ENUMERATIONS OF BARGRAPHS WITH RESPECT TO CORNER STATISTICS

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We study the enumeration of bargraphs with respect to some corner statistics. We find generating functions for the number of bargraphs that track the corner statistics of interest, the number of cells, and the number of columns. We also consider bargraph representation of set partitions and obtain some explicit formulas for the number of specific types of corners in such representations.

1. Introduction

Combinatorial analysis of certain geometric cluster models such as polygons, polycubes, polyominos is an important research endeavor for understanding many statistical physics models [**8, 9, 15**]. A finite connected union of unit squares on two dimensional integer lattice is called a *polyomino*, and a *bargraph* is a column-convex polyomino in the first quadrant of the lattice such that its lower boundary lies on the x-axis. A *bargraph* can also be considered as a self-avoiding path in the integer lattice $\mathbb{L} = \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ with steps $u = (0, 1)$, $h = (1, 0)$ and $d = (0, -1)$ that starts at the origin, ends on the x-axis and never touches the x-axis except at the endpoints. The steps u, h and d are called *up, horizontal* and *down* steps respectively. Enumerations of bargraphs with respect to some statistics have been an active area of research recently [**8, 10, 14**]. Bosquet-Melóu and Rechnitzer [**6**] obtain the site-perimeter generating function for bargraphs, and also show that it is not D-finite. Blecher et al. investigated the generating functions for bargraphs with respect to some statistics such as the number of levels [**1**], descents [**2**], peaks

[**3**], and walls [**5**]. Deutsch and Elizalde [**7**] used a bijection between bargraphs and cornerless Motzkin paths, and determined more than twenty generating functions for bargraphs according to the number of up steps, the number of horizontal steps, and the statistics of interest such as the number of double rises and double falls, the length of the first descent, the least column height. Bargraphs are also used in statistical physics to model vesicles or polymers [**12, 13, 14**].

We shall study the enumerations of bargraphs and set partitions with respect to some corner statistics. We shall first introduce some definitions. A unit square in the lattice $\mathbb{L}$ is called a *cell*. We identify a bargraph with a sequence of numbers $\pi = \pi_1 \pi_2 \cdots \pi_m$ where m is the number of horizontal steps of the bargraph and π_j is the number of *cells* beneath the jth horizontal step which is also called the *height* of the jth *column*. A vertex on a bargraph is called a *corner* if it is at the intersection of two different types of steps. A corner is called an (a, b)-*corner* if it is formed by maximum number a of one type of consecutive steps followed by maximum number b of another type of consecutive steps. A corner is called of *type A* if it is formed by down steps followed by horizontal steps ($\llcorner$). Similarly, a corner is of *type B* if it is formed by horizontal steps followed by down steps ($\urcorner$), see Figure 1. We use $\mathcal{B}_n$ and $\mathcal{B}_{n,k}$ to denote the set of all bargraphs with n cells, and the set of all bargraphs with n cells and k columns respectively.

Bargraphs are also related to the set partitions. Recall that a partition of set $[n] := \{1, 2, \cdots, n\}$ is any collection of nonempty, pairwise disjoint subsets whose union is $[n]$. Each subset in a partition is called a *block* of the partition. A partition p of $[n]$ with k blocks is said to be in the *standard form* if it is written as $p = A_1/A_2/\cdots/A_k$ where $\min(A_1) < \min(A_2) < \cdots < \min(A_k)$. There is also a unique *canonical sequential representation* of a partition p as a word of length n over the alphabet $[k]$ denoted by $\pi = \pi_1 \pi_2 \cdots \pi_n$ where $\pi_i = j$ if $i \in A_{\pi_j}$ which can be considered a bargraph representation. For instance, the partition $\pi = \{1, 3, 6\}/\{2, 5\}/\{4, 7\}/\{8\}$ has the canonical sequential representation $\pi = 12132134$. Mansour [**10**] studied the generating functions for the number of set partitions of $[n]$ represented as bargraphs according to the number of interior vertices. For some other enumeration results, see also [**4, 11**]. Henceforth, we shall represent set partitions as bargraphs corresponding to their canonical sequential representations.

The rest of the paper is organized as follows. In section 2, we find the generating function for the number of bargraphs according to the number of cells, the number of columns, and the number of (a, b)-corners of type A for any given positive integers a, b. As a corollary, we determine the total number of (a, b)-corners of type A, and the total number of type A corners over all bargraphs having n cells. In section 2.3 and section 2.4, we extend these results to the restricted bargraphs in which the height of each column is restricted to be a maximum of N for any given positive integer N, and to the set partitions respectively. We obtain similar results for corners of type B in section 3. One of the main results of the paper, Theorem 6, shows that the total number of corners of type A over the set partitions of $[n + 1]$
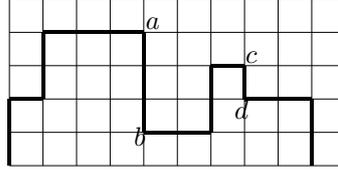
Figure 1: The bargraph $\pi = 244411322$. Type A corners b and d are $(3,2)$ and $(1,2)$-corners respectively. Type B corners a and c are $(3,3)$ and $(1,1)$-corners respectively.

with k blocks is given by

$$\frac{n}{2}S_{n+1,k} - \frac{1}{4}S_{n+2,k} - \frac{n}{2}S_{n,k} + \frac{1}{4}S_{n+1,k} + S_{n,k-2},$$

where $S_{n,k}$ is the Stirling number of second kind. Similarly, Theorem 11, shows that the total number of corners of type B over the set partitions of $[n+1]$ with k blocks is given by

$$\frac{n}{2}S_{n+1,k} - \frac{1}{4}S_{n+2,k} - \frac{n}{2}S_{n,k} + \frac{5}{4}S_{n+1,k} + S_{n,k-2}.$$

2. Counting Corners of type A

Let $H := H(x, y, \mathbf{q})$ be the generating function for the number of bargraphs π according to the number of cells in π, the number of columns of π, and the number of (a, b)-corners of type A in π corresponding to the variables x, y and $\mathbf{q} = (q_{a,b})_{a,b \geq 1}$ respectively. That is,

$$H = \sum_{n \geq 0} \sum_{\pi \in \mathcal{B}_n} x^n y^{\mathrm{col}(\pi)} \prod_{a,b \geq 1} q_{a,b}^{\Lambda_{(a,b)}(\pi)},$$

where $\Lambda_{(a,b)}(\pi)$ is the number of (a, b)-corners of type A in π, and $\mathrm{col}(\pi)$ denotes the number of columns of π.

From the definitions, we have

$$(1) \qquad\qquad H = 1 + \sum_{a \geq 1} H_a,$$

where 1 counts the empty bargraph, and $H_{a_1 a_2 \cdots a_s} := H_{a_1 a_2 \cdots a_s}(x, y, \mathbf{q})$ is the generating function for the number of bargraphs $\pi = a_1 a_2 \cdots a_s \pi'$ in which the height of the jth column is a_j, where $j = 1, 2, \ldots, s$. Since each bargraph $\pi = a\pi'$

can be decomposed as either a, $aj\pi''$ with $j \geq a$ or $ab\pi''$ with $1 \leq b \leq a-1$, we have

$$(2) \qquad H_a = x^a y + x^a y \sum_{j \geq a} H_j + \sum_{b=1}^{a-1} H_{ab}.$$

Note that each bargraph $\pi = ab\pi''$, $1 \leq b \leq a-1$, can be written as either ab^m (where we define b^m to be the word $bb \cdots b$), $ab^m j\pi'$ with $j \geq b+1$, or $ab^m j\pi'$ with $j \leq b-1$. Thus, for all $1 \leq b \leq a-1$, we have

$$H_{ab} = \sum_{m \geq 1} x^{a+bm} y^{m+1} q_{a-b,m} + \sum_{m \geq 1} \left(x^{a+bm} y^{m+1} q_{a-b,m} \sum_{j \geq b+1} H_j \right)$$
$$+ \sum_{m \geq 1} \left(x^{a+b(m-1)} y^m q_{a-b,m} \sum_{c=1}^{b-1} H_{bc} \right),$$

which is equivalent to

$$(3) \qquad H_{ab} = \sum_{m \geq 1} x^{a+bm} y^{m+1} q_{a-b,m}$$
$$+ \sum_{m \geq 1} \left(x^{a+b(m-1)} y^m q_{a-b,m} \left(x^b y \sum_{j \geq b+1} H_j + \sum_{c=1}^{b-1} H_{bc} \right) \right).$$

Thus, by (2), we have that $H_a - x^a y - x^a y H_a = x^a y \sum_{j \geq a+1} H_j + \sum_{b=1}^{a-1} H_{ab}$, which, by (3), leads to

$$(4) \qquad H_{ab} = \alpha_{ab}(1 - x^b y) H_b \quad \text{with} \quad \alpha_{ab} = \sum_{m \geq 1} x^{a+b(m-1)} y^m q_{a-b,m}.$$

Therefore, by (1) and (2), we can write

$$(5) \qquad H_a = x^a y H + \sum_{b=1}^{a-1} \beta_{ab} H_b$$

with $\beta_{ab} = \alpha_{ab}(1 - x^b y) - x^a y$.

Lemma 1. *For all $a \geq 1$,*

$$H_a = H \left(x^a y + \sum_{j=1}^{a} \left(x^j y \sum_{s \geq 0} L_a(j,s) \right) \right),$$

where $L_a(j,s) = \sum_{j=i_{s+1} < i_s < \cdots < i_1 < i_0 = a} \prod_{\ell=0}^{s} \beta_{i_\ell i_{\ell+1}}$.

Proof. We prove it by induction on a. For $a = 1$, this gives $H_1 = xyH$ as expected (by removing the leftmost column of the bargraph $1\pi'$). Assume that the claim holds for $1, 2, \cdots, a$, and let us prove it for $a + 1$. By (5), we have

$$H_{a+1} = x^{a+1}yH + \sum_{b=1}^{a} \beta_{(a+1)b} H_b.$$

Thus, by induction assumption, we obtain

$$H_{a+1} = x^{a+1}yH + \sum_{b=1}^{a} \beta_{(a+1)b} H\left(x^b y + \sum_{j=1}^{b} x^j y \left(\sum_{s \geq 0} L_b(j,s)\right)\right)$$

$$= H\left(x^{a+1}y + \sum_{b=1}^{a} \beta_{(a+1)b} x^b y + \sum_{b=1}^{a}\sum_{j=1}^{b} x^j y \beta_{(a+1)b}\left(\sum_{s \geq 0} L_b(j,s)\right)\right)$$

$$= H\left(x^{a+1}y + \sum_{b=1}^{a} \beta_{(a+1)b} x^b y\right)$$

$$+ H\left(\sum_{j=1}^{a}\sum_{b=j}^{a} x^j y \left(\sum_{s \geq 0} \sum_{j=i_{s+1}<i_s<\cdots<i_1<i_0=b<i_{-1}=a+1} \prod_{\ell=-1}^{s} \beta_{i_\ell i_{\ell+1}}\right)\right)$$

$$= H\left(x^{a+1}y + \sum_{b=1}^{a} x^b y \sum_{i_1=b<i_0=a+1} \beta_{i_0 i_1}\right)$$

$$+ H\sum_{j=1}^{a} x^j y \left(\sum_{s \geq 0} \sum_{j=i_{s+1}<i_s<\cdots<i_1<i_0<i_{-1}=a+1} \prod_{\ell=-1}^{s} \beta_{i_\ell i_{\ell+1}}\right)$$

$$= H\left(x^{a+1}y + \sum_{j=1}^{a} x^j y \left(\sum_{s \geq 0} L_{a+1}(j,s)\right)\right),$$

which completes the proof.   □

By (1) and Lemma 1, we can state our first main result.

Theorem 2. *The generating function $H(x, y, \mathbf{q})$ is given by*

$$H(x, y, \mathbf{q}) = \frac{1}{1 - \frac{xy}{1-x} - \sum_{j \geq 1}\left(x^j y \sum_{s \geq 0} L(j,s)\right)},$$

where $L(j,s) = \sum_{j=i_{s+1}<i_s<\cdots<i_1<i_0} \prod_{\ell=0}^{s} \beta_{i_\ell i_{\ell+1}}$.

For instance, if $q_{a,b} = 1$ for all $a, b \geq 1$, then $\alpha_{ab} = \sum_{m \geq 1} x^{a+b(m-1)}y^m = \frac{x^a y}{1-x^b y}$, which yields $\beta_{ab} = \alpha_{ab}(1 - x^b y) - x^a y = 0$. Thus, in this case, Theorem 2 shows that $H(x, y, 1, 1, \ldots) = \frac{1-x}{1-x-xy}$, as expected.

2.1. Counting all corners of type A

Let $q_{a,b} = q$ for all $a, b \geq 1$. From the definitions, we have $\alpha_{ab} = q\frac{x^a y}{1-x^b y}$ and $\beta_{ab} = (q-1)x^a y$. Therefore,

$$
\begin{aligned}
L(j,s) &= \sum_{j=i_{s+1}<i_s<\cdots<i_1<i_0} \prod_{\ell=0}^{s} (q-1)x^{i_\ell} y \\
&= \sum_{j=i_{s+1}<i_s<\cdots<i_1<i_0} (q-1)^{s+1} y^{s+1} x^{i_0+i_1+\cdots+i_s} \\
&= (q-1)^{s+1} y^{s+1} \frac{x^{(s+1)j+\binom{s+2}{2}}}{(1-x)(1-x^2)\cdots(1-x^{s+1})}.
\end{aligned}
$$

Thus the generating function $F = H(x, y, q, q, \cdots)$ is given by

$$
\begin{aligned}
F &= \cfrac{1}{1 - \frac{xy}{1-x} - \sum_{j\geq 1} x^j y \sum_{s\geq 0} \frac{(q-1)^{s+1} y^{s+1} x^{(s+1)j+\binom{s+2}{2}}}{(1-x)(1-x^2)\cdots(1-x^{s+1})}} \\
&= \cfrac{1}{1 - \frac{xy}{1-x} - \sum_{s\geq 0} \frac{(q-1)^{s+1} y^{s+2} x^{(s+2)+\binom{s+2}{2}}}{(1-x)(1-x^2)\cdots(1-x^{s+2})}} \\
&= \cfrac{1}{1 - \frac{xy}{1-x} - \sum_{s\geq 1} \frac{(q-1)^{s} y^{s+1} x^{\binom{s+2}{2}}}{(1-x)(1-x^2)\cdots(1-x^{s+1})}}.
\end{aligned}
$$

Let $\mathrm{cor}_A(\pi)$ be the number of corners of type A in π. We define $g_{n,k} = \sum_{\pi\in\mathcal{B}_{n,k}} \mathrm{cor}_A(\pi)$ and $g_n = \sum_{k\geq 1} g_{n,k}$. Let $G(x,y) = \sum_{n,k\geq 1} g_{n,k} x^n y^k$ be the generating function for the total number of type A corners over all bargraphs according to the number of cells and columns. Then, it follows that

$$
G(x,y) = \frac{\partial F}{\partial q}\bigg|_{q=1} = \frac{y^2 x^3}{(1-x-xy)^2(1+x)}.
$$

Note that $G(x,1) = \frac{x^3}{(1-2x)(1+x)}$ is the generating function for the total number of type A corners over all bargraphs according to the number of cells. Hence,

$$
(6) \qquad g_n = \left(\frac{n+1}{12} - \frac{2}{9}\right) 2^n - \frac{1}{9}(-1)^n.
$$

2.2. Counting (v, w)-corners of type A

Fix $v, w \geq 1$. Define $q_{v,w} = q$ and $q_{a,b} = 1$ for all $(a,b) \neq (v,w)$. Then we have

$$
\alpha_{ab} = \sum_{m \geq 1} x^{a+b(m-1)} y^m q_{a-b,m} = \sum_{m \geq 1} x^{a+b(m-1)} y^m + x^{a+b(w-1)} y^w (q_{a-b,w} - 1)
$$

$$
= \frac{x^a y}{1 - x^b y} + x^{a+b(w-1)} y^w (q - 1) \delta_{a-b=v},
$$

where $\delta_\chi = 1$ if χ holds, and $\delta_\chi = 0$ otherwise. Hence,

$$
(7) \qquad \beta_{ab} = \alpha_{ab}(1 - x^b y) - x^a y = x^{a+b(w-1)} y^w (q-1) \delta_{a-b=v}(1 - x^b y).
$$

Recall that $L(j,s) = \sum_{j=i_{s+1} < i_s < \cdots < i_1 < i_0} \prod_{\ell=0}^s \beta_{i_\ell i_{\ell+1}}$. By using (7), we have

$L(j,s)$

$$
= \sum_{j=i_{s+1} < i_s < \cdots < i_1 < i_0} \prod_{\ell=0}^s \left(x^{i_\ell + i_{\ell+1}(w-1)} y^w (q-1) \delta_{i_\ell - i_{\ell+1} = v}(1 - x^{i_{\ell+1}} y) \right)
$$

$$
= \sum_{j=i_{s+1} < i_s < \cdots < i_1 < i_0} (q-1)^{s+1} y^{w(s+1)} x^{\sum_{\ell=0}^s i_\ell + (w-1) i_{\ell+1}} \prod_{\ell=0}^s \left(\delta_{i_\ell - i_{\ell+1} = v}(1 - x^{i_{\ell+1}} y) \right)
$$

$$
= (q-1)^{s+1} y^{w(s+1)} x^{wj(s+1) + v\binom{s+2}{2} + (w-1)v\binom{s+1}{2}} \prod_{\ell=0}^s (1 - x^{j+(s-l)v} y).
$$

From Theorem 2, we obtain that the generating function $F = H(x, y, \mathbf{q})$ is given by

$$
F = \cfrac{1}{1 - \frac{xy}{1-x} - \sum_{j \geq 1} x^j y \sum_{s \geq 0} (q-1)^{s+1} y^{w(s+1)} x^{wj(s+1) + v\binom{s+2}{2} + (w-1)v\binom{s+1}{2}} \prod_{\ell=0}^s (1 - x^{j+\ell v} y)}.
$$

Recall that $\Lambda_{(v,w)}(\pi)$ denotes the number of (v,w)-corners of type A in π. We define $t_{n,k} = \sum_{\pi \in \mathcal{B}_{n,k}} \Lambda_{(v,w)}(\pi)$ and $t_n = \sum_{k \geq 1} t_{n,k}$. Let $T(x,y) = \sum_{n,k \geq 1} t_{n,k} x^n y^k$ be the generating function for the total number of (v,w)-corners of type A over all bargraphs according to the number of cells and columns. Then, it follows that

$$
T(x,y) = \frac{\partial F}{\partial q}\bigg|_{q=1} = \frac{x^{v+w+1} y^{w+1}}{(1 - \frac{xy}{1-x})^2} \frac{1 - xy - x^{w+2}(1-y)}{(1 - x^{w+1})(1 - x^{w+2})},
$$

which leads to

$$
T(x,1) = \frac{x^{v+w+1}}{(1-2x)^2} \frac{(1-x)^3}{(1 - x^{w+1})(1 - x^{w+2})};
$$

the generating function for the total number of (v,w)-corners of type A over all bargraphs according to the number of cells. As a consequence, we have the following result.

Corollary 3. *The total number of (v,w)-corners of type A over all bargraphs having n cells is given by $t_n = \frac{n}{(2^{w+1}-1)(2^{w+2}-1)} 2^{w-v+n-1}$.*

2.3. Restricted bargraphs

Theorem 2 can be refined as follows. Fix $N \geq 1$. Let $H^{(N)} := H^{(N)}(x, y, \mathbf{q})$ be the generating function for the number of bargraphs π such that the height of each column is at most N according to the number of cells in π, the number of columns of π, and the number of (a, b)-corners of type A in π corresponding to the variables x, y and $\mathbf{q} = (q_{a,b})_{a,b \geq 1}$ respectively. Then by using similar arguments as in the proof of (5), we obtain

$$\text{(8)} \qquad H_a^{(N)} = x^a y H^{(N)} + \sum_{b=1}^{a-1} \beta_{ab} H_b^{(N)},$$

where $H_a^{(N)} := H_a^{(N)}(x, y, \mathbf{q})$ is the generating function for the number of bargraphs $\pi = a\pi'$ such that the height of each column is a maximum of N. Clearly, $H^{(N)} = 1 + \sum_{a=1}^{N} H_a^{(N)}$. By the proof of Theorem 2, we can state its extension as follows.

Theorem 4. *The generating function $H^{(N)}(x, y, \mathbf{q})$ is given by*

$$H^{(N)}(x, y, \mathbf{q}) = \frac{1}{1 - y \sum_{j=1}^{N} x^j - \sum_{j=1}^{N} \left(x^j y \sum_{s \geq 0} L(j, s) \right)},$$

where

$$L(j, s) = \sum_{j = i_{s+1} < i_s < \cdots < i_1 < i_0 \leq N} \prod_{\ell=0}^{s} \beta_{i_\ell i_{\ell+1}}.$$

Moreover, for all $a = 1, 2, \ldots, N$, we have

$$H_a^{(N)} = H^{(N)} \left(x^a y + \sum_{j=1}^{a} \left(x^j y \sum_{s \geq 0} L_a(j, s) \right) \right),$$

where $L_a(j, s) = \sum_{j = i_{s+1} < i_s < \cdots < i_1 < i_0 = a} \prod_{\ell=0}^{s} \beta_{i_\ell i_{\ell+1}}$.

For instance, Theorem 4 for $N = 1, 2$ gives $H^{(1)}(x, y, \mathbf{q}) = \frac{xy}{1 - xy}$ and

$$H^{(2)}(x, y, \mathbf{q}) = \frac{1}{1 - (x + x^2)y - x^2(1 - xy) \sum_{m \geq 1} x^m y^m q_{1,m} + x^3 y^2}.$$

2.4. Counting corners of type A in set partitions

Recall that we represent any set partition as a bargraph corresponding to its canonical sequential representation. Let $P_k(x, y, \mathbf{q})$ be the generating function for the number of set partitions π of $[n]$ with exactly k blocks according to the number of cells in π, the number of columns of π (which is n), and the number of

(a, b)-corners of type A in π corresponding to the variables x, y and $\mathbf{q} = (q_{a,b})_{a,b \geq 1}$ respectively.

Note that each set partition with exactly k blocks can be decomposed as $1\pi^{(1)} \cdots k\pi^{(k)}$ such that $\pi^{(j)}$ is a word over alphabet $[j]$. Thus, by Theorem 4, we have the following result.

Theorem 5. *The generating function* $P_k(x, y, \mathbf{q})$ *is given by*

$$P_k(x, y, \mathbf{q}) = \prod_{N=1}^{k} H_N^{(N)}(x, y, \mathbf{q}) = \prod_{N=1}^{k} \frac{x^N y + \sum_{j=1}^{N} \left(x^j y \sum_{s \geq 0} L_N(j, s) \right)}{1 - y \sum_{j=1}^{N} x^j - \sum_{j=1}^{N} \left(x^j y \sum_{s \geq 0} L(j, s) \right)},$$

where

$$L(j, s) = \sum_{j=i_{s+1}<i_s<\cdots<i_0 \leq N} \prod_{\ell=0}^{s} \beta_{i_\ell i_{\ell+1}} \ and \ L_N(j, s) = \sum_{j=i_{s+1}<i_s<\cdots<i_0 = N} \prod_{\ell=0}^{s} \beta_{i_\ell i_{\ell+1}}.$$

Now, we consider counting all corners of type A in set partitions. Let $q_{a,b} = q$ for all $a, b \geq 1$, and $Q_k(x, y) = \frac{\partial}{\partial q} P_k(x, y, \mathbf{q}) \big|_{q=1}$. Note that for any $s \geq 0$ and $1 \leq j \leq N - 1$,

$$L_N(j, s) = (q - 1)^{s+1} y^{s+1} \sum_{j=i_{s+1}<i_s<\cdots<i_0 = N} x^{\sum_{\ell=0}^{s} x_\ell}.$$

We have a similar expression for $L(j, s)$. From Theorem 10, we have the generating function $Q_k(x, y)$ given by

$$Q_k(x, y)$$
$$= \prod_{N=1}^{k} \frac{x^N y}{1 - y \sum_{j=1}^{N} x^j} \sum_{N=1}^{k} \frac{\sum_{j=1}^{N-1} \left(x^j y (1 - y \sum_{j=1}^{N} x^j) + x^j y^2 (x^{j+1} + \cdots + x^N) \right)}{1 - y \sum_{j=1}^{N} x^j}.$$

Let $\phi(t) = \frac{t^k}{(1-t)(1-2t)\cdots(1-kt)}$. Then we have $\phi'(t) = \frac{t^{k-1}}{(1-t)\cdots(1-kt)} \sum_{j=1}^{k} \frac{1}{1-jt}$.

Note that

$$
\begin{aligned}
Q_k(1,t) &= \phi(t) \sum_{N=2}^{k} \left((N-1)t + \frac{t^2 \sum_{j=1}^{N-1}(N-j)}{1-Nt} \right) \\
&= \phi(t)t \sum_{N=2}^{k} \left((N-1) + \frac{t\binom{N}{2}}{1-Nt} \right) \\
&= \phi(t)t \left(\binom{k}{2} + \frac{1}{2} \sum_{N=2}^{k} \frac{tN(N-1)}{1-Nt} \right) \\
&= \phi(t)t \left(\frac{1}{2}\binom{k}{2} + \frac{1}{2} \sum_{N=1}^{k} \frac{N-1}{1-Nt} \right) \\
&= \frac{1}{2}\binom{k}{2}\phi(t)t + \frac{1}{2}\phi(t) \sum_{N=1}^{k} \left(-1 - t\frac{1}{1-Nt} + \frac{1}{1-Nt} \right) \\
&= \frac{1}{2}\binom{k}{2}t\phi(t) - \frac{1}{2}k\phi(t) + \frac{1}{2}t\phi'(t) - \frac{1}{2}t^2\phi'(t).
\end{aligned}
$$

Let $q_{n,k}$ be the coefficient of t^n in $Q_k(1,t)$. Define $\tilde{Q}_k(t) = \sum_{n \geq k} q_{n,k} \frac{t^n}{n!}$ to be the exponential generating function for $q_{n,k}$. Recall that the ordinary and exponential generating functions for Stirling numbers of the second kind $S_{n,k}$ are given by $\phi(t)$ and $\frac{(e^t-1)^k}{k!}$, respectively.

Thus,

$$
\begin{aligned}
\tilde{Q}_k(t) &= \frac{1}{2}\binom{k}{2} \int_0^t \frac{(e^r-1)^k}{k!} dr - \frac{k(e^t-1)^k}{2k!} + \frac{kt(e^t-1)^{k-1}e^t}{2k!} \\
&\quad - \int_0^t \frac{rk(e^r-1)^{k-1}e^r}{2k!} dr.
\end{aligned}
$$

Hence, the exponential generating function $\tilde{Q}(t,y) = \sum_{k \geq 0} \tilde{Q}_k(t)y^k$ for the total number of corners over set partitions of $[n]$ with k blocks is given by

$$
\begin{aligned}
\tilde{Q}(t,y) &= \frac{y^2}{4} \int_0^t (e^r-1)^2 e^{y(e^r-1)} dr + \frac{yt}{2} e^{t+y(e^t-1)} \\
&\quad - \frac{y}{2}(e^t-1)e^{y(e^t-1)} - \frac{y}{2} \int_0^t re^{r+y(e^r-1)} dr.
\end{aligned}
$$

In particular, we have

$$
\begin{aligned}
\frac{\partial}{\partial t}\tilde{Q}(t,y) &= \frac{y^2}{4}\left(2te^{2t+y(e^t-1)} - e^{2t+y(e^t-1)} + e^{y(e^t-1)} \right) \\
&= \frac{2t-1}{4}\frac{\partial^2}{\partial t^2}e^{y(e^t-1)} - \frac{2t-1}{4}\frac{\partial}{\partial t}e^{y(e^t-1)} + \frac{y^2}{4}e^{y(e^t-1)}.
\end{aligned}
$$

Hence, we can state the following result.

Theorem 6. *The total number of corners of type A over set partitions of $[n+1]$ with k blocks is given by*

$$\frac{n}{2}S_{n+1,k} - \frac{1}{4}S_{n+2,k} - \frac{n}{2}S_{n,k} + \frac{1}{4}S_{n+1,k} + S_{n,k-2}.$$

Moreover, the total number of corners of type A over set partitions of $[n+1]$ is given by

$$\frac{2n+1}{4}B_{n+1} - \frac{1}{4}B_{n+2} - \frac{n-2}{2}B_n,$$

where B_n is the nth Bell number.

3. Counting Corners of type B

Let $J := J(x, y, \mathbf{p})$ be the generating function for the number of bargraphs π according to the number of cells in π, the number of columns of π, and the number of (a, b)-corners of type B in π, corresponding to the variables x, y and $\mathbf{p} = (p_{a,b})_{a,b \geq 1}$ respectively, that is,

$$J = \sum_{n \geq 0} \sum_{\pi \in \mathcal{B}_n} x^n y^{\text{col}(\pi)} \prod_{a,b \geq 1} p_{a,b}^{\Lambda_{(a,b)}(\pi)},$$

where $\Lambda_{(a,b)}(\pi)$ denotes the number of (a, b)-corners of type B in π. From the definitions, we have

$$(9) \qquad\qquad J = 1 + \sum_{a \geq 1} J_a,$$

where J_a is the generating function for the number of bargraphs $\pi = a\pi'$ in which the height of the first column is a. Since each bargraph $\pi = a\pi'$ can be decomposed as either $\pi = a^m$, $\pi = a^m b\pi''$ with $b \geq a + 1$, or $\pi = a^m b\pi''$ with $1 \leq b \leq a - 1$, we have

$$J_a = \sum_{m \geq 1} x^{am} y^m p_{m,a} + \sum_{m \geq 1} x^{am} y^m (J_{a+1} + J_{a+2} + \cdots)$$

$$(10) \qquad\qquad + \sum_{m \geq 1} \sum_{b=1}^{a-1} x^{am} y^m p_{m,a-b} J_b.$$

Define $\gamma_a := \sum_{m \geq 1} x^{am} y^m p_{m,a}$. It follows from (9) that

$$J - 1 - \sum_{b=1}^{a} J_b = \sum_{b \geq a+1} J_b.$$

Then we obtain

$$
\begin{aligned}
J_a &= \gamma_a + \frac{x^a y}{1 - x^a y}\left(J - 1 - \sum_{b=1}^{a} J_b\right) + \sum_{m \geq 1}\left(x^{am} y^m \sum_{b=1}^{a-1} p_{m,a-b} J_b\right) \\
&= \gamma_a + \frac{x^a y}{1 - x^a y}(J - 1) - \frac{x^a y}{1 - x^a y} J_a + \sum_{m \geq 1}\left(x^{am} y^m \sum_{b=1}^{a-1}(p_{m,a-b} - 1) J_b\right),
\end{aligned}
$$

which, by solving for J_a, gives

$$
J_a = x^a y(J - 1) + (1 - x^a y)\gamma_a + (1 - x^a y)\sum_{m \geq 1}\left(x^{am} y^m \sum_{b=1}^{a-1}(p_{m,a-b} - 1) J_b\right).
$$

If we define

$$
\theta_a := x^a y(J - 1) + (1 - x^a y)\gamma_a \text{ and } \mu_{a,b} := (1 - x^a y)\sum_{m \geq 1}\left(x^{am} y^m (p_{m,a-b} - 1)\right),
$$

then we obtain

(11)
$$
J_a = \theta_a + \sum_{b=1}^{a-1} \mu_{a,b} J_b.
$$

By similar techniques as in the proof of Lemma 1, we can state the following result.

Lemma 7. *For all $a \geq 1$,*

$$
J_a = \theta_a + \sum_{j=1}^{a-1} \Gamma_{a,j} \theta_j,
$$

where $\Gamma_{a,j} = \sum_{s \geq 0} \sum_{j=i_{s+1} < i_s < \cdots < i_0 = a} \prod_{\ell=0}^{s} \mu_{i_\ell i_{\ell+1}}$.

Theorem 8. *The generating function $J(x, y, \mathbf{p})$ is given by*

$$
J(x, y, \mathbf{p}) = 1 + \frac{\sum_{m \geq 1} \sum_{j \geq 1}(1 + \Gamma_j)(1 - x^j y) x^{jm} y^m p_{m,j}}{1 - \frac{xy}{1-x} - \sum_{j \geq 1} x^j y \Gamma_j},
$$

where $\Gamma_j = \sum_{s \geq 0} \sum_{j=i_{s+1} < i_s < \cdots < i_0} \prod_{\ell=0}^{s} \mu_{i_\ell i_{\ell+1}}$.

For instance, if $p_{a,b} = 1$ for all $a, b \geq 1$, then $\mu_{a,b} = 0$ which implies that $\Gamma_a = 0$. Thus Theorem 8 shows that $J(x, y, 1, 1, \cdots) = \frac{1-x}{1-x-xy}$.

3.1. Counting all corners of type B

Let $p_{a,b} = p$ for all $a, b \geq 1$. From the definitions, we have $\mu_{a,b} = (p-1)x^a y$ which yields

$$\Gamma_j = \sum_{s \geq 0} \left((p-1)^{s+1} \sum_{j = i_{s+1} < i_s < \cdots < i_0} x^{i_0 + \cdots + i_s} \right)$$

(12)
$$= \sum_{s \geq 0} \frac{(p-1)^{s+1} y^{s+1} x^{(s+1)j + \binom{s+2}{2}}}{(1-x)(1-x^2) \cdots (1-x^{s+1})}.$$

From Theorem 8 and (12), the generating function $F = J(x, y, p, p, \cdots)$ is given by

$$F = 1 + \frac{p\frac{xy}{1-x} + p\sum_{j \geq 1} \Gamma_j x^j y}{1 - \frac{xy}{1-x} - \sum_{j \geq 1} \Gamma_j x^j y} = 1 + \frac{p\frac{xy}{1-x} + p\sum_{s \geq 0} \frac{(p-1)^{s+1} y^{s+2} x^{\binom{s+3}{2}}}{(1-x)(1-x^2) \cdots (1-x^{s+2})}}{1 - \frac{xy}{1-x} - \sum_{s \geq 0} \frac{(p-1)^{s+1} y^{s+2} x^{\binom{s+3}{2}}}{(1-x)(1-x^2) \cdots (1-x^{s+2})}}.$$

Let $\mathrm{cor}_B(\pi)$ be the number of corners of type B in π. Define $h_{n,k} = \sum_{\pi \in \mathcal{B}_{n,k}} \mathrm{cor}_B(\pi)$ and $h_n = \sum_{k \geq 1} h_{n,k}$. Let $H(x, y) = \sum_{n,k \geq 1} h_{n,k} x^n y^k$ be the generating function for the total number of type B corners over all bargraphs according to the number of cells and columns. Then, it follows that

(13)
$$H(x, y) = \left. \frac{\partial F}{\partial p} \right|_{p=1} = \frac{xy(1 - x - xy + x^2 y^2)}{(1 - x - xy)^2}.$$

Note that $H(x, 1) = \frac{x(x-1)^2}{(1-2x)^2}$ is the generating function for the total number of type B corners over all bargraphs according to the number of cells.

3.2. Counting (v, w)-corners of type B

Fix $v, w \geq 1$. Define $p_{v,w} = p$ and $p_{a,b} = 1$ for all $(a, b) \neq (v, w)$. Then we have $\mu_{a,b} = (1 - x^a y)x^{av} y^v (p-1)\delta_{a-b=w}$ which yields

$$\Gamma_j = \sum_{s \geq 0} \sum_{j = i_{s+1} < i_s < \cdots < i_0} \prod_{\ell=0}^{s} \left((1 - x^{i_\ell} y)x^{i_\ell v} y^v (p-1)\delta_{i_{\ell+1} - i_\ell = w} \right)$$

(14)
$$= \sum_{s \geq 0} (p-1)^{s+1} y^{v(s+1)} x^{vj(s+1) + vw\binom{s+1}{2}} \prod_{\ell=0}^{s} (1 - x^{j + (\ell+1)w} y).$$

From Theorem 8 and (14), the generating function $F = J(x, y, \mathbf{p})$ is given by

$$F = 1 + \frac{\frac{yx}{1-x} + (1 + \Gamma_w)(1 - x^w y)x^{wv} y^v (p-1) + y\sum_{j \geq 1} x^j \Gamma_j}{1 - \frac{xy}{1-x} - y\sum_{j \geq 1} x^j \Gamma_j}.$$

Recall that $\Lambda_{(v,w)}(\pi)$ denotes the number of (v,w)-corners of type B in π. We define $t_{n,k} = \sum_{\pi \in \mathcal{B}_{n,k}} \Lambda_{(v,w)}(\pi)$ and $t_n = \sum_{k \geq 1} t_{n,k}$. Let $T(x,y) = \sum_{n,k \geq 1} t_{n,k} x^n y^k$ be the generating function for the total number of (v,w)-corners of type B over all bargraphs according to the number of cells and columns. Then, it follows that

$$
\begin{aligned}
T(x,y) &= \frac{\partial F}{\partial p}\bigg|_{p=1} \\
&= \frac{(1 - x^w y) x^{vw} y^w}{\left(1 - \frac{xy}{1-x}\right)^2} + \frac{y^{v+1} x^{2v+3}(1 - x^w y) + (yx)^{v+1}(1 - x^{w+1} y)}{(1 - x^{v+1})(1 - x^{v+2}) \left(1 - \frac{xy}{1-x}\right)^2},
\end{aligned}
$$

which leads to

$$
T(x,1) = \frac{(1 - x^w) x^{vw}}{\left(1 - \frac{x}{1-x}\right)^2} + \frac{x^{2v+3}(1 - x^w) + x^{v+1}(1 - x^{w+1})}{(1 - x^{v+1})(1 - x^{v+2}) \left(1 - \frac{x}{1-x}\right)^2};
$$

this latter is the generating function for the total number of (v,w)-corners of type B over all bargraphs according to the number of cells.

3.3. Restricted bargraphs

Theorem 8 can be refined as follows. For $N \geq 1$, let $J^{(N)} := J^{(N)}(x,y,\mathbf{p})$ be the generating function for the number of bargraphs π such that the height of each column is at most N according to the number of cells in π, the number of columns of π, and the number of (a,b)-corners of type B in π corresponding to the variables x, y and $\mathbf{p} = (p_{a,b})_{a,b \geq 1}$ respectively. Then by using similar arguments as in the proof of (9) and (11), we obtain that

$$
J^{(N)} = 1 + \sum_{a=1}^{N} J_a^{(N)} \text{ and } J_a^{(N)} = \theta_a + \sum_{b=1}^{a-1} \mu_{a,b} J_b^{(N)},
$$

for all $a = 1, 2, \ldots, N$, where $J_a^{(N)} := J_a^{(N)}(x,y,\mathbf{p})$ is the generating function for the number of bargraphs $\pi = a\pi'$ such that the height of each column is at most N. From the proof of Theorem 8, we can state its extension as follows.

Theorem 9. *The generating function $J^{(N)} = J^{(N)}(x,y,\mathbf{p})$ is given by*

$$
J^{(N)} = 1 + \frac{\sum_{j=1}^{N}(1 + \Gamma_j)(1 - x^j y)\gamma_j}{1 - y\sum_{j=1}^{N} x^j - \sum_{j=1}^{N} x^j y \Gamma_j},
$$

where

$$
\Gamma_j = \sum_{s \geq 0} \sum_{j = i_{s+1} < i_s < \cdots < i_0 \leq N} \prod_{\ell=0}^{s} \mu_{i_\ell i_{\ell+1}}.
$$

Moreover, for all $a = 1, 2, \ldots, N$, we have

$$J_a^{(N)} = \left(x^a y + \sum_{j=1}^{a-1} x^j y \Gamma_{a,j} \right) (J^{(N)} - 1) + (1 - x^a y) \gamma_a + \sum_{j=1}^{a-1} \Gamma_{a,j} (1 - x^j y) \gamma_j,$$

where $\Gamma_{N,j} = \sum_{s \geq 0} \sum_{j=i_{s+1} < i_s < \cdots < i_0 = N} \prod_{\ell=0}^{s} \mu_{i_\ell i_{\ell+1}}$.

For instance, Theorem 9 for $N = 1$ gives

$$J^{(1)}(x, y, \mathbf{p}) = 1 + \frac{(1 - xy)\gamma_1}{1 - xy} = 1 + \sum_{m \geq 1} x^m y^m p_{m,1}.$$

3.4. Counting corners of type B in set partitions

Recall that we represent any set partition as a bargraph corresponding to its canonical sequential representation. Let $P_k(x, y, \mathbf{p})$ be the generating function for the number of set partitions π of $[n]$ with exactly k blocks according to the number of cells in π, the number of columns of π (which is n), and the number of (a,b)-corners of type B in π corresponding to the variables x, y and $\mathbf{p} = (p_{a,b})_{a,b \geq 1}$ respectively.

Note that each set partition with exactly k blocks can be decomposed as $1\pi^{(1)} \cdots k\pi^{(k)}$ such that $\pi^{(j)}$ is a word over alphabet $[j]$. Thus, by Theorem 9, we have the following result.

Theorem 10. *Let $p_{a,b} = p$ for all $a, b \geq 1$. Then the generating function $P_k(x, y, \mathbf{p})$ is given by*

$$P_k(x, y, \mathbf{p}) = p^{1-k} \prod_{N=1}^{k} J_N^{(N)}(x, y, \mathbf{p}),$$

where $J_N^{(N)}$ is given in statement Theorem 9.

Now, we consider counting all corners of type B in set partitions. Let $p_{a,b} = p$ for all $a, b \geq 1$, and $Q_k(x, y) = \left. \frac{\partial}{\partial p} P_k(x, y, \mathbf{p}) \right|_{p=1}$. By Theorem 9, we have that $J^{(N)}(x, y, \mathbf{1}) = \frac{1}{1 - y \sum_{j=1}^{N} x^j}$ and

$$\frac{\partial}{\partial p} J^{(N)}(x, y, \mathbf{p}) \mid_{p=1} = \frac{y \sum_{j=1}^{N} x^j - \left(y \sum_{j=1}^{N} x^j \right)^2 + y^2 \sum_{j=1}^{N} x^j \frac{x^{j+1} - x^{N+1}}{1-x}}{\left(1 - y \sum_{j=1}^{N} x^j \right)^2}.$$

Moreover, Theorem 9 gives that $J_N^{(N)}(x, y, \mathbf{1}) = \frac{x^N y}{1 - y \sum_{j=1}^{N} x^j}$, and

$$\frac{\partial}{\partial p} J_N^{(N)}(x, y, \mathbf{p}) \mid_{p=1} = x^N y \left(\frac{\partial}{\partial p} J^{(N)}(x, y, \mathbf{p}) \mid_{p=1} + \frac{1 - x^N y}{1 - \sum_{j=1}^{N} x^j y} \right).$$

Hence, by Theorem 10, we have

$$
Q_k(x, y) = \prod_{N=1}^{k} \frac{x^N y}{1 - y \sum_{j=1}^{N} x^j} \left(\sum_{N=1}^{k} \frac{\frac{\partial}{\partial p} J_N^{(N)}(x, y, \mathbf{p}) \mid_{p=1}}{\frac{x^N y}{1-y \sum_{j=1}^{N} x^j}} - k + 1 \right).
$$

In particular, the generating function for the total number of corners of type B over all set partitions of $[n]$ with k blocks is given by

$$
Q_k(1, t) = \frac{t^k}{(1-t)(1-2t)\cdots(1-kt)} \left(\sum_{N=1}^{k} \frac{\frac{\partial}{\partial p} J_N^{(N)}(1, t, \mathbf{p}) \mid_{p=1}}{\frac{t}{1-Nt}} - k + 1 \right),
$$

which, by $\frac{\partial}{\partial p} J_N^{(N)}(1, t, \mathbf{p}) \mid_{p=1} = t \left(\frac{Nt}{1-Nt} + \frac{t^2 N(N-1)}{2(1-Nt)^2} + \frac{1-t}{1-Nt} \right)$, is equivalent to

$$
Q_k(1, t)
$$
$$
= \frac{t^k}{(1-t)(1-2t)\cdots(1-kt)} \left(1 - k + \sum_{N=1}^{k} \left(1 + (N-1)t + \frac{t^2 N(N-1)}{2(1-Nt)} \right) \right).
$$

Hence,

$$
Q_k(1, t) = \frac{t^k}{(1-t)(1-2t)\cdots(1-kt)} \left(1 + \frac{t}{2} \binom{k}{2} + \frac{t}{2} \sum_{N=1}^{k} \frac{N-1}{1-Nt} \right).
$$

Define $\tilde{Q}_k(t)$ to be the corresponding exponential generating function to $Q_k(1, t)$, that is $\tilde{Q}_k(t) = \sum_{n \geq 0} q_{n,k} \frac{t^n}{n!}$ where $q_{n,k}$ is the coefficient of t^n in $Q_k(1, t)$. Similarly to Section 2.4, we have

$$
\tilde{Q}_k(t) = \frac{(e^t - 1)^k}{k!} + \binom{k}{2} \int_0^t \frac{(e^r - 1)^k}{2k!} dr + \frac{kt(e^t - 1)^{k-1} e^t}{2k!} - \frac{k(e^t - 1)^k}{2k!}
$$
$$
- \int_0^t \frac{rk(e^r - 1)^{k-1} e^r}{2k!} dt.
$$

Define $\tilde{Q}(t, y) = \sum_{k \geq 0} \tilde{Q}_k(t) y^k$; thus by multiplying by y^k and summing over $k \geq 1$, we obtain

$$
\tilde{Q}(t, y) = e^{y(e^t - 1)} - 1 + \frac{y^2}{4} \int_0^t (e^r - 1)^2 e^{y(e^r - 1)} dr
$$
$$
+ \frac{ty}{2} e^{t + y(e^t - 1)} - \frac{y}{2} (e^t - 1) e^{y(e^t - 1)} - \frac{y}{2} \int_0^t r e^{r + y(e^r - 1)} dr.
$$

In particular,

$$
\frac{\partial}{\partial t} \tilde{Q}(t, y) = \frac{y}{4} \left(y(2t - 1) e^{2t + y(e^t - 1)} + y e^{y(e^t - 1)} + 4 e^{t + y(e^t - 1)} \right),
$$

which is equivalent to

$$\frac{\partial}{\partial t}\tilde{Q}(t,y) = \frac{2t-1}{4}\frac{\partial^2}{\partial t^2}e^{y(e^t-1)} - \frac{2t-5}{4}\frac{\partial}{\partial t}e^{y(e^t-1)} + \frac{y^2}{4}e^{y(e^t-1)}.$$

Since $e^{y(e^t-1)} = \sum_{n\geq 0}\sum_{k=0}^{n}S_{n,k}\frac{x^n y^k}{n!}$, $S_{n,k}$ is the Stirling number of the second kind, we obtain the following result.

Theorem 11. *The total number of corners of type B over set partitions of $[n+1]$ with k blocks is given by*

$$\frac{n}{2}S_{n+1,k} - \frac{1}{4}S_{n+2,k} - \frac{n}{2}S_{n,k} + \frac{5}{4}S_{n+1,k} + S_{n,k-2}.$$

Moreover, the total number of corners of type B over set partitions of $[n+1]$ is given by

$$\frac{2n+5}{4}B_{n+1} - \frac{1}{4}B_{n+2} - \frac{n-2}{2}B_n,$$

where B_n is the nth Bell number.

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