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SOME SHARP CIRCULAR AND HYPERBOLIC BOUNDS OF $\exp(-x^2)$ WITH APPLICATIONS

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This article is devoted to the determination of sharp lower and upper bounds for $\exp(-x^2)$ over the interval $(-\epsilon, \epsilon)$. The bounds are of the type $\left[\frac{a+f(x)}{a+1}\right]^{\alpha}$, where $f(x)$ denotes either cosine or hyperbolic cosine. The results are then used to obtain and refine some known Cusa-Huygens type inequalities. In particular, a new simple proof of Cusa-Huygens type inequalities is presented as an application. For other interesting applications of the main results, sharp bounds of the truncated Gaussian sine integral and error functions are established. They can be useful in probability theory.

1. INTRODUCTION

Bounds of the exponential function $\exp(-x^2)$ can be useful in many areas of mathematics where it appears, mainly to evaluate analytically or numerically complex integrals involving it. Recent studies show that there is still a room of improvements; sharp and tractable bounds for this function remain an actual challenge for any contemporary mathematician. In this regard, Chesneau [**8, 9**] gave tight lower bounds of $\exp(x^2)$ over the real line. For some other sharp bounds, see [**3, 4**], where the bounds are obtained over $(0, 1)$ by the use of circular and hyperbolic functions. This type of bounds can in fact be obtained naturally over $(0, \pi/2)$(see [**10**]). Interested readers are referred to [**2, 8, 9, 14, 20**], and the references therein.

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The aim of this paper is to present more tight bounds for $\exp(-x^2)$ in the interval $(-\pi/2, \pi/2)$. Some bounds are obtained on $(-\epsilon, \epsilon)$. For applications, these bounds are then used to refine some known Cusa-Huygens type inequalities and to exhibit new sharp bounds for Gaussian type integrals, including the so-called error function, opening new perspectives in many applied areas, including statistics, probability, physics and engineering.

This paper is organized, as follows. Section 2 presents main results of the paper, with graphical and numerical evidences. Then, with the aim of providing the complete proofs of them, some auxiliary results are discussed in Section 3. The full proofs of the main results are available in Section 4. Finally, some applications are given in Section 5.

2. RESULTS

This section contains the two main results of the paper.

2.1 First Result

We state the first main result of this paper as follows:

Theorem 1. *For $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we have*

$$\left(\frac{1+\cos x}{2}\right)^a \leqslant \exp(-x^2) \leqslant \left(\frac{1+\cos x}{2}\right)^b \tag{2.1}$$

and

$$\left(\frac{2+\cos x}{3}\right)^c \leqslant \exp(-x^2) \leqslant \left(\frac{2+\cos x}{3}\right)^d, \tag{2.2}$$

with the best possible constants $a = 4, b = \frac{-(\pi/2)^2}{\ln(1/2)} \approx 3.559707, c = \frac{-(\pi/2)^2}{\ln(2/3)} \approx 6.08536$ and $d = 6$, and the inequalities hold as equalities at $x = 0$.

Note: The right inequality in (2.2) has been proved in [**23**, Theorem 2]. In fact, it holds for $x \in (0, \infty)$. However, it is not sharp for large values of x. Again, our proof will use different method.

Some graphical and numerical illustrations: The inequalities (2.1) are illustrated in Figures 1 and 2. We clearly observe the sharpness of the obtained bounds. In particular, with this graphical investigation, the inequalities (2.2) seem more sharp; the curves of the functions of the bounds are almost visually confounded, even with a reasonable zoom. In order to illustrate this point, let us investigate the global L_2 error defined by: $e(h) = \int_{-\pi/2}^{\pi/2}(\exp(-x^2) - h(x))^2 dx$, where $h(x)$ denotes any function in the bounds (2.1) and (2.2). The obtained numerical results are collected in Table 1. From this numerical point of view, we see that the bounds in (2.2) are sharper to those in (2.1).

Table 1: Global L_2 errors $e(h)$ for the functions $h(x)$ in the bounds of (2.1) and (2.2).

	Inequality (2.1)		Inequality (2.2)	
$h(x)$	$\left(\frac{1+\cos x}{2}\right)^a$	$\left(\frac{1+\cos x}{2}\right)^b$	$\left(\frac{2+\cos x}{3}\right)^c$	$\left(\frac{2+\cos x}{3}\right)^d$
$e(h)$	≈ 0.000629229	≈ 0.001120559	$\approx 2.791112 \times 10^{-5}$	$\approx 4.605539 \times 10^{-6}$

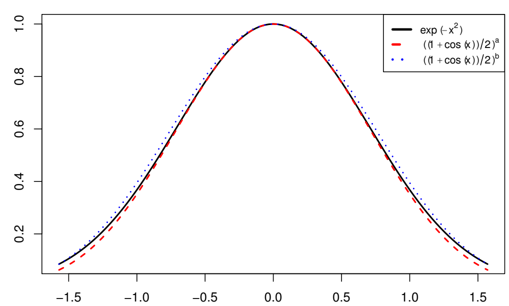

Figure 1: Graphs of the functions of bounds (2.1) for $x \in (-\pi/2, \pi/2)$.

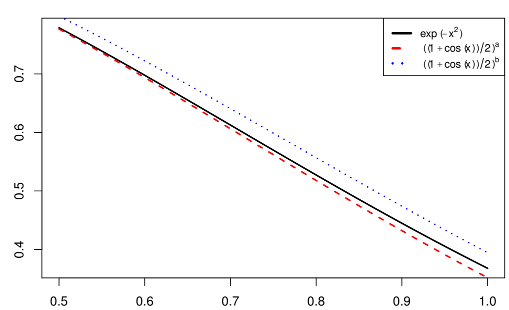

Figure 2: Graphs of the functions of bounds (2.1) for $x \in (0.5, 1)$.

2.2 Second Result

The hyperbolic variants are given in the following theorem.

Theorem 2. *For $x \in (-\pi/2, \pi/2)$, we have*

$$\left(\frac{1+\cosh x}{2}\right)^{\alpha} \leqslant \exp(x^2) \leqslant \left(\frac{1+\cosh x}{2}\right)^{\beta} \tag{2.3}$$

and

$$\left(\frac{2+\cosh x}{3}\right)^{\theta} \leqslant \exp(x^2) \leqslant \left(\frac{2+\cosh x}{3}\right)^{\gamma} \tag{2.4}$$

with the best possible constants $\alpha = 4, \beta = \frac{(\pi/2)^2}{\ln[(1+\cosh(\pi/2))/2]} \approx 4.38856, \theta = 6$ and $\gamma = \frac{(\pi/2)^2}{\ln[(2+\cosh(\pi/2))/3]} \approx 6.054932$, and the inequalities hold as equalities at $x = 0$.

The bounds of $\exp(x^2)$ given in (2.4) are very sharp. Moreover they are simple and better than the corresponding bounds of $\exp(x^2)$ given in [**8, 9**] as far as $x \in (-\pi/2, \pi/2)$.

Some graphical and numerical illustrations: The inequalities (2.3) are illustrated in Figures 3 and 4, showing the sharpness of the obtained bounds. After a graphical investigation, the inequalities (2.4) seem more sharp. In order to illustrate this point, as the previous numerical study, let us consider the global L_2 error defined by: $e_*(h) = \int_{-\pi/2}^{\pi/2}(\exp(x^2) - h(x))^2 dx$, where $h(x)$ denotes any function in the bounds of (2.3) and (2.4). The results are set in Table 2. From this numerical point of view, we then see that the bounds in (2.4) are sharper to those in (2.3).

Table 2: Global L_2 errors $e_*(h)$ for the functions $h(x)$ in the bounds of (2.3) and (2.4).

	Inequality (2.3)		Inequality (2.4)	
$h(x)$	$\left(\frac{1+\cosh x}{2}\right)^{\alpha}$	$\left(\frac{1+\cosh x}{2}\right)^{\beta}$	$\left(\frac{2+\cosh x}{3}\right)^{\theta}$	$\left(\frac{2+\cosh x}{3}\right)^{\gamma}$
$e_*(h)$	≈ 1.011738	≈ 0.05904132	≈ 0.01013854	≈ 0.001456429

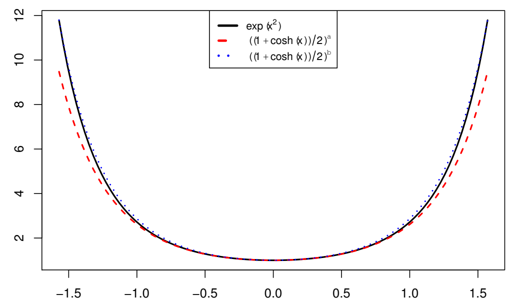

Figure 3: Graphs of the functions of bounds (2.3) for $x \in (-\pi/2, \pi/2)$.

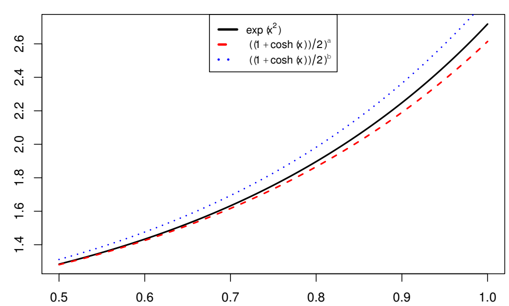

Figure 4: Graphs of the functions of bounds (2.3) for $x \in (0.5, 1)$.

Note: It follows from Theorem 2 that, for $x \in (-\pi/2, \pi/2)$, we have

$$\left(\frac{2+\cosh x}{3}\right)^{-\gamma} \leqslant \exp(-x^2) \leqslant \left(\frac{2+\cosh x}{3}\right)^{-\theta}. \tag{2.5}$$

It is natural to address the following question: what are the best bounds for $\exp(-x^2)$ between those in (2.2) and (2.5) ? An element of answer can be given numerically. By considering again the global L_2 error, i.e., $e(h) = \int_{-\pi/2}^{\pi/2}(\exp(-x^2) - h(x))^2 dx$, where $h(x)$ denotes any function in the bounds (2.2) and (2.5). The results are set in Table 3. From this numerical point of view, we then see that the bounds in (2.5) are near twice sharper to those in (2.2).

Table 3: Global L_2 errors $e(h)$ for the functions $h(x)$ in the bounds of (2.2) and (2.5).

	Inequality (2.2)		Inequality (2.5)	
$h(x)$	$\left(\frac{2+\cos x}{3}\right)^c$	$\left(\frac{2+\cos x}{3}\right)^d$	$\left(\frac{2+\cosh x}{3}\right)^{-\gamma}$	$\left(\frac{2+\cosh x}{3}\right)^{-\theta}$
$e(h)$	$\approx 2.791112 \times 10^{-5}$	$\approx 4.605539 \times 10^{-6}$	$\approx 1.068113 \times 10^{-5}$	$\approx 2.338449 \times 10^{-6}$

3. PRELIMINARIES AND LEMMAS

We now present two lemmas which will be useful for the proofs of our theorems.

Lemma 1. *The following inequalities hold:*

$$\frac{\sin x}{x} > \frac{1+2\cos x}{2+\cos x};\ x \in (0,\pi) \tag{3.6}$$

and

$$\frac{x}{\sinh x} + \cosh x > 2;\ x \neq 0. \tag{3.7}$$

Proof: For (3.6), let $f(x) = \sin x(2+\cos x) - x\,(1+2\cos x)$. Then, a simple computation yields

$$\begin{aligned} f'(x) = -\sin^2 x + \cos^2 x + 2x\sin x - 1 &= 2x\sin x - 2\sin^2 x \\ &= 2\sin x(x - \sin x) > 0, \end{aligned}$$

for $x \in (0,\pi)$. Hence, $f(x)$ is strictly increasing in $(0,\pi)$. Thus $f(x) > f(0)$ for any $x \in (0,\pi)$, implying that

$$\sin x\,(2+\cos x) > x\,(1+2\cos x).$$

For (3.7), by symmetry of the function, we need to consider only positive values of x. In this regard, let us set

$$g(x) = 2\sinh x - \sinh x\cosh x - x.$$

Differentiation gives

$$\begin{aligned} g'(x) = 2\cosh x - \sinh^2 x - \cosh^2 x - 1 &= 2\cosh x - 2\cosh^2 x \\ &= 2\cosh x(1-\cosh x) < 0. \end{aligned}$$

Therefore, $g(x)$ is strictly decreasing in $(0,\infty)$. So, $g(x) < 0$ for every $x \in (0,\infty)$, meaning that $x + \sinh x\cosh x > 2\sinh x$. This completes the proof. $\square$

Note: For hyperbolic version of (3.6), one can see [**12**, Remark 1].

Lemma 2. *(The L'Hospital's monotonicity rule [**1**]) : Let $f, g : [p, q] \to \mathbb{R}$ be two continuous functions which are derivable on (p, q) and $g'(x) \neq 0$ for any $x \in (p, q)$. If f'/g' is increasing (or decreasing) on (p, q), then the functions $\frac{f(x)-f(p)}{g(x)-g(p)}$ and $\frac{f(x)-f(q)}{g(x)-g(q)}$ are also increasing (or decreasing) on (p, q). If f'/g' is strictly monotone, then the monotonicity in the conclusion is also strict.*

Lemma 3. *(Theorem 4)[**16**] If $f : (a, b) \to \mathbb{R}$ is a real analytic function such that*

$$f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k$$

where $c_k \in \mathbb{R}$ and $c_k \geqslant 0$ for all $k \in \mathbb{N} \cup \{0\}$, then

$$f(0+) \leqslant f(x) \leqslant f(b-).$$

For general form of Lemma 3 and its applications to analytical inequalities we refer reader to [**16, 17**].

4. PROOFS OF THE THEOREMS

In this section we prove our main results.

Proof of Theorem 1: Clearly, for $x = 0$, equalities hold. We need to consider only positive values of x in $(-\pi/2, \pi/2)$ as bounds and $\exp(-x^2)$ are even functions. For (2.1) , let

$$f(x) = \frac{-x^2}{\ln\left(\frac{1+\cos x}{2}\right)} = \frac{f_1(x)}{f_2(x)},$$

where $f_1(x) = -x^2$ and $f_2(x) = \ln\left(\frac{1+\cos x}{2}\right)$ with $f_1(0) = f_2(0) = 0$. Upon differentiation, we get

$$\frac{f_1'(x)}{f_2'(x)} = \frac{2\,x\,(1+\cos x)}{\sin x} = 2\frac{x}{\sin x}\,(1+\cos x) = 2\,F(x),$$

where $F(x) = \frac{x}{\sin x}\,(1+\cos x)$. Again, upon differentiation,

$$\begin{aligned}
F'(x) &= -x + \frac{(\sin x - x\cos x)}{\sin^2 x}(1+\cos x) \\
&= \frac{1}{\sin^2 x}\left[-x\sin^2 x + \sin x + \sin x\cos x - x\cos x - x\cos^2 x\right] \\
&= \frac{1}{\sin^2 x}\left[-x(1+\cos x) + \sin x(1+\cos x)\right] \\
&= \frac{1}{\sin^2 x}\left[(1+\cos x)(\sin x - x)\right] < 0,
\end{aligned}$$

since $\sin x - x < 0$ in $(0, \pi/2)$. Therefore, $F(x)$ is strictly decreasing in $(0, \pi/2)$ and so is $f(x)$ by Lemma 2. Consequently, $a = f(0+) = 4$ by L'Hospital's rule and $b = f(\pi/2) = \frac{-(\pi/2)^2}{\ln(1/2)} \approx 3.559707$.

Similarly to (2.2), let us consider

$$g(x) = \frac{-x^2}{\ln\left(\frac{2+\cos x}{3}\right)} = \frac{g_1(x)}{g_2(x)},$$

where $g_1(x) = -x^2$ and $g_2(x) = \ln\left(\frac{2+\cos x}{3}\right)$ with $g_1(0) = g_2(0) = 0$. Then,

$$\frac{g_1'(x)}{g_2'(x)} = \frac{2\,x\,(2+\cos x)}{\sin x} = 2\,G(x),$$

where $G(x) = \frac{x}{\sin x}(2+\cos x)$. Differentiation gives

$$\begin{aligned} G'(x) &= -x + \frac{(\sin x - x\cos x)(2+\cos x)}{\sin^2 x} \\ &= \frac{1}{\sin^2 x}\left[\sin x(2+\cos x) - x(1+2\cos x)\right] > 0, \end{aligned}$$

by virtue of Lemma 1, i.e., (3.6) and thus, $G(x)$ is strictly increasing in $(0, \pi/2)$. Therefore $g(x)$ is strictly increasing in $(0, \pi/2)$ by Lemma 2. Thus, $c = g(\pi/2) = \frac{-(\pi/2)^2}{\ln(2/3)} \approx 6.08536$ and $d = g(0+) = 6$. This completes the proof. □

Remark 1. *For $x \in [-\epsilon, \epsilon]$ where $\epsilon \in (0, \pi)$, we can actually see that, the inequalities in Theorem* 1 *hold with the best possible constants $a = 4, b = \frac{-\epsilon^2}{\ln\left(\frac{1+\cos\epsilon}{2}\right)}, c = \frac{-\epsilon^2}{\ln\left(\frac{2+\cos\epsilon}{3}\right)}$ and $d = 6$. The same proof as given above is applicable as* (3.6) *is valid in* $(0, \pi)$. *There is also an alternative method to prove Theorem* 1 *by using algorithm presented in [***6, 15***].*

Proof of Theorem 2: Equalities hold for $x = 0$. As in the proof of Theorem 1, we need to consider only positive values of x in $(-\pi/2, \pi/2)$. For (2.3), let

$$f(x) = \frac{x^2}{\ln\left(\frac{1+\cosh x}{2}\right)} = \frac{f_1(x)}{f_2(x)},$$

where $f_1(x) = x^2$ and $f_2(x) = \ln\left(\frac{1+\cosh x}{2}\right)$ with $f_1(0) = 0 = f_2(0)$. Upon differentiation, we get

$$\frac{f_1'(x)}{f_2'(x)} = \frac{2x(1+\cosh x)}{\sinh x} = \frac{f_3(x)}{f_4(x)},$$

where $f_3(x) = 2x(1+\cosh x)$ and $f_4(x) = \sinh x$ with $f_3(0) = f_4(0) = 0$. Differentiation yields

$$\begin{aligned} \frac{f_3'(x)}{f_4'(x)} &= 2\left[\frac{x\sinh x + 1 + \cosh x}{\cosh x}\right] \\ &= 2\left[x\tanh x + \operatorname{sech} x + 1\right] \\ &= 2\,F(x), \end{aligned}$$

where $F(x) = x\tanh x + \operatorname{sech} x + 1$. Again, by differentiating

$$\begin{aligned} F'(x) &= x\operatorname{sech}^2 x + \tanh x - \operatorname{sech} x \tanh x \\ &= \tanh x \operatorname{sech} x \left[\frac{x}{\sinh x} + \cosh x - 1\right] > 0, \end{aligned}$$

since $\frac{x}{\sinh x} + \cosh x > 2$ by (3.7) of Lemma 1. Therefore, $F(x)$ is strictly increasing, which implies that $f(x)$ is also strictly increasing by Lemma 2. Thus, $\alpha = f(0+) = 4$ and $\beta = f(\pi/2) = \frac{(\pi/2)^2}{\ln\left[\frac{1+\cosh(\pi/2)}{2}\right]} \approx 4.38856$.

Next, let us prove (2.4). We set

$$g(x) = \frac{x^2}{\ln\left(\frac{2+\cosh x}{3}\right)} = \frac{g_1(x)}{g_2(x)},$$

where $g_1(x) = x^2$ and $g_2(x) = \ln\left(\frac{2+\cosh x}{3}\right)$ with $g_1(0) = g_2(0) = 0$. Differentiation gives

$$\frac{g_1'(x)}{g_2'(x)} = \frac{2x\,(2+\cosh x)}{\sinh x} = \frac{g_3(x)}{g_4(x)},$$

where $g_3(x) = 2x(2+\cosh x)$ and $g_4(x) = \sinh x$ with $g_3(0) = g_4(0) = 0$. Therefore,

$$\begin{aligned} \frac{g_3'(x)}{g_4'(x)} &= 2\left[\frac{x\sinh x + 2 + \cosh x}{\cosh x}\right] \\ &= 2\,G(x), \end{aligned}$$

where $G(x) = x\tanh x + 2\operatorname{sech} x + 1$. By differentiation, we get

$$\begin{aligned} G'(x) &= x\operatorname{sech}^2 x + \tanh x - 2\operatorname{sech} x \tanh x \\ &= \tanh x \operatorname{sech} x \left[\frac{x}{\sinh x} + \cosh x - 2\right] > 0, \end{aligned}$$

due to second inequality (3.7) of Lemma 1. So, $G(x)$ is strictly increasing and hence $g(x)$ in $(0, \pi/2)$ by Lemma 2. Therefore, $\theta = g(0+) = 6$ and $\gamma = g(\pi/2) = \frac{(\pi/2)^2}{\ln\left[\frac{2+\cosh(\pi/2)}{3}\right]} \approx 6.054932$. This proves Theorem 2. □

Remark 2. *For $x \in (-\epsilon, \epsilon)$ where $\epsilon > 0$, it is easy to see that, the inequalities in Theorem 2 hold with the best possible constants $\alpha = 4, \beta = \frac{\epsilon^2}{\ln\left(\frac{1+\cos h\epsilon}{2}\right)}, \gamma = \frac{\epsilon^2}{\ln\left(\frac{2+\cosh \epsilon}{3}\right)}$ and $\theta = 6$. Again, the same proof can be given in this case also or, as mentioned in Remark 1, an alternative method can be applied to prove Theorem 2 by using algorithm presented in [**6, 15**].*

It is quite interesting to see that the inequality in (2.1) can also be generalized and proved by the method described in the recent papers [**16, 17**] by B. Malešević et al. We state and prove the generalized statement.

Theorem 3. *For $x \in [-\epsilon, \epsilon]$ where $\epsilon \in (0, \pi)$ it is true that:*

$$\left(\frac{1+\cos x}{2}\right)^4 \leqslant \exp(-x^2) \leqslant \left(\frac{1+\cos x}{2}\right)^\eta$$

with the best possible constants 4 *and* $\eta = \frac{-\epsilon^2}{\ln\left(\frac{1+\cos\epsilon}{2}\right)}$.

New proof of Theorem 3: It suffices to prove the theorem in $(0, \epsilon]$. Consider the function

$$h(x) = -\frac{\ln\left(\frac{1+\cos x}{2}\right)}{x^2} = -\frac{2\ln\left(\cos(x/2)\right)}{x^2}.$$

Using logarithmic series expansion (Formula 1.518/2)[**11**] we get

$$h(x) = \frac{1}{4} + \frac{x^2}{96} + \frac{x^4}{1440} + \frac{17x^6}{322560} + \frac{31x^8}{7257600} + \frac{691x^{10}}{1916006400} \cdots$$

which is analytic in $(0, \epsilon]$ and $c_k \geqslant 0$ for all $k \in \mathbb{N} \cup \{0\}$. Therefore by Lemma 3, we have

$$h(0+) \leqslant h(x) \leqslant h(\epsilon-).$$

Lastly $h(0+) = \frac{1}{4}$ and $\eta = \frac{1}{h(\epsilon-)} = \frac{-\epsilon^2}{\ln\left(\frac{1+\cos\epsilon}{2}\right)}$ complete the proof.

5. SOME APPLICATIONS

Three applications of Theorems 1 and 2 are presented below. Applications of general cases can also be given accordingly.

5.3 Application 1: On Cusa-Huygens Type Inequalities

The famous Cusa-Huygen's inequality [**7, 13, 18, 19, 21**] is known as

$$\frac{\sin x}{x} < \frac{2+\cos x}{3};\ x \in \left(0, \frac{\pi}{2}\right) \tag{5.8}$$

and its hyperbolic version, sometimes called hyperbolic Cusa-Huygen's inequality [**19**] is stated as follows:

$$\frac{\sinh x}{x} < \frac{2+\cosh x}{3};\ x \neq 0. \tag{5.9}$$

Some researchers have tried to obtain extended sharp versions of the inequalities (5.8) and (5.9) in recent years. In [**7, 21**] the following inequalities have been established:

$$\left(\frac{2+\cos x}{3}\right)^\lambda < \frac{\sin x}{x} < \frac{2+\cos x}{3};\ x \in \left(0, \frac{\pi}{2}\right), \tag{5.10}$$

with the best possible constants $\lambda \approx 1.11374$ and 1.
The authors of [**7, 21**] proved double inequality (5.10) in a complex way. In 2013, a simple proof of it was claimed by Sun and Zhu [**22**]; but later it was found that the proof was logically incorrect [**5**]. We present here very simple and lucid proof of (5.10).

Simple Proof of Inequality (5.10)**:** Using [**3**, Theorem 2] and [**10**, Proposition 3], we have

$$\exp(-kx^2) < \frac{\sin x}{x} < \exp(-x^2/6)\,;x \in \left(0, \frac{\pi}{2}\right),$$

where $k = \frac{-\ln(2/\pi)}{(\pi/2)^2}$. Hence, we can write

$$\left(\frac{\sin x}{x}\right)^6 < \exp(-x^2) < \left(\frac{\sin x}{x}\right)^{1/k} ;x \in \left(0, \frac{\pi}{2}\right), \tag{5.11}$$

where $k = \frac{-4\ln(2/\pi)}{\pi^2}$. From (2.2) and (5.11), it is clear that

$$\left(\frac{2+\cos x}{3}\right)^{\lambda} < \frac{\sin x}{x} < \frac{2+\cos x}{3},$$

where $\lambda = kc = \frac{-4\ln(2/\pi)}{\pi^2} \cdot \frac{-\pi^2}{4\ln(2/3)} = \frac{\ln(2/\pi)}{\ln(2/3)} \approx 1.11374$. Moreover, λ and 1 are the best possible constants, because k and c are. The proof of (5.10) is complete. □

Sándor [**21**] proved that the best positive constants m and n such that

$$\left(\frac{1+\cosh x}{2}\right)^m < \frac{\sinh x}{x} < \left(\frac{1+\cosh x}{2}\right)^n ;\; x > 0 \tag{5.12}$$

are 2/3 and 1, respectively.
In the following corollary, we refine the right inequality of (5.12) over the interval $(0, \pi/2)$.

Corollary 1. *For $x \in (0, \pi/2)$ one has*

$$\frac{\sinh x}{x} < \left(\frac{1+\cosh x}{2}\right)^{\mu}, \tag{5.13}$$

where $\mu = \frac{\pi^2}{24\ln\left[\frac{1+\cosh(\pi/2)}{2}\right]} \approx 0.731427$ is the best possible constant.

Proof: Using [**3**, Theorem 3], [**10**] we can actually see that

$$e^{-x^2/6} < \frac{x}{\sinh x}; x \in x \in \left(0, \frac{\pi}{2}\right),$$

which implies that

$$\left(\frac{\sinh x}{x}\right)^{6} < \exp(x^2);\ x \in \left(0, \frac{\pi}{2}\right). \tag{5.14}$$

Now, by virtue of (2.3) and (5.14), we obtain

$$\frac{\sinh x}{x} < \left(\frac{1+\cosh x}{2}\right)^{\mu},$$

where $\mu = \frac{\beta}{6} = \frac{\pi^2}{24 \ln\left[\frac{1+\cosh(\pi/2)}{2}\right]} \approx 0.731427$ is the best possible constant. □

Other useful applications of (2.1) and (2.2) include the sharp bounds of Gaussian type integrals, with simple analytical expressions. Both of them are described below.

5.4 Application 2: Simple Bounds for a Truncated Sine Gaussian Integral

In Corollary 2, we determine simple bounds for the truncated Gaussian sine integral defined by $\int_0^y \sin x \exp(-x^2)dx$. This function has some connection with the Dawson type integrals.

Corollary 2. *For $y \in (0, \pi/2)$, it is true that*

$$\frac{3}{c+1}\left[1-\left(\frac{2+\cos y}{3}\right)^{c+1}\right] \leqslant \int_0^y \sin x \exp(-x^2)dx \leqslant \frac{3}{d+1}\left[1-\left(\frac{2+\cos y}{3}\right)^{d+1}\right],$$

with the best possible constants $c \approx 6.08536$ and $d = 6$.

Proof: By utilizing (2.2), we can write

$$\int_0^y \sin x \left(\frac{2+\cos x}{3}\right)^{c} dx \leqslant \int_0^y \sin x\ \exp(-x^2)dx \leqslant \int_0^y \sin x \left(\frac{2+\cos x}{3}\right)^{d} dx.$$

By remarking that $\int_0^y \sin x \left(\frac{2+\cos x}{3}\right)^{c} dx = \frac{3}{c+1}\left[1-\left(\frac{2+\cos y}{3}\right)^{c+1}\right]$, with the same for d in place of c, we end the assertion. □

5.5 Application 3: Simple Bounds for the Error Function erf

For the last application, we consider the well known error function defined by

$$\operatorname{erf}(y) = \frac{2}{\sqrt{\pi}}\int_0^y \exp(-x^2)dx.$$

For this function also we give sharp explicit bounds in Corollary 3.

Corollary 3. *For $y \in (0, \pi/2)$, it holds that*

$$\frac{\left(6\cos^3 y + 32\cos^2 y + 81\cos y + 160\right)\sin y + 105y}{384} \leqslant \frac{\sqrt{\pi}\,\mathrm{erf}(y)}{2} \leqslant$$

(5.15)

$$\frac{(40\cos y + 576)\sin^5 y - (3730\cos y + 14720)\sin^3 y + (37965\cos y + 87360)\sin y + 49635y}{174960}$$

Proof: Using (2.1) and (2.2), we have

$$\left(\frac{1+\cos x}{2}\right)^4 \leqslant \exp(-x^2) \leqslant \left(\frac{2+\cos x}{3}\right)^6.$$

Therefore,

$$\int_0^y \left(\frac{1+\cos x}{2}\right)^4 dx \leqslant \frac{\sqrt{\pi}\,\mathrm{erf}(y)}{2} \leqslant \int_0^y \left(\frac{2+\cos x}{3}\right)^6 dx.$$

Using the expansions : $(1+\cos x)^4 = \frac{1}{8}(56\cos x + 28\cos(2x) + 8\cos(3x) + \cos(4x) + 35)$ and $(2+\cos x)^6 = \frac{1}{32}(10224\cos x + 4815\cos(2x) + 1400\cos(3x) + 246\cos(4x) + 24\cos(5x) + \cos(6x) + 6618)$ and by integration, we obtain required result. $\square$

Some graphical and numerical illustrations: The sharpness of the bounds in (5.15) are illustrated in Figures 5 and 6. Let us now investigate the global L_2 error : $e_o(h) = \int_0^{\pi/2} \left(\frac{\sqrt{\pi}\,\mathrm{erf}(x)}{2} - h(x)\right)^2 dx$, where $h(x)$ denotes any function in the bounds of (5.15). The results are set in Table 4. We see that the error is negligible, attesting the interest of our findings.

Table 4: Global L_2 errors $e_o(h)$ for the functions $h(x)$ in the bounds of (5.15) ; Boundinf is for the lower bound and Boundsup is for the upper bound.

$h(x)$	Boundinf	Boundsup
$e_o(h)$	$\approx 6.930623 \times 10^{-5}$	$\approx 2.314179 \times 10^{-7}$

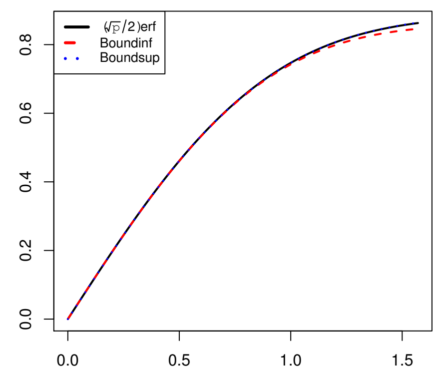

Figure 5: Graphs of the functions of bounds (5.15) for $x \in (0, \pi/2)$.

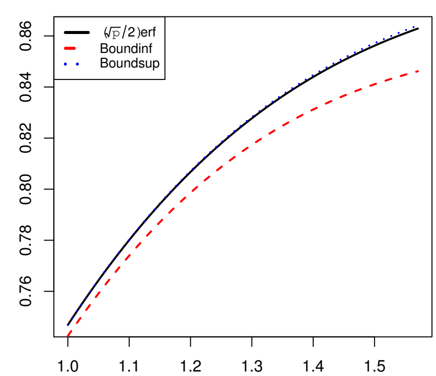

Figure 6: Graphs of the functions of bounds (5.15) for $x \in (1, \pi/2)$.

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