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BLOCKING SETS FOR CYCLES AND PATHS DESIGNS

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In this paper, we study blocking sets for C_4, P_3 and P_5-designs. In the case of C_4-designs and P_3-designs we determine the cases in which the blocking sets have the largest possible range of cardinalities. These designs are called *largely blocked*. Moreover, a blocking set T for a G-design is called perfect if in any block the number of edges between elements of T and elements in the complement is equal to a constant. In this paper, we consider *perfect blocking sets* for C_4-designs and P_5-designs.

1. INTRODUCTION

Let K_v be the complete undirected graph defined on a vertex set X. Given a graph with n vertices, a G-design of order v (briefly a $G(v)$-*design*), for $v \geq n$, is a pair $\Sigma = (X, \mathcal{B})$, where $\mathcal{B}$ is a partition of the edge set of K_v into classes generating graphs all isomorphic to G. The classes of $\mathcal{B}$ are said to be the *blocks* of Σ.

Let $\Sigma = (X, \mathcal{B})$ be a G-design of order v. A *transversal* T of Σ is a subset of X intersecting every block of Σ. A *blocking set* T of Σ is a transversal such that also its complement $C_X(T)$ is a transversal of Σ. So T is a blocking sets if and only if every block of Σ contains elements of T and elements of $C_X(T)$. In what follows, we will indicate by $B(\Sigma)$ the set of all possible $p \in \mathbb{N}$ for which there exist in Σ blocking sets of cardinality p.

The existence of possible blocking sets has been studied in numerous papers (see [2, 3, 5–16]) for t-designs, projective planes, symmetric designs, block designs, balanced and almost balanced path designs and G-designs when G has fewer than 5 edges.

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In [10] the notion of *largely blocked* C_k-designs was introduced, for $k \geq 4$. The idea is that the set $B(\Sigma)$ has the maximum possible cardinality. Indeed, Gionfriddo and Milazzo in [10] proved that:

Theorem 1. *If* $\Sigma = (X, \mathcal{B})$ *is a* $C_k(v)$*-design and* B *is a blocking set of cardinality* p*, then* $k \geq 4$ *and:*

$$\beta_1 = \lceil \frac{v}{2} - \frac{\sqrt{kv[(k-4)v+4]}}{2k} \rceil \leq p \leq \lfloor \frac{v}{2} + \frac{\sqrt{kv[(k-4)v+4]}}{2k} \rfloor = \beta_2.$$

So, a C_k-design Σ is called *largely blocked* if $B(\Sigma) = [\beta_1, \beta_2]$ (the closed interval of integers). In this paper we prove that for any $v \equiv 1 \mod 8$ there exists a largely blocked C_4-design of order v and we determine the spectrum of largely blocked P_3-designs. We study also *perfect blocking sets*, analizing the idea of a blocking set distributed in an optimal and homogeneous way. The notion was introduced in [4] and it requires that any block contains a constant number of edges between vertices of the blocking set T and vertices of $C_X(T)$. In [4], the spectrum of P_3-designs with a perfect blocking set is determined. In this paper, we easily determine the spectrum of C_4-designs having a perfect blocking set, as it is follows from the result on the spectrum of largely blocked C_4-designs. Moreover, we study the problem for P_5-designs, determining the spectrum in the case that the constant is 2, by using a peculiar construction for P_3-designs with a perfect blocking set.

2. LARGELY BLOCKED C_4-DESIGNS

A 4-cycle on the vertices $\{x, y, z, t\}$ with edges $\{x, y\}$, $\{y, z\}$, $\{z, t\}$ and $\{x, t\}$ is denoted by (x, y, z, t). The spectrum of C_4-designs is known:

Theorem 2. *There exists a* C_4*-designs of order* v *if and only if* $v \equiv 1 \mod 8$*,* $v \geq 9$.

By Theorem 1 we determine the bound on the cardinality of a possible blocking set for a C_4-design.

Proposition 3. *Let* $\Sigma = (X, \mathcal{B})$ *be a* C_4*-design of order* v *and let* $T \subset X$ *be a blocking set of cardinality* t*. Then:*

$$\lceil \tfrac{v}{2} - \tfrac{\sqrt{v}}{2} \rceil \leq t \leq \lfloor \tfrac{v}{2} + \tfrac{\sqrt{v}}{2} \rfloor.$$

In this way (see [10]), we get two parameters $\beta_1 = \lceil \frac{v}{2} - \frac{\sqrt{v}}{2} \rceil$ and $\beta_2 = \lfloor \frac{v}{2} + \frac{\sqrt{v}}{2} \rfloor = v - \beta_1$ such that $B(\Sigma) \subseteq [\beta_1, \beta_2]$ (closed interval of integers). So, a natural definition is the following:

Definition 4 ([10])**.** A C_4-design Σ is largely blocked if $B(\Sigma) = [\beta_1, \beta_2]$.

We want to prove that for any $v \equiv 1 \mod 8$ there exists a largely blocked C_4-design. In order to do that we need the following constructions.

Proposition 5. *If there exists a C_4-design $\Sigma = (X, \mathcal{B})$ of order v with blocking sets $T_1, \ldots, T_s$ of cardinalities, respectively, $p_1, \ldots, p_s$ such that $T_1 \subset \cdots \subset T_s$, then there exists a C_4-design $\Sigma' = (X', \mathcal{B}')$ of order $v + 8$ with blocking sets $T'_1, \ldots, T'_s$ of cardinalities, respectively, $p_1 + 4, \ldots, p_s + 4$ such that $T'_1 \subset \cdots \subset T'_s$. Moreover, if in any block $(a, b, c, d) \in \mathcal{B}$ there exists a vertex $a \in T_1$ such that either $b, d \notin T_s$ or $c \notin T_s$, then also in any block $(a, b, c, d) \in \mathcal{B}'$ there exists a vertex $a \in T'_1$ such that either $b, d \notin T'_s$ or $c \notin T'_s$.*

Proof. The proof follows by the construction given in [9, Theorem 4.1]. Indeed, let $\Sigma_1 = (X_1, \mathcal{B}_1)$ be a C_4-design of order $v = 1 + 8k$, $k \in \mathbb{N}$, $k \geq 1$, with $T_1, \ldots, T_s$ blocking sets of cardinality $p_1, \ldots, p_s$ and $X_1 = \{0, x_1, \ldots, x_{8k}\}$. Let $X_2 = \{0, 1, \ldots, 8\}$ so that $X_1 \cap X_2 = \{0\}$ and $0 \notin T_s$. Consider the 4-cycle system $\Sigma_2 = (X_2, \mathcal{B}_2)$ having as blocks $(0, 1, 5, 2)$ and all its translates. Then $\{1, 2, 3, 4\}$ is a blocking set for Σ_2. Consider also the family $\mathcal{B}_3$ of blocks:

$$(i, x_j, i + 4, x_{j+4k})$$

for $i = 1, 2, 3, 4$ and $j = 1, \ldots, 4k$. Then $\Sigma = (X_1 \cup X_2, \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3)$ is a 4-cycle system of order $v + 8$ having $T'_1 = T_1 \cup \{1, 2, 3, 4\}$, $\ldots$, $T'_s = T_s \cup \{1, 2, 3, 4\}$ as blocking sets. This proves the statement. □

Lemma 6. *Let X and Y be disjoint sets, with $|X| = |Y| = 8$. Let $T_X \subset X$ and $T_Y \subset Y$ such that $|T_X| = 4$ and $|T_Y| = 3$ and let $y \in Y \setminus T_Y$. Then there exists a decomposition of $K_{X,Y}$ in a family $\mathcal{B}$ of 4-cycles having $T = T_X \cup T_Y$ and $T \cup \{y\}$ as blocking sets. Moreover, in any block $(a, b, c, d) \in \mathcal{B}$ there exists a vertex $a \in T$ such that $c \notin T \cup \{y\}$.*

Proof. Let $X = \{x_i \mid i = 1, \ldots, 8\}$ and $Y = \{y_i \mid i = 1, \ldots, 8\}$. Let:

$$\mathcal{B} = \{(x_i, y_j, x_{i+4}, y_{j+4}) \mid i, j = 1, 2, 3, 4\}.$$

Then the blocks of $\mathcal{B}$ decompose $K_{X,Y}$ in 4-cycles and it is easy to get statement by taking $T_X = \{x_1, x_2, x_3, x_4\}$, $T_Y = \{y_1, y_2, y_3\}$ and $y = y_4$. □

Lemma 7. *Let X, Y and Z be pairwise disjoint sets, with $|X| = |Y| = |Z| = 8$. Let $\infty \notin X \cup Y \cup Z$, $T_X \subset X$, $T_Y \subset Y$ and $T_Z \subset Z$, with $|T_X| = 4$, $|T_Y| = 3$ and $|T_Z| = 3$. Then there exists a decomposition of $K_{X,Y,Z} \cup K_{X \cup \{\infty\}}$ in a family $\mathcal{B}$ of 4-cycles having $T = T_X \cup T_Y \cup T_Z$, $T \cup \{y\}$ and $T \cup \{y, z\}$ as blocking sets, for some $y \in Y \setminus T_Y$ and $z \in Z \setminus T_Z$. Moreover, in any block $(a, b, c, d) \in \mathcal{B}$ there exists a vertex $a \in T$ such that either $b, d \notin T \cup \{y, z\}$ or $c \notin T \cup \{y, z\}$.*

Proof. Let $X = \{x_i \mid i = 1, \ldots, 8\}$, $Y = \{y_i \mid i = 1, \ldots, 8\}$, $Z = \{z_i \mid i = 1, \ldots, 8\}$ and $T_X = \{x_1, x_2, x_3, x_4\}$, $T_Y = \{y_1, y_2, y_3\}$, $T_Z = \{z_1, z_2, z_3\}$. Let us consider the C_4-design $\Sigma = (X \cup \{\infty\}, \mathcal{B})$ such that $\mathcal{B}$ is given by the blocks:

$$(x_1, x_5, x_2, x_6),\ (\infty, x_1, x_2, x_8),\ (\infty, x_2, x_3, x_7),\ (\infty, x_3, x_8, x_6),$$
$$(\infty, x_4, x_7, x_5),\ (x_2, x_4, x_8, x_7),\ (x_8, x_5, x_3, x_1),\ (x_1, x_4, x_6, x_7),\ (x_3, x_4, x_5, x_6).$$

Then $\{x_1, x_2, x_3, x_4\}$ is a blocking set for this C_4-design.

Consider now the decomposition of $K_{X,Y,Z}$ in 4-cycles given by the following family $\mathcal{B}'$ of blocks:

$$(x_i, y_j, x_{i+4}, y_{j+4}),\ (x_i, z_j, x_{i+4}, z_{j+4}),\ (y_i, z_j, y_{i+4}, z_{j+4}) \quad \text{for } i, j = 1, 2, 3, 4.$$

So $\mathcal{C} = \mathcal{B} \cup \mathcal{B}'$ provides us a decomposition of $K_{X,Y,Z} \cup K_{X\cup\{\infty\}}$ in 4-cycles such that $T = T_X \cup T_Y \cup T_Z$ is a blocking set for any block in $\mathcal{C}$, with the exception of (y_4, z_4, y_8, z_8). So, take the blocks:

$$(y_4, z_4, y_8, z_8),\ (x_1, x_5, x_2, x_6),\ (x_1, y_4, x_5, y_8),\ (x_2, y_4, x_6, y_8)$$

and replace them with:

$$(x_1, y_4, z_4, y_8),\ (y_4, x_5, x_1, x_6),\ (z_8, y_4, x_2, y_8),\ (y_8, x_5, x_2, x_6).$$

Denoted by $\mathcal{C}'$ the family of blocks that we obtain, it is easy to see that T is a blocking set for any block in $\mathcal{C}'$ and that these blocks provide a decomposition of $K_{X,Y,Z} \cup K_{X\cup\{\infty\}}$ in 4-cycles. It is also easy to see that $T \cup \{y_4\}$ and $T \cup \{y_4, z_5\}$ are blocking sets. Moreover, we can easily see that each 4-cycle of $\mathcal{C}'$ can be decomposed in two 3-paths in such a way that T, $T \cup \{y_4\}$ and $T \cup \{y_4, z_5\}$ are blocking sets also for these 3-paths. □

Now we can prove the following:

Theorem 8. *For any* $v \equiv 1 \mod 8$ *there exists a largely blocked* C_4*-design of order* v.

Proof. **Case v is a square.** Let $v = (2r+1)^2$ for some $r \in \mathbb{N}$, so that $v \equiv 1 \mod 8$. We want, first, to prove the statement in this case.

Suppose that $r = 1$, so that $v = 9$. Then the statement follows by [10, Theorem 3.2]. More precisely, in [10, Theorem 3.2] it is proved that there exists a C_4-design of order 9 largely blocked with T and T' blocking sets of cardinality 3 and 4 such that $T \subset T'$. Moreover, we see that in any block (a, b, c, d) there exists a vertex $a \in T$ such that either $b, d \notin T'$ or $c \notin T'$.

Suppose now that $r > 1$. Then $v = 4r^2 + 4r + 1$ and by Proposition 3 for any C_4-design Σ we have $B(\Sigma) \subseteq \{\lceil \frac{v}{2} - \frac{\sqrt{v}}{2} \rceil, \ldots, \lfloor \frac{v}{2} + \frac{\sqrt{v}}{2} \rfloor\} = \{2r^2 + r, \ldots, 2r^2 + 3r + 1\}$. Consider $X_1, \ldots, X_{\frac{r^2+r}{2}}$ disjoint sets, each of cardinality 8, and take an element $\infty \notin \bigcup_i X_i$. For any $i = 1, \ldots, \frac{r^2+r}{2}$ take a subset $T_i \subset X_i$ such that:

$$|T_i| = \begin{cases} 3 & \text{for } i = 1, \ldots, r \\ 4 & \text{for } i = r+1, \ldots, \frac{r^2+r}{2}. \end{cases}$$

So $T = \bigcup_i T_i$ is a set of cardinality $2r^2 + r$. For any $i = 1, \ldots, r$ take $x_i \in X_i$ such that $x_i \notin T_i$. Consider any bijection:

$$\phi\colon \{\{i, j\} \mid i, j = 1, \ldots, r,\ i \neq j\} \to \{r+1, \ldots, \tfrac{r^2+r}{2}\}.$$

By Lemma 7 for any $i, j = 1, \dots, r$, $i \neq j$, there exists a decomposition of

$$K_{X_i, X_j, X_{\phi(\{i,j\})}} \cup K_{X_{\phi(\{i,j\})} \cup \{\infty\}}$$

in a family $\mathcal{B}_{ij}$ of 4-cycles such that $T_i \cup T_j \cup T_{\phi(\{i,j\})}$, $T_i \cup T_j \cup T_{\phi(\{i,j\})} \cup \{x_i\}$ and $T_i \cup T_j \cup T_{\phi(\{i,j\})} \cup \{x_i, x_j\}$ are blocking sets for $\mathcal{B}_{ij}$.

Then, by the case $v = 9$, for $i = 1, \dots, r$ there exists a C_4-design $\Sigma_i = (X_i \cup \{\infty\}, \mathcal{C}_i)$ having T_i and $T_i \cup \{x_i\}$ as blocking sets.

At last, for any $i, j = 1, \dots, \frac{r^2+r}{2}$, $i \neq j$, such that both i, j are not simultaneously in some of the triples of $\{\{p, q, \phi(\{p, q\})\} \mid p, q = 1, \dots, r,\ p \neq q\}$ by Lemma 6 we can consider a decomposition of K_{X_i, X_j} in a family D_{ij} of 4-cycles having $T_i \cup T_j$ and $T_i \cup T_j \cup \{x_i\}$ if $i = 1, \dots, r$ as blocking sets.

If we call $\mathcal{E} = \bigcup \mathcal{B}_{ij} \cup \bigcup \mathcal{C}_i \cup \bigcup \mathcal{D}_{ij}$, then $\Sigma = (\bigcup X_i \cup \{\infty\}, \mathcal{E})$ is a 4-cycle system of order $v = 4r^2 + 4r + 1$ having as blocking sets T and $T \cup \{x_1, \dots, x_s\}$ for any $s = 1, \dots, r$. So there exist for Σ blocking sets of cardinality $2r^2 + r, \dots, 2r^2 + 2r$. This immediately implies that there exist for Σ blocking sets of cardinality $2r^2 + r, \dots, 2r^2 + 3r + 1$, because, if T is a blocking set, also its complement is a blocking set. This completely proves the statement in the case $v = (2r+1)^2$.

General v. Take any $v \in \mathbb{N}$ such that $v \equiv 1 \mod 8$. Then we write:

$$v = (2r+1)^2 + 8k,$$

for some $r, k \in \mathbb{N}$, in such a way that $(2r+1)^2 \leq v < (2r+3)^2$. This means that $0 \leq k \leq r$ and that, of course, $2r + 1 \leq \sqrt{v} < 2r + 3$. This implies that:

$$\{\lceil \tfrac{v}{2} - \tfrac{\sqrt{v}}{2} \rceil, \dots, \lfloor \tfrac{v}{2} + \tfrac{\sqrt{v}}{2} \rfloor\} = \{2r^2 + r + 4k, \dots, 2r^2 + 3r + 1 + 4k\}.$$

So by what we have just proved for the orders of type $(2r+1)^2$, for any $r \in \mathbb{N}$, and by iteratively using Proposition 5 we easily get the statement. $\square$

The following remark will be used in the next section.

Remark 9. By the previous construction, by the case $v = 9$ and by Proposition 5, Lemma 6 and Lemma 7 it follows that for any $v \equiv 1 \mod 8$ there exists a largely blocked 4-cycle design $\Sigma = (X, \mathcal{B})$ with some blocking sets $T_1, \dots, T_r$ with $T_1 \subset \cdots \subset T_r$ and $|T_1| = \beta_1, |T_2| = \beta_1 + 1, \dots, |T_r| = \frac{v-1}{2}$. Moreover, in any block $(a, b, c, d) \in \mathcal{B}$ there exists a vertex $a \in T_1$ such that either $b, d \notin T_r$ or $c \notin T_r$.

3. LARGELY BLOCKED P_3-DESIGNS

Now we want to study largely blocked P_3-designs. In general, the k-path on the vertices $\{x_1, \dots, x_k\}$ with edges $\{x_i, x_{i+1}\}$ for $i = 1, \dots, k-1$ is denoted by $[x_1, x_2, \dots, x_k]$. The spectrum of P_3-designs is known:

Theorem 10. *A P_3-design of order v exists if and only if $v \equiv 0, 1 \mod 4$, $v \geq 4$.*

Note that for a P_3-design $\Sigma = (X, \mathcal{B})$ of order v with a blocking set of cardinality t we have the clear condition that:

$$|\mathcal{B}| = \frac{v(v-1)}{4} \leq p \cdot (v-p) \Rightarrow \beta_1 = \lceil \tfrac{v}{2} - \tfrac{\sqrt{v}}{2} \rceil \leq p \leq \lfloor \tfrac{v}{2} + \tfrac{\sqrt{v}}{2} \rfloor = \beta_2.$$

So we can give the following definition for P_3-designs:

Definition 11. A P_3-design Σ of order v is called *largely blocked* if $B(\Sigma) = [\beta_1, \beta_2]$.

In this section we want to determine the spectrum of largely blocked P_3-designs. First, we need some technical lemmas.

Lemma 12. *Let $X = \{x_i \mid i = 1, \ldots, 4\}$ and $Y = \{y_i \mid i = 1, \ldots, 4\}$ be disjoint sets, with $|X| = |Y| = 4$. Then there exists a P_3-decomposition of $K_{X,Y}$ having $\{x_1, x_2, y_1\}$, $\{x_1, x_2, y_1, y_2\}$, $\{x_2, x_3, y_1\}$, $\{x_2, x_3, y_1, y_2\}$ and $\{x_2, x_3, y_2, y_3\}$ as blocking sets.*

Proof. Let:

$$\mathcal{B} = \{[x_i, y_j, x_{i+2}] \mid i = 1, 2,\ j = 1, 2, 3, 4\}.$$

Then the blocks of $\mathcal{B}$ decompose $K_{X,Y}$ in 3-paths and satisfy the conditions of the statement. □

Lemma 13. *Let $X = \{x_1, x_2, x_3, x_4\}$, $Y = \{y_1, y_2, y_3, y_4\}$, $Z = \{z_1, z_2, z_3, z_4\}$ and $T = \{t_1, t_2, t_3, t_4\}$ be pairwise disjoint sets. Then there exists a P_3-decomposition of $K_{X,Y,Z,T} \cup K_X \cup K_Y$ having $W = \{x_1, x_2, y_1, y_2, z_1, t_1\}$, $W \cup \{z_2\}$ and $(W \setminus \{y_1\}) \cup \{y_3, z_2, t_2\}$ as blocking sets.*

Proof. Let us consider the family $\mathcal{B}$ of blocks:

- $[x_1, z_j, t_3]$, $[x_2, z_j, t_4]$, $[y_1, z_j, t_2]$, $[x_3, z_j, t_1]$, $[x_1, t_j, x_3]$, $[x_2, t_j, x_4]$, $[y_1, t_j, y_3]$, $[y_2, t_j, y_4]$ and $[x_4, z_j, y_2]$ for $j = 1, 2, 3, 4$
- $[z_2, y_3, y_1]$, $[z_3, y_3, y_2]$, $[z_4, y_3, x_1]$, $[z_2, y_4, x_1]$, $[z_3, y_4, x_2]$, $[z_4, y_4, z_1]$, $[y_4, y_3, z_1]$, $[y_4, y_1, y_2]$, $[y_3, x_2, y_1]$, $[y_3, x_3, y_1]$, $[y_3, x_4, y_1]$, $[x_3, x_1, y_1]$, $[x_4, x_1, y_2]$, $[y_2, x_2, x_3]$, $[x_1, x_2, x_4]$, $[y_2, x_3, y_4]$, $[y_2, x_4, x_3]$ and $[x_4, y_4, y_2]$.

Then it is easy to verify that the blocks of $\mathcal{B}$ give us the statement. □

Now we can determine the spectrum of largely blocked P_3-designs.

Theorem 14. *For any $v \equiv 0, 1 \mod 4$, $v \geq 4$, there exists a largely blocked P_3-design of order v.*

Proof. **Case 1.** Suppose, first, that $v \equiv 0 \mod 4$. If $v = 4$, let $X = \{1, 2, 3, 4\}$ and let:

$$\mathcal{B} = \{[1, 2, 3], [1, 3, 4], [1, 4, 2]\}.$$

Then $\Sigma = (X, \mathcal{B})$ is a P_3-design having as blocking sets $\{1\}$, $\{1, 2\}$ and $\{2, 3, 4\}$. This proves the statement for $v = 4$.

Now let $v = 4r^2+4k$, $v \geq 8$, for some $r, k \in \mathbb{N}$ such that $(2r)^2 \leq v < (2r+2)^2$. In this way, we have $0 \leq k \leq 2r$ and $[\beta_1, \beta_2] = [2r^2 - r + 2k, 2r^2 + r + 2k]$.

Let $X_1, \ldots, X_{r^2+k}$ be pairwise disjoint sets with $|X_i| = 4$ for any $i = 1, \ldots, r^2+k$ and let $X = \bigcup_i X_i$. By what we just proved we can consider $\Sigma_i = (X_i, \mathcal{B}_i)$ largely blocked P_3-designs with blocking sets T_i such that:

$$|T_i| = \begin{cases} 1 & \text{for } i = 1, \ldots, r \\ 2 & \text{for } i = r+1, \ldots, r^2+k. \end{cases}$$

Let $T = \bigcup_i T_i$, so that $|T| = 2r^2 - r + 2k$. Consider also $x_i \in X_i$, $x_i \notin T_i$, for $i = 1, \ldots, r$, and $T'_i \subset X_i$ for $i = \frac{r^2+r}{2}+1, \ldots, r^2$ such that $|T'_i| = 2$ and $|T_i \cap T'_i| = 1$. By what we just proved we can suppose that $T_i \cup \{x_i\}$ for $i = 1, \ldots, r$ is still a blocking set for Σ_i.

Consider any bijection:

$$\varphi\colon \{\{i,j\} \mid i,j = 1, \ldots, r,\, i \neq j\} \to \{r+1, \ldots, \tfrac{r^2+r}{2}\}$$

and let:

$$\psi(i,j) = \varphi(i,j) + \binom{r}{2}.$$

By Lemma 13 for $i, j = 1, \ldots, r$, $i < j$, we can consider a family $\mathcal{C}_{i,j}$ of blocks decomposing

$$K_{X_i, X_j, X_{\varphi(i,j)}, X_{\psi(i,j)}} \cup K_{X_{\varphi(i,j)}} \cup K_{X_{\psi(i,j)}}$$

in P_3 paths such that:

- $T_i \cup T_j \cup T_{\varphi(i,j)} \cup T_{\psi(i,j)}$,
- $T_i \cup T_j \cup T_{\varphi(i,j)} \cup T_{\psi(i,j)} \cup \{x_i\}$,
- $T_i \cup T_j \cup T_{\varphi(i,j)} \cup T'_{\psi(i,j)} \cup \{x_i, x_j\}$

are blocking sets for this decomposition.

Let $i, j = 1, \ldots, r^2+k$, $i \neq j$, such that both i, j are not simultaneously in some of the quadruples:

$$\{\{p, q, \varphi(p,q), \psi(p,q)\} \mid p, q = 1, \ldots, r,\, p \neq q\}.$$

Then by Lemma 12 let $\mathcal{D}_{i,j}$ a family of blocks decomposing K_{X_i, X_j} such that:

- $T_i \cup T_j$,
- $T_i \cup T_j \cup \{x_i\}$ if $i = 1, \ldots, r$,
- $T_i \cup T'_j$ if $j = \frac{r^2+r}{2} + 1, \ldots, r^2$
- $T_i \cup T'_j \cup \{x_i\}$ if $i = 1, \ldots, r$ and $j = \frac{r^2+r}{2} + 1, \ldots, r^2$,

- $T'_i \cup T'_j$ if $i,j = \frac{r^2+r}{2}+1,\dots,r^2$

are all blocking sets for the blocks of $\mathcal{D}_{i,j}$.

Let:

$$\mathcal{B} = \bigcup_{i=1}^{r} \mathcal{B}_i \cup \bigcup_{i=r^2+1}^{r^2+2k} \mathcal{B}_i \cup \bigcup \mathcal{C}_{i,j} \cup \bigcup \mathcal{D}_{i,j}.$$

Then $\Sigma = (X, \mathcal{B})$ is a P_3-design having:

- T
- $T \cup \{x_1\}$
- $\bigcup_{i \notin I_s} T_i \cup \bigcup_{i \in I_s} T'_i \cup \{x_1, \dots, x_s\}$ for $s = 2,\dots,r$, where $I_s = \{\psi(i,j) \mid i,j = 1,\dots,s,\ i \neq j\}$,

as blocking sets. So there exist for Σ blocking sets of cardinality $2r^2-r+2k,\dots,2r^2+2k$. This immediately implies that there exist for Σ blocking sets of cardinality $2r^2+2k+1,\dots,2r^2+2k+r$, because the complement of a blocking set is a blocking set. This proves the statement for $v \equiv 0 \mod 4$.

Case 2. Let $v \equiv 1 \mod 8$, $v \geq 9$. In this case the statement follows by Theorem 8. Indeed, there exists a largely blocked C_4-design $\Sigma = (X, \mathcal{B})$ of order v with the same interval of integers

$$\left[\left\lceil \frac{v-\sqrt{v}}{2} \right\rceil, \left\lfloor \frac{v+\sqrt{v}}{2} \right\rfloor\right] = [\beta_1, \beta_2].$$

Moreover, as noted in Remark 9 called $T_1, \dots T_r$ the blocking sets of cardinality:

$$\left\lceil \frac{v-\sqrt{v}}{2} \right\rceil, \dots, \tfrac{v-1}{2}$$

given in the construction, in any 4-cycle (x,y,z,t) of $\mathcal{B}$ we have a vertex $x \in T_i$ for any i such that either $y,t \notin T_i$ for any i or $z \notin T_i$ for any i. This implies that from this 4-cycle we get the paths $[x,y,z]$ and $[x,t,z]$ in order to obtain a P_3-design of order v having $T_1, \dots T_r$ and their complements as blocking sets. This proves the statement in the case $v \equiv 1 \mod 8$, $v \geq 9$.

Case 3. Let $v \equiv 5 \mod 8$. If $v = 5$, we have $[\beta_1, \beta_2] = [2,3]$. Consider on $\{0,1,2,3,4\}$ the P_3-design Σ having as base block $[1,0,2]$. Then $\{0,2\}$ (and consequently also its complement $\{1,3,4\}$) is a blocking set for Σ. This proves the statement for $v = 5$.

Let $v \equiv 5 \mod 8$, with $v \geq 13$. Then let $v = (2r+1)^2 + 4(2k+1)$ for some $r,k \in \mathbb{N}$, $r \geq 1$, such that $(2r+1)^2 + 4 \leq v < (2r+3)^2 + 4$. So $0 \leq k \leq r$ and $v < (2r+3)^2$. Let $v' = v-4$. Then by what we just proved we can take $\Sigma = (X, \mathcal{B})$ a largely blocked P_3-design of order v'. Let T be a blocking set for Σ of cardinality p. It is easy to see that:

$$\beta_1(v) = \left\lceil \frac{v-\sqrt{v}}{2} \right\rceil = 2r^2+r+4k+2 \quad \text{and} \quad \beta_2(v) = \left\lceil \frac{v+\sqrt{v}}{2} \right\rceil = 2r^2+3r+4k+3$$

and

$$\beta_1(v') = \left\lceil \frac{v' - \sqrt{v'}}{2} \right\rceil = 2r^2+r+4k \quad \text{and} \quad \beta_2(v') = \left\lceil \frac{v' + \sqrt{v'}}{2} \right\rceil = 2r^2+3r+4k+1.$$

So $[\beta_1(v), \beta_2(v)] = [\beta_1(v') + 2, \beta_2(v') + 2]$. Let $X = \{x_i \mid i = 1, \ldots, v'\}$ and $Y = \{y_1, y_2, y_3, y_4\}$. Let $\mathcal{B}'$ be the following family of blocks:

$$[y_1, y_2, y_3],\ [y_1, y_3, y_4],\ [y_1, y_4, y_2],\ [y_{2i+1}, x_j, y_{2i+2}]$$

for $i = 0, 1$ and $j = 1, \ldots, v'$. Then $\Sigma' = (X \cup Y, \mathcal{B} \cup \mathcal{B}')$ is a P_3-design of order v having as blocking set $T \cup \{y_1, y_3\}$. Since Σ is largely blocked and $[\beta_1(v), \beta_2(v)] = [\beta_1(v') + 2, \beta_2(v') + 2]$, we immediately get the statement for $v \equiv 5 \mod 8$, $v \geq 13$. □

4. PERFECT BLOCKING SETS

In general, when we have a blocking set T for a G-design $\Sigma = (X, \mathcal{B})$ we might want that the elements of T are distributed in an optimal and homogeneous way in the blocks of $\mathcal{B}$. So in [4] the following definition is given:

Definition 15. Let $\Sigma = (X, \mathcal{B})$ be a G-design. A blocking set T of Σ is called *perfect* if there exists $C \in \mathbb{N}$ such that any block $B \in \mathcal{B}$ contains exactly C edges joining vertices of T and of $C_X(T)$.

This definition in general forces a strict condition on the order of the G-design:

Proposition 16. *If $\Sigma = (X, \mathcal{B})$ is a C_4-design of order v and T is a perfect blocking set for Σ of cardinality p, then:*

$$p = \frac{v \pm \sqrt{v}}{2}$$

and v is a square.

Proof. Since $|\mathcal{B}| = \frac{v(v-1)}{8}$ and T is a perfect blocking set, then any block $B \in \mathcal{B}$ contains exactly 2 edges joining vertices of T and $C_X(T)$ and:

$$p \cdot (v - p) = 2 \cdot \frac{v(v-1)}{8}.$$

So $p = \frac{v \pm \sqrt{v}}{2}$ and v is a square, because p is a positive integer. □

By Theorem 2, Theorem 8 and Proposition 16 we get immediately the following:

Theorem 17. *There exist C_4-designs of order v with a perfect blocking set if and only if $v = (2r+1)^2$ for some $r \in \mathbb{N}$, $r \geq 1$.*

5. PERFECT BLOCKING SETS IN P_5-DESIGNS

In [4] the spectrum of P_3-designs having a perfect blocking set is determined. So it is proved that:

Theorem 18 ([4])**.** *If T is a perfect blocking set of any P_3-design of order v, then $c = 1$, v is a square and*

$$|T| = \frac{v \pm \sqrt{v}}{2}.$$

If we consider a P_3-design of order v having a perfect blocking set, since $v \equiv 0$ or $1 \mod 4$, then there exists a positive integer k such that $v = (2k)^2$ or $v = (2k+1)^2$. So in [4] it is proved that:

Theorem 19 ([4])**.** *There exist P_3-designs of order v having perfect blocking sets if and only if v is a square.*

In this section we provide a construction that will be useful in studying P_5-designs with perfect blocking sets. So, let $\Sigma = (X, \mathcal{B})$ be a P_3-design of order v with a perfect blocking set T. For any $x \in T$ we consider the set:

$$E(x) = \{\{y, y'\} \mid y, y' \in X,\ y \neq y',\ [x, y, y'] \text{ or } [x, y', y] \in \mathcal{B}\}$$

and the graph $G(x) = (X \setminus \{x\}, E(x))$.

Remark 20. Note that in a P_3-design with a perfect blocking set T, any block B is a path $[x_1, x_2, x_3]$ where $x_1 \in T$ and $x_3 \in C_X(T)$.

Given a graph $G = (X, E)$, we denote by $\Delta(G)$ the maximum degree of the vertices of G. The chromatic index $\chi'(G)$ of G is the minimum number of colors needed for a proper edge coloring of G. The following construction will be used in the proof of the main result of this section:

Theorem 21. *For any $k \in \mathbb{N}$ there exists a P_3-design Σ of order v with a perfect blocking set T such that one of the following conditions holds:*

1. *$v = (2k+1)^2$, $|E(x)|$ is even for any $x \in T$ and*

$$\chi'(G(x)) \leq \frac{|E(x)|}{2};$$

2. *$v = (2k)^2$, $k \geq 2$, $|E(x)|$ is odd for any $x \in T$ and there exists $b \in C_X(T)$ such that for any $x \in T$ there exists $a_x \in C_X(T)$ satisfying the conditions:*
   - *$[x, a_x, b] \in \mathcal{B}$,*

- $a_x \neq a_y$ *for any* $x, y \in T$, $x \neq y$,
- $\chi'(G(x) - \{a_x, b\}) \leq \dfrac{|E(x)| - 1}{2}$,

where $G(x) - \{a_x, b\} = (X \setminus \{x\}, E(x) \setminus \{\{a_x, b\}\})$.

Proof. Let $\Sigma = (X, \mathcal{B})$ be a P_3-design with a perfect blocking set T of cardinality p. Let $x \in T$ and:

- let a_x be the number of blocks of type $[x, x_1, x_2]$, with $x_1 \in T$ and $x_2 \in C_X(T)$
- let b_x be the number of blocks of type $[x, x_1, x_2]$, with $x_1, x_2 \in C_X(T)$
- let c_x be the number of blocks of type $[x_1, x, x_2]$, $x_1 \in T$ and $x_2 \in C_X(T)$.

Then:

$$\begin{cases} a_x + c_x = p - 1 \\ b_x + c_x = v - p \end{cases} \Rightarrow b_x - a_x = v - 2p + 1.$$

Since $|E(x)| = a_x + b_x$, we easily see that if v is odd, then $|E(x)|$ is even, and, conversely, if v is even, $|E(x)|$ is odd.

Next, to simplify the proof let us make the following position. If $v = (2k+1)^2$, let $p = k(2k+1)$ and $q = 2k+1$. If $v = (2k)^2$, let $p = 2k^2 - k$ and $q = 2k$.

Let us consider X_1, X_2 and X_3, pairwise disjoint, such that $|X_1| = p$, $|X_2| = p$ and $|X_3| = q$. We will construct a P_3-design Σ of order v with vertex set $X = X_1 \cup X_2 \cup X_3$ and $T = X_1$. Let:

$$X_1 = \{a_1, a_2, ..., a_p\}$$
$$X_2 = \{b_1, b_2, ..., b_p\}$$
$$X_3 = \{c_1, c_2, ..., c_q\}.$$

For any $i = 1, \dots, q-1$ and $j = 1, \dots, q-i$ we define:

$$\varphi(i, j) = \begin{cases} j & \text{for } i = 1 \\ \sum_{r=1}^{i-1}(q - r) + j & \text{for } i \geq 2. \end{cases}$$

Note that $\varphi(i,1) - \varphi(i-1, q-i+1) = 1$, $\varphi(1,1) = 1$ and $\varphi(q-1,1) = p$. This implies that for any $s \in \{1, \dots, p\}$ there exist unique $i \in \{1, \dots, q-1\}$ and $j \in \{1, \dots, q-i\}$ such that $\varphi(i, j) = s$.

Define in X the following families of paths P_3:

$$\mathcal{F}_1 = \{[a_i, a_{i+j}, b_j] \mid i = 1, \dots, p-1,\ j = 1, \dots, p-i\}$$
$$\mathcal{F}_2 = \{[b_i, b_{i+j}, a_j] \mid i = 1, \dots, p-1,\ j = 1, \dots, p-i\}$$
$$\mathcal{F}_3 = \{[c_{i+j}, c_i, a_{\varphi(i,j)}] \mid i = 1, \dots, q-1,\ j = 1, \dots, q-i\}$$
$$\mathcal{F}_4 = \left\{[a_{\varphi(i,j)}, b_{\varphi(i,j)}, c_i] \mid i = 1, \dots, q-1,\ j = 1, \dots, q-i\right\}$$
$$\mathcal{F}_5 = \left\{[a_s, c_i, b_{\varphi(i,1)-s}] \mid i = 2, \dots, q-1,\ s = 1, \dots, \varphi(i,1) - 1\right\} \cup$$
$$\cup \left\{[a_s, c_i, b_{p-s+1+\varphi(i,q-i)}] \mid i = 1, \dots, q-2,\ s = \varphi(i, q-i) + 1, \dots, p\right\} \cup$$
$$\cup \{[a_s, c_q, b_{p+1-s}] \mid s = 1, \dots, p\}.$$

It is possible to verify that $\Sigma = (X, \bigcup_{i=1}^{5} \mathcal{F}_i)$ is a P_3-design of order $(2k+1)^2$ such that X_1 is a perfect blocking set satisfying the condition of the statement.

Indeed, for any $i = 1, \ldots, p$ let $\bar{i} \in \{1, \ldots, q-1\}$ and $\bar{j} \in \{1, \ldots, q-i\}$ be such that $\varphi(\bar{i}, \bar{j}) = i$. Moreover, for any a_i, $i = 1, \ldots, p-1$ in $E(a_i)$ we have:

- from $\mathcal{F}_1$ $p-i$ edges, with vertices in $\{a_{i+1}, \ldots, a_p\} \cup \{b_1, \ldots, b_{p-i}\}$
- from $\mathcal{F}_2$ $p-i$ edges, which are $\{b_j, b_{i+j}\}$ for $j = 1, \ldots, p-i$
- from $\mathcal{F}_3$ just one edge, $\{c_{\bar{i}}, c_{\bar{i}+\bar{j}}\}$
- from $\mathcal{F}_4$ we have just one edge $\{c_{\bar{i}}, b_{\varphi(\bar{i},\bar{j})}\} = \{c_{\bar{i}}, b_i\}$
- from $\mathcal{F}_5$ we have the edges $\{c_q, b_{p+1-i}\}$, $\{\{c_{i'}, b_{\varphi(i',1)-i}\} \mid i' = \bar{i}+1, \ldots, q-1\}$ for $\bar{i} \leq q-2$, and $\{\{c_{i'}, b_{p-i+1+\varphi(i',q-i')}\} \mid i' = 1, \ldots, \bar{i}-1\}$ for $\bar{i} \geq 2$.

Instead, in $E(a_p)$ we have:

- from $\mathcal{F}_3$ just one edge, $\{c_{q-1}, c_q\}$
- from $\mathcal{F}_4$ we have just one edge $\{c_{q-1}, b_p\}$
- from $\mathcal{F}_5$ we have the edges $\{c_q, b_1\}$ and $\{\{c_i, b_{1+\varphi(i,q-i)}\} \mid i = 1, \ldots, q-2\}$.

This implies that:

$$\Delta(G(a_i)) = \begin{cases} 4 & \text{for } i \leq \dfrac{p}{2} \\ 3 & \text{for } \dfrac{p}{2} < i \leq p-1 \\ 2 & \text{for } i = p. \end{cases}$$

Since $|E(G(a_i))| = 2p - 2i + 1 + q$ and $\chi'(G(a_i)) \leq \Delta(G(a_i)) + 1$, the statement follows if $v = (2k+1)^2$ and $k \geq 2$. If $k = 1$, then $p = q = 3$ and it is easy to verify that:

$$\Delta(G(a_1)) = 4 = \chi'(G(a_1)) = \frac{|E(a_1)|}{2}$$

$$\Delta(G(a_2)) = 3 = \chi'(G(a_1)) = \frac{|E(a_2)|}{2}$$

$$\Delta(G(a_3)) = 2 = \chi'(G(a_1)) = \frac{|E(a_3)|}{2}.$$

So also for $v = (2k+1)^2$ with $k = 1$ the statement holds.

If $v = (2k)^2$, in the statement take $b = b_1$ and the paths $[a_i, b_{i+1}, b_1]$ for $i = 1, \ldots, p-1$ and $[a_p, c_q, b_1]$. Then, we have $|E(G(a_i))| - 1 = 2p - 2i + q$ and

$\chi'(G(a_i)) \leq \Delta(G(a_i)) + 1$ for any $i = 1, \ldots, p$. Then the statement follows for $v = (2k)^2$ and $k \geq 3$. If $k = 2$, then $p = 6$ and $q = 4$ and we see that:

$$\chi'(G(a_i)) \leq \Delta(G(a_i)) + 1 \leq \frac{|E(G(a_i))| - 1}{2}$$

for $i = 1, 2, 3, 4$. Moreover it is not difficult to see that we also have:

$$\Delta(G(a_5) - \{b_1, b_6\}) = 2 = \chi'(G(a_5) - \{b_1, b_6\}) < \frac{|E(a_5)| - 1}{2}$$

$$\Delta(G(a_6) - \{c_4, b_1\}) = 2 = \chi'(G(a_6) - \{c_4, b_1\}) = \frac{|E(a_6)| - 1}{2}.$$

So also for $v = (2k)^2$ with $k = 2$ the statement holds. □

The next result is the key to the proof of the main result of this section:

Lemma 22 ([1, Lemma 2])**.** *For every graph $G = (X, E)$ and for every $t > 1$, $tK_2 \big| G$ if and only if $t \big| |E|$ and $\chi'(G) \leq \frac{|E|}{t}$.*

Recall now the following:

Theorem 23. *A P_5-design of order v exists if and only if $v \equiv 0$ or 1 mod 8, $v \geq 5$.*

Now we determine the spectrum of P_5-designs having a perfect blocking set with constant $C = 2$:

Theorem 24. *There exists a P_5-design of order v having a perfect blocking set with constant $C = 2$ if and only if either $v = (2k+1)^2$ or $v = 16k^2$, for some $k \in \mathbb{N}$.*

Proof. If $\Sigma = (X, \mathcal{B})$ is a P_5-design of order v with a perfect blocking set T of cardinality p and constant $C = 2$, then:

$$p \cdot (v - p) = 2 \cdot |\mathcal{B}| \Rightarrow p = \frac{v \pm \sqrt{v}}{2}.$$

So v is a square and by Theorem 23 either $v = (2k+1)^2$ or $v = 16k^2$, for some $k \in \mathbb{N}$.

Suppose, now, that $v = (2k+1)^2$ for some $k \in \mathbb{N}$. Let $\Sigma = (X, \mathcal{B})$ be a P_3-design satisfying the conditions of Theorem 21. Then, by Lemma 22 we see that, for any $x \in T$ such that $|E(x)| > 0$, $2K_2 | G(x)$. So if $\{y_1, y_2\}$, $\{y_3, y_4\}$ is one of these copies of $2K_2$, we can join the two paths $[x, y_1, y_2]$ and $[x, y_3, y_4]$ in the path $[y_4, y_3, x, y_1, y_2]$. By Remark 20 this gives us a P_5-design of order $v = (2k+1)^2$ having T as a perfect blocking set with constant $C = 2$.

Suppose that $v = 16k^2$ for some $k \in \mathbb{N}$. Let $\Sigma = (X, \mathcal{B})$ be a P_3-design satisfying the conditions of Theorem 21, where now the perfect blocking set T has even cardinality. Keeping the notation of the theorem, for any $x \in T$ such that $|E(x)| > 1$ we can proceed as we have just done: by Lemma 22 we see that

$2K_2|(G(x)-\{a_x,b\})$. So, again, if $\{y_1,y_2\}$, $\{y_3,y_4\}$ is one of these copies of $2K_2$, we can join the two paths $[x,y_1,y_2]$ and $[x,y_3,y_4]$ in the path $[y_4,y_3,x,y_1,y_2]$. Then there an even number of blocks $[x,a_x,b]$, one for each $x \in T$, where $a_x \in C_X(T)$. If $T = \{x_1,\dots,x_p\}$, then we can consider the following paths:

$$[x_{2i+1}, a_{x_{2i+1}}, b, a_{x_{2i+2}}, x_{2i+2}]$$

for $i = 0,\dots,\frac{p}{2}-1$. In this way, by Remark 20 we get a P_5-design of order $v = 16k^2$ having T as a perfect blocking set with constant $C = 2$. □

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