari, Amir Ghasem: *A Grüss type inequality for vector-valued functions in Hilbert C^*-modules*, J. Inequal. Appl., Journal of Inequalities and Applications, 2014, (16), 2014
6. Jocić, Danko R.: *Cauchy-Schwarz norm inequalities for weak*-integrals of operator valued functions*, J. Funct. Anal., Journal of Functional Analysis, (218:2), 2005, pp. 318-346
7. Jocić, Danko R.: *Cauchy-Schwarz and means inequalities for elementary operators into norm ideals*, Proc. Amer. Math. Soc., Proceedings of the American Mathematical Society, (126:9), 1998, pp. 2705-2711
8. Jocić, Danko R.: *The Cauchy-Schwarz norm inequality for elementary operators in Schatten ideals*, J. London Math. Soc., Journal of the London Mathematical Society, (60:3), 1999, pp. 925-934
9. Jocić, Danko R. and Krtinić, Đorđe and Lazarević, Milan and Melentijević, Petar and Milošević, Stefan: *Refinements of inequalities related to Landau-Grüss inequalities for elementary operators acting on ideals associated to p-modified unitarily invariant norms*, Complex Anal. Oper. Theory, Complex Analysis and Operator Theory, (12:1), 2018, pp. 195-205
10. Jocić, Danko R. and Milošević, Stefan and Đurić, Vladimir: *Norm inequalities for elementary operators and other inner product type integral transformers with the spectra contained in the unit disc*, Filomat, Filomat, (31:2), 2017, pp. 197-206
11. Milošević, Stefan: *Norm inequalities for elementary operators related to contractions and operators with spectra contained in the unit disk in norm ideals*, Adv. Oper. Theory, Advances in Operator Theory, (1:2), 2016, pp. 147-159

12. Lumer, Ginter and Rosenblum, Marvin: *Linear operator equations*, Proc. Amer. Math. Soc., Proceedings of the American Mathematical Society, (10:1), 1959, pp. 32-41

13. Takesaki, Masamichi: *Theory of Operator Algebras I*, Springer, Encyclopaedea of Mathematical Sciences, (124), Berlin, Heidelberg, etc., 2001

14. Lance, E. Christopher: *Hilbert C^*-Modules: A toolkit for operator algebraists*, PUBLISHER = Cambridge University Press, 1995

15. Fack, Thierry and Kosaki, Hideki: *Generalized s-numbers of τ-measurable operators*, Pacific J Math., Pacific Journal of Mathematics, (123:2), 1986, pp. 269-300

16. Manuilov, Vladimir Markovich and Troitsky, Evgeniy Vadimovich: *Hilbert C^*-modules*, AMS, Translations of mathematical monographs, (226), Providence, Rhode Island, 2005

17. Haagerup, Uffe: *L^p-spaces associated with an arbitrary von Neumann algebra*, Colloques Internationaux, C.N.R.S., Colloques Internationaux, C.N.R.S., (274), 1977, pp. 175-184

@misc Terp Terp, Marianne: *L^p-spaces associated with von Neumann algebras*, Copenhagen Univ., 1981

18. Takesaki, Masamichi: *Tomita's theory of modular Hilbert algebras and its applications*, Springer, Lecture Notes Math., (128), 1970

19. Mitrinović, Dragoslav S.: *Analytic inequalities*, Springer, Die Grundlehren der Matematischen Wissenschaften in Einzeldarstellungen Band, (165), 1970

20. Dragomir, Sever Silvestru: *Generalization of Grüss's Inequality in Inner Product Spaces and Applications*, J Math. Anal. Appl., Journal of Mathematical Analysis and Applications, (237:1), 1999, pp. 74-82

21. Jocić, Danko R. and Milošević, Stefan: *Refinements of operator Cauchy-Schwarz and Minkowski inequalities for p-modified norms and related norm inequalities*, Linear Algebra Appl., Linear Algebra and its Applications, (488:1), 2016, pp. 284-301

22. Gohberg, Israel C. and Kreĭn, Mark G.: *Itroduction to the theory of linear nonselfadjoint operators*, American Mathematical Society, ADDRESS = Providence RI, Translations of Mathematical Monographs, (018), 1st eddition, 1969

23. Kosaki, Hideki: *Applications of the Complex Interpolation Method to a von Neumann Algebra (Non-commutative L^p-Spaces)*, J. Funct. Anal., Journal of Functional Analysis, (56:1), 1984, pp. 29-78

Dragoljub J. Kečkić
University of Belgrade,
Faculty of Mathematics,
Studentski trg 16-18,
11000 Beograd,
Serbia E-mail: *keckic@matf.bg.ac.rs*

(Received 23.01.2018)
(Revised 05.04.2020)