Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **14** (2020), 255–271.
https://doi.org/10.2298/AADM190924020W

ANSWERS TO THREE CONJECTURES ON CONVEXITY OF THREE FUNCTIONS INVOLVING COMPLETE ELLIPTIC INTEGRALS OF THE FIRST KIND

*Miao-Kun Wang, Hong-Hu Chu, Yong-Min Li and Yu-Ming Chu**

In the article, we prove that the function $x \to (1-x)^p \mathcal{K}(\sqrt{x})$ is logarithmically concave on $(0,1)$ if and only if $p \geq 7/32$, the function $x \to \mathcal{K}(\sqrt{x})/\log(1+4/\sqrt{1-x})$ is convex on $(0,1)$ and the function

$$x \to \frac{d^2}{dx^2}\left[\mathcal{K}(\sqrt{x}) - \log\left(1+\frac{4}{\sqrt{1-x}}\right)\right]$$

is absolutely monotonic on $(0,1)$, where $\mathcal{K}(x) = \int_0^{\pi/2}\left(1-x^2\sin^2 t\right)^{-1/2} dt$ $(0<x<1)$ is the complete elliptic integral of the first kind.

1. INTRODUCTION

In the past few centuries, the complete elliptic integrals of the first and second kinds $\mathcal{K}(r)$ and $\mathcal{E}(r)$, defined on $[0,1]$ by

$$\mathcal{K}(r) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-r^2\sin^2\theta}}, \quad \mathcal{K}(0) = \frac{\pi}{2}, \quad \mathcal{K}(1) = \infty,$$

$$\mathcal{E}(r) = \int_0^{\pi/2} (1-r^2\sin^2\theta)^{1/2} d\theta, \quad \mathcal{E}(0) = \frac{\pi}{2}, \quad \mathcal{E}(1) = 1,$$

*Corresponding author. Yu-Ming Chu.
2010 Mathematics Subject Classification. 33E05, 33F05, 26A51.
Keywords and Phrases. Complete elliptic integrals, monotonicity, convexity, logarithmically concave.

have been found that they have many important applications in mathematics as well as in physics and engineering, including the evaluation of the length of curves [**1, 5, 9, 10, 22, 28, 39, 42, 46, 55**], the algorithm of the circumference ration π [**11, 13, 21, 41**], the computations of electromagnetic field and the study of the period of the simple pendulum [**12, 15, 19, 25, 30**].

In 1990s, the complete elliptic integrals $\mathcal{K}(r)$ and $\mathcal{E}(r)$ appeared in geometric function theory frequently, especially in conformal and quasiconformal mappings [**3, 4, 6, 17, 27, 31, 38, 54, 59**]. Many conformal invariants and distortion functions in quasiconformal mappings can be expressed by $\mathcal{K}(r)$ and $\mathcal{E}(r)$. Because of the great importance of these integrals, Vourinen and his collaborators initiated an independent subject to study the complete elliptic integrals and their related special functions, numerous new properties and inequalities involving $\mathcal{K}(r)$ and $\mathcal{E}(r)$ have been obtained in recent years [**16, 18, 24, 26, 29, 33, 40, 43, 44, 50, 51, 52, 53**].

In 1992, in order to refine the following well-known asymptotic formula [**14**]

$$\lim_{r\to 1^-}\left[\mathcal{K}(r)-\log\left(\frac{4}{\sqrt{1-r^2}}\right)\right]=0,$$

Anderson, Vamanamurthy and Vuorinen [**7**] conjectured that the inequality

$$\mathcal{K}(r)<\log\left(1+\frac{4}{\sqrt{1-r^2}}\right)-\left(\log 5-\frac{\pi}{2}\right)(1-r)$$

holds for all $r\in(0,1)$. Later, the conjecture was proved by Qiu et al. in [**37**].

Very recently, Yang and Tian [**56**] studied the convexity of the function $x\mapsto \mathcal{K}(\sqrt{x})-\log\left(1+4/\sqrt{1-x}\right)$ on $(0,1)$ and provided a better upper bound for $\mathcal{K}(r)$. In fact, the authors proved that

Theorem 1.1. (See [**56**, Theorem 3]) The function

$$F(x)=\mathcal{K}(\sqrt{x})-\log\left(1+\frac{4}{\sqrt{1-x}}\right) \tag{1.1}$$

is strictly convex on $(0,1)$.

Corollary 1.2. (See [**56**, Remark 7]) The function $x\mapsto[\mathcal{K}(\sqrt{x})-\log(1+4/\sqrt{1-x})+(\log 5-\pi/2)]/x$ is strictly increasing from $(0,1)$ onto $(\pi/8-2/5,\log 5-\pi/2)$. Consequently, the double inequality

$$\begin{aligned}\log\left(1+\frac{4}{\sqrt{1-r^2}}\right)-\left(\log 5-\frac{\pi}{2}\right)+\alpha r^2<\mathcal{K}(r)\\ <\log\left(1+\frac{4}{\sqrt{1-r^2}}\right)-\left(\log 5-\frac{\pi}{2}\right)+\beta r^2\end{aligned} \tag{1.2}$$

holds for all $r\in(0,1)$ with the best possible constants $\alpha=\pi/8-2/5=-0.0073\cdots$ and $\beta=\log 5-\pi/2=0.0386\cdots$.

Besides, Yang and Tian [**56**] also investigated the monotonicity of the function $x \mapsto \mathcal{K}(\sqrt{x})/\log\left(1+4/\sqrt{1-x}\right)$ on $(0,1)$, established Theorem1.3 and proposed Conjectures 1.4-1.6 as follows.

Theorem 1.3. (See [**56**, Theroem 2]) The function

$$G(x) = \frac{\mathcal{K}(\sqrt{x})}{\log\left(1+4/\sqrt{1-x}\right)}$$

is strictly increasing from $(0,1)$ onto $(\pi/(2\log 5), 1)$. In particular, the double inequality

$$\frac{\pi}{2\log 5}\log\left(1+\frac{4}{\sqrt{1-r^2}}\right) < \mathcal{K}(r) < \log\left(1+\frac{4}{\sqrt{1-r^2}}\right)$$

holds for all $r \in (0,1)$.

Conjecture 1.4. (See [**56**, Conjecture 1]) The function $H(x) = (1-x)^p\mathcal{K}(\sqrt{x})$ is logarithmically concave on $(0,1)$ if and only if $p \geq 7/32$.

Conjecture 1.5. (See [**56**, Conjecture 2]) The function $G(x) = \mathcal{K}(\sqrt{x})/\log(1+4/\sqrt{1-x})$ is convex on $(0,1)$.

Conjecture 1.6. (See [**56**, Conjecture 3]) Let $F(x)$ be defined by (1.1). Then $F''(x)$ is absolutely monotonic on $(0,1)$.

The main purpose of this paper is to give positive answers to Conjectures 1.4-1.6. Our results are the following Theorems 1.7-1.9.

Theorem 1.7. *The function* $H(x) = (1-x)^p\mathcal{K}(\sqrt{x})$ *is logarithmically concave on* $(0,1)$ *if and only if* $p \leq 0$ *and logarithmically concave on* $(0,1)$ *if and only if* $p \geq 7/32$.

Theorem 1.8. *The function* $G(x) = \mathcal{K}(\sqrt{x})/\log(1+4/\sqrt{1-x})$ *is convex on* $(0,1)$. *In particular, the inequality*

$$\mathcal{K}(r) < \log\left(1+\frac{4}{\sqrt{1-r^2}}\right)\left[\frac{\pi}{2\log 5}+\left(1-\frac{\pi}{2\log 5}\right)r^2\right]$$

holds for all $r \in (0,1)$.

Theorem 1.9. *Let* $F(x)$ *be defined by* (1.1). *Then* $F''(x)$ *is absolutely monotonic on* $(0,1)$.

Remark 1.10. According to Theorem 1.9, we can find better bounds for $\mathcal{K}(r)$ than (1.2). For example, Theorem 1.9 implies that the function $x \mapsto [\mathcal{K}(\sqrt{x}) - \log(1+4/\sqrt{1-x}) - (\pi/2 - \log 5) - (\pi/8 - 2/5)x]/x^2$ is strictly increasing from $(0,1)$ onto $(9\pi/128 - 11/50, 2/5 + \log 5 - 5\pi/8)$. Consequently, the inequality

$$\begin{aligned}(1.3)\qquad &\log\left(1+\frac{4}{\sqrt{1-r^2}}\right) - \left(\log 5 - \frac{\pi}{2}\right) + \left(\frac{\pi}{8}-\frac{2}{5}\right)r^2 + \alpha^* r^4 < \mathcal{K}(r) \\ &< \log\left(1+\frac{4}{\sqrt{1-r^2}}\right) - \left(\log 5 - \frac{\pi}{2}\right) + \left(\frac{\pi}{8}-\frac{2}{5}\right)r^2 + \beta^* r^4\end{aligned}$$

holds for all $r \in (0,1)$ with the best possible constants $\alpha^* = 9\pi/128 - 11/50 = 0.000893\cdots$ and $\beta^* = 2/5 + \log 5 - 5\pi/8 = 0.0459\cdots$.

Throughout this paper, for convenience, we denote $r' = \sqrt{1-r^2}$ for $r \in (0,1)$. The following formulas involving $\mathcal{K}(r)$ and $\mathcal{E}(r)$ can be found in [**8**, (3.13), Appendix E]:

$$\frac{d\mathcal{K}(r)}{dr} = \frac{\mathcal{E}(r) - r'^2\mathcal{K}(r)}{rr'^2}, \qquad \frac{d\mathcal{E}(r)}{dr} = \frac{\mathcal{E}(r) - \mathcal{K}(r)}{r},$$

$$\frac{d[\mathcal{E}(r) - r'^2\mathcal{K}(r)]}{dr} = r\mathcal{K}(r), \qquad \frac{d[\mathcal{K}(r) - \mathcal{E}(r)]}{dr} = \frac{r\mathcal{E}(r)}{r'^2},$$

$$\mathcal{K}(r) = \frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2};1;r^2\right), \quad \mathcal{E}(r) = \frac{\pi}{2}F\left(-\frac{1}{2},\frac{1}{2};1;r^2\right),$$

where $F(a,b;c;x)$ is the Gaussian hypergeometric function [**2, 47, 48**] given by

$$F(a,b;c;x) = {}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!} \quad (-1 < x < 1)$$

with the Pochhammer symbol $(a)_0 = 1$, $(a)_n = \prod_{k=0}^{n-1}(a+k)$ for $n = 1, 2, \cdots$.

2. LEMMAS

In order to prove our main results in the next section, we need several lemmas which we present in this section. For the sake of simplification, in what follows we use $\mathcal{K}$ and $\mathcal{E}$ to represent $\mathcal{K}(r)$ and $\mathcal{E}(r)$, respectively.

Lemma 2.11. [**8, 34, 35**] Let $-\infty < a < b < \infty$, $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and be differentiable on (a,b) such that $g'(x) \neq 0$ on (a,b). Then both the functions $[f(x) - f(a)]/[g(x) - g(a)]$ and $[f(x) - f(b)]/[g(x) - g(b)]$ are (strictly) increasing (decreasing) on (a,b) if $f'(x)/g'(x)$ is (strictly) increasing (decreasing) on (a,b).

The following Lemma 2.12 can be found in the literatures [**8**, Thereom 3.21(1), (7), (8), Exercises 3.43(32), (46)] and [**45**, Lemma 2.4].

Lemma 2.12. The following statements are true:

(1) The function $r \mapsto r'^c\mathcal{K}$ is strictly decreasing from $(0,1)$ onto $(0,\pi/2)$ if $c \in [1/2, +\infty)$;

(2) The function $r \mapsto r'^c\mathcal{E}$ is strictly increasing from $(0,1)$ onto $(\pi/2, +\infty)$ if and only if $c \leq -1/2$;

(3) The function $r \mapsto (\mathcal{E} - r'^2\mathcal{K})/r^2$ is strictly increasing from $(0,1)$ onto $(\pi/4, 1)$;

(4) The function $r \mapsto (\mathcal{K} - \mathcal{E})/(r^2\mathcal{K})$ is strictly increasing from $(0,1)$ onto $(1/2, 1)$;

(5) The function $r \mapsto (\mathcal{E} - r'^2\mathcal{K})/(r^2\mathcal{K})$ is strictly decreasing from $(0,1)$ onto $(0,1/2)$;
(6) The function $r \mapsto [\mathcal{E} - r'^2\mathcal{K} - r'^2(\mathcal{K} - \mathcal{E})]/(\mathcal{E} - r'^2\mathcal{K})^2$ is strictly increasing from $(0,1)$ onto $(3/\pi, 1)$.

Lemma 2.13. *The function*

$$f(r) = \frac{4r^2r'^2\mathcal{K} - (r'^2 - r^2)(\mathcal{E} - r'^2\mathcal{K})}{r^2(\mathcal{E} + r'^2\mathcal{K})}$$

is strictly decreasing from $(0,1)$ *onto* $(1, 7/4)$.

Proof. Let $f_1(r) = 4r^2r'^2\mathcal{K} - (r'^2 - r^2)(\mathcal{E} - r'^2\mathcal{K})$ and $f_2(r) = r^2(\mathcal{E} + r'^2\mathcal{K})$. Then elaborated computations lead to

$$\begin{aligned} f_1'(r) =& 8rr'^2\mathcal{K} - 8r^3\mathcal{K} + 4r^2r'^2\left(\frac{\mathcal{E} - r'^2\mathcal{K}}{rr'^2}\right) + 4r(\mathcal{E} - r'^2\mathcal{K}) - (r'^2 - r^2)r\mathcal{K} \\ =& 7r(r'^2 - r^2)\mathcal{K} + 8r(\mathcal{E} - r'^2\mathcal{K}) = r[7(r'^2 - r^2)\mathcal{K} + 8(\mathcal{E} - r'^2\mathcal{K})], \end{aligned}$$

$$\begin{aligned} f_2'(r) =& 2r(\mathcal{E} + r'^2\mathcal{K}) + r^2\left(-\frac{\mathcal{K} - \mathcal{E}}{r} - 2r\mathcal{K} + \frac{\mathcal{E} - r'^2\mathcal{K}}{r}\right) \\ =& 2r(\mathcal{E} + r'^2\mathcal{K}) - r(\mathcal{K} - \mathcal{E}) - 2r^3\mathcal{K} + r(\mathcal{E} - r'^2\mathcal{K}) \\ =& r(4\mathcal{E} - 3r^2\mathcal{K}) \end{aligned}$$

and

$$\begin{aligned} r^3(\mathcal{E} + r'^2\mathcal{K})^2 f'(r) =& [7(r'^2 - r^2)\mathcal{K} + 8(\mathcal{E} - r'^2\mathcal{K})]r^2(\mathcal{E} + r'^2\mathcal{K}) \\ & - [4r^2r'^2\mathcal{K} - (r'^2 - r^2)(\mathcal{E} - r'^2\mathcal{K})](4\mathcal{E} - 3r^2\mathcal{K}) \\ =& 7r^2(1 - 2r^2)\mathcal{K}(\mathcal{E} + r'^2\mathcal{K}) + 8r^2(\mathcal{E}^2 - r'^4\mathcal{K}^2) \\ & - 4r^2(1 - r^2)\mathcal{K}(4\mathcal{E} - 3r^2\mathcal{K}) + (1 - 2r^2)(\mathcal{E} - r'^2\mathcal{K})(4\mathcal{E} - 3r^2\mathcal{K}) \\ =& -2(-r^2r'^2\mathcal{K}^2 - 2\mathcal{E}^2 + 2\mathcal{K}\mathcal{E}) \\ =& -2[2\mathcal{E}(\mathcal{K} - \mathcal{E}) - r^2r'^2\mathcal{K}^2] \\ =& -2r^2r'^2\mathcal{K}^2\left[2\left(\frac{\mathcal{E}}{r'}\right)\left(\frac{1}{r'\mathcal{K}}\right)\left(\frac{\mathcal{K} - \mathcal{E}}{r^2\mathcal{K}}\right) - 1\right]. \end{aligned}$$

Lemma 2.12(1), (2) and (4) imply that the function $r \mapsto 2\mathcal{E}(\mathcal{K} - \mathcal{E})/(r^2r'^2\mathcal{K}^2)$ is strictly increasing from $(0,1)$ onto $(1, +\infty)$. Hence $f'(r) < 0$ for all $r \in (0,1)$, so that $f(r)$ is strictly decreasing on $(0,1)$. Clearly $f(1^-) = 1$, and by Lemma 2.12(3),

$$\lim_{r\to 0^+} f(r) = \lim_{r\to 0^+} \frac{4r'^2\mathcal{K} - (r'^2 - r^2)(\mathcal{E} - r'^2\mathcal{K})/r^2}{\mathcal{E} + r'^2\mathcal{K}} = \frac{4\cdot\pi/2 - \pi/4}{\pi} = \frac{7}{4}.$$

□

Lemma 2.14. *The function*

$$g(r)=\frac{r^2r'^2\mathcal{K}^2-(\mathcal{E}-r'^2\mathcal{K})^2-2(r'^2-r^2)\mathcal{K}(\mathcal{E}-r'^2\mathcal{K})}{r^4\mathcal{K}^2}$$

is strictly decreasing from $(0,1)$ *onto* $(0,7/8)$.

Proof. Let $g_1(r)=r^2r'^2\mathcal{K}^2-(\mathcal{E}-r'^2\mathcal{K})^2-2(r'^2-r^2)\mathcal{K}(\mathcal{E}-r'^2\mathcal{K})$ and $g_2(r)=r^4\mathcal{K}^2$. Then $g(r)=g_1(r)/g_2(r)$ and $g_1(0)=g_2(0)=0$. Differentiations lead to

$$\begin{aligned}g_1'(r)=&2rr'^2\mathcal{K}^2-2r^3\mathcal{K}^2+2r\mathcal{K}(\mathcal{E}-r'^2\mathcal{K})-2(\mathcal{E}-r'^2\mathcal{K})r\mathcal{K}\\&+8r\mathcal{K}(\mathcal{E}-r'^2\mathcal{K})-2(r'^2-r^2)\frac{(\mathcal{E}-r'^2\mathcal{K})^2}{rr'^2}-2(r'^2-r^2)r\mathcal{K}^2\\=&8r\mathcal{K}(\mathcal{E}-r'^2\mathcal{K})-2(r'^2-r^2)\frac{(\mathcal{E}-r'^2\mathcal{K})^2}{rr'^2}\\=&\frac{2(\mathcal{E}-r'^2\mathcal{K})}{rr'^2}\left[4r^2r'^2\mathcal{K}-(r'^2-r^2)(\mathcal{E}-r'^2\mathcal{K})\right]\end{aligned}$$

and

$$g_2'(r)=4r^3\mathcal{K}^2+2r^3\mathcal{K}\left(\frac{\mathcal{E}-r'^2\mathcal{K}}{r'^2}\right)=\frac{2r^3\mathcal{K}}{r'^2}(\mathcal{E}+r'^2\mathcal{K}).$$

Thus we derive that

$$\frac{g_1'(r)}{g_2'(r)}=\left(\frac{\mathcal{E}-r'^2\mathcal{K}}{r^2\mathcal{K}}\right)\left[\frac{4r^2r'^2\mathcal{K}-(r'^2-r^2)(\mathcal{E}-r'^2\mathcal{K})}{r^2(\mathcal{E}+r'^2\mathcal{K})}\right]. \tag{2.4}$$

It follows from (2.4), Lemma 2.12(5) and Lemma 2.13 that $g_1'(r)/g_2'(r)$ is strictly decreasing on $(0,1)$, and so is $g(r)$ by application of Lemma 2.11. The limiting value $g(1^-)$ is clear, and making use of l'Hôpital's rule, Lemma 2.12(5) and Lemma 2.13, one has

$$\begin{aligned}\lim_{r\to0^+}g(r)=\lim_{r\to0^+}\frac{g_1'(r)}{g_2'(r)}=&\lim_{r\to0^+}\left(\frac{\mathcal{E}-r'^2\mathcal{K}}{r^2\mathcal{K}}\right)\\\times\lim_{r\to0^+}&\left[\frac{4r^2r'^2\mathcal{K}-(r'^2-r^2)(\mathcal{E}-r'^2\mathcal{K})}{r^2(\mathcal{E}+r'^2\mathcal{K})}\right]=\frac{7}{8}.\end{aligned}$$

□

Lemma 2.15. *Let* $p\in\mathbb{R}$, $r\in(0,1)$ *and the function* $J_p(r)$ *be defined by*

$$J_p(r)=\frac{\mathcal{E}-r'^2\mathcal{K}}{2r^2r'^2\mathcal{K}(r)}-\frac{p}{r'^2}. \tag{2.5}$$

Then J_p *is strictly increasing on* $(0,1)$ *if and only if* $p\le0$ *and strictly decreasing on* $(0,1)$ *if and only if* $p\ge7/32$. *If* $0<p<7/32$, *then* J_p *is piecewise monotone on* $(0,1)$.

Proof. It follows from (2.5) that

$$(2.6)\quad J_p'(r) = \frac{1}{2}\left(\frac{r^3r'^2\mathcal{K}^2-(\mathcal{E}-r'^2\mathcal{K})[2rr'^2\mathcal{K}-2r^3\mathcal{K}+r(\mathcal{E}-r'^2\mathcal{K})]}{r^4r'^4\mathcal{K}^2}\right)-\frac{2pr}{r'^4}$$
$$=\frac{2r}{r'^4}\left[\frac{r^2r'^2\mathcal{K}^2-(\mathcal{E}-r'^2\mathcal{K})^2-2(r'^2-r^2)\mathcal{K}(\mathcal{E}-r'^2\mathcal{K})}{4r^4\mathcal{K}^2}-p\right].$$

Combining Lemma 2.14 and (2.6), we clearly see that Lemma 2.15 holds true. □

Lemma 2.16. *The function*

$$h(r)=\frac{(4+r')\log(1+4/r')-4r^2\mathcal{K}/(\mathcal{E}-r'^2\mathcal{K})}{r'}$$

is strictly increasing and positive on $(0,1)$.

Proof. Differentiations yield

$$\frac{d\,[(4+r')\log(1+4/r')]}{dr}=\frac{r}{r'^2}\left[4-r'\log(1+4/r')\right],$$

$$\frac{d\left[r^2\mathcal{K}/(\mathcal{E}-r'^2\mathcal{K})\right]}{dr}=\frac{\left(2r\mathcal{K}+r^2\cdot\frac{\mathcal{E}-r'^2\mathcal{K}}{rr'^2}\right)(\mathcal{E}-r'^2\mathcal{K})-r^2\mathcal{K}\cdot r\mathcal{K}}{(\mathcal{E}-r'^2\mathcal{K})^2}$$
$$=\frac{r[(\mathcal{E}+r'^2\mathcal{K})(\mathcal{E}-r'^2\mathcal{K})-r^2r'^2\mathcal{K}^2]}{r'^2(\mathcal{E}-r'^2\mathcal{K})^2}=\frac{r(\mathcal{E}^2-r'^2\mathcal{K}^2)}{r'^2(\mathcal{E}-r'^2\mathcal{K})^2},$$

and thereby

(2.7)

$$r'^2h'(r)=\frac{r}{r'}\left[4-r'\log(1+\frac{4}{r'})\right]-\frac{4r(\mathcal{E}^2-r'^2\mathcal{K}^2)}{r'(\mathcal{E}-r'^2\mathcal{K})^2}+\frac{r}{r'}\left[(4+r')\log\left(1+\frac{4}{r'}\right)-\frac{4r^2\mathcal{K}}{\mathcal{E}-r'^2\mathcal{K}}\right]$$
$$=\frac{4r}{r'}+\frac{4r}{r'}\log\left(1+\frac{4}{r'}\right)-\frac{4r(\mathcal{E}^2-r'^2\mathcal{K}^2)}{r'(\mathcal{E}-r'^2\mathcal{K})^2}-\frac{4r^3\mathcal{K}}{r'(\mathcal{E}-r'^2\mathcal{K})}$$
$$=\frac{4r}{r'(\mathcal{E}-r'^2\mathcal{K})^2}\left[(\mathcal{E}-r'^2\mathcal{K})^2\log\left(1+\frac{4}{r'}\right)+(\mathcal{E}-r'^2\mathcal{K})^2-(\mathcal{E}^2-r'^2\mathcal{K}^2)\right.$$
$$\left.-r^2\mathcal{K}(\mathcal{E}-r'^2\mathcal{K})\right]$$
$$=\frac{4r}{r'(\mathcal{E}-r'^2\mathcal{K})^2}\left[(\mathcal{E}-r'^2\mathcal{K})^2\log\left(1+\frac{4}{r'}\right)+2r'^2\mathcal{K}^2-2r'^2\mathcal{K}\mathcal{E}-r^2\mathcal{K}\mathcal{E}\right]$$
$$=\frac{4r}{r'(\mathcal{E}-r'^2\mathcal{K})^2}\left[(\mathcal{E}-r'^2\mathcal{K})^2\log\left(1+\frac{4}{r'}\right)-\mathcal{K}[\mathcal{E}-r'^2\mathcal{K}-r'^2(\mathcal{K}-\mathcal{E})]\right]$$
$$=\frac{4r\mathcal{K}}{r'}\left[\frac{\log(1+4/r')}{\mathcal{K}}-\frac{\mathcal{E}-r'^2\mathcal{K}-r'^2(\mathcal{K}-\mathcal{E})}{(\mathcal{E}-r'^2\mathcal{K})^2}\right].$$

Theorem 1.3 and Lemma 2.12(6) show that the function $r \mapsto \log(1+4/r')/\mathcal{K} - [\mathcal{E}-r'^2\mathcal{K}-r'^2(\mathcal{K}-\mathcal{E})]/(\mathcal{E}-r'^2\mathcal{K})^2$ is strictly decreasing from $(0,1)$ onto $(0,(2\log 5-3)/\pi)$. This in conjunction with (2.7) leads to the conclusion that $h'(r)>0$ for all $r\in(0,1)$, so that $h(r)$ is strictly increasing on $(0,1)$. Since $h(0^+)=5\log 5-8=0.0471\cdots>0$, we clearly see that $h(r)>0$ for all $r\in(0,1)$, which completes the proof. □

Lemma 2.17. *Let $n\in\mathbb{N}$ and W_n be the Wallis ratio defined by*

$$W_n=\frac{(\frac{1}{2})_n}{n!}=\frac{\Gamma(n+1/2)}{\Gamma(1/2)\Gamma(n+1)}, \tag{2.8}$$

where $\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt$ is the Euler gamma function [**23, 36, 49, 60**]. *Then one has*

$$\frac{\pi}{8}\frac{(2n+1)^2}{n+1}{W_n}^2+\frac{1}{8}\left[1-\frac{1}{15(2n-1)}\right]W_n-\frac{1}{2}-\frac{9}{16\cdot 15^{n+1}}>0 \tag{2.9}$$

for all $n\ge 5$.

Proof. It was proved in [**58**] that inequality

$$W_n>\frac{1}{\sqrt{\pi n\left(1+\frac{2}{8n-1}\right)}}$$

holds for all $n\ge 1$. Hence it is enough to prove that

$$\frac{(2n+1)^2(8n-1)}{8n(n+1)(8n+1)}+\frac{1}{8}\left[1-\frac{1}{15(2n-1)}\right]\sqrt{\frac{(8n-1)}{n\pi(8n+1)}}-\frac{1}{2}-\frac{9}{16\cdot 15^{n+1}}>0$$

for all $n\ge 5$, which is equivalent to

$$\frac{1}{8}\left[1-\frac{1}{15(2n-1)}\right]\sqrt{\frac{(8n-1)}{n\pi(8n+1)}}>\frac{1}{2}+\frac{9}{16\cdot 15^{n+1}}-\frac{(2n+1)^2(8n-1)}{8n(n+1)(8n+1)}. \tag{2.10}$$

Taking the squares of two sides of inequality (2.10) leads to

$$\frac{1}{64}\left[\frac{30n-16}{15(2n-1)}\right]^2\frac{(8n-1)}{n\pi(8n+1)}-\left[\frac{1}{2}+\frac{9}{16\cdot 15^{n+1}}-\frac{(2n+1)^2(8n-1)}{8n(n+1)(8n+1)}\right]^2 \tag{2.11}$$

$$=\frac{(15n-8)^2(8n-1)}{3600\pi n(8n+1)(2n-1)^2}$$

$$-\left[\frac{40\cdot 15^n\cdot n(n+1)(8n+1)+3n(n+1)(8n+1)-10\cdot 15^n(2n+1)^2(8n-1)}{80\cdot 15^n\cdot n(n+1)(8n+1)}\right]^2$$

$$= \frac{D_n}{57600\pi n^2(2n-1)^2(n+1)^2(8n+1)^2 15^{2n}},$$

where

$$\begin{aligned} D_n =& 16n(8n-1)(8n+1)(n+1)^2(15n-8)^2 15^{2n} - 9\pi(2n-1)^2 \\ & \times \left[10(8n^2+1)15^n + 3n(n+1)(8n+1)\right]^2. \end{aligned} \tag{2.12}$$

Note that

$$\begin{aligned} & \left[16n(8n-1)(8n+1)(n+1)^2(15n-8)^2 15^{2n}\right]^{1/2} \\ & - \left\{9\pi(2n-1)^2\left[10(8n^2+1)15^n + 3n(n+1)(8n+1)\right]^2\right\}^{1/2} \\ >& 4\cdot 16n\cdot(n+1)(15n-8)15^n - 3\sqrt{\pi}(2n-1)\left[10(8n^2+1)15^n + 3n(n+1)(8n+1)\right] \\ =& 64n(n+1)(15n-8)15^n - 30\sqrt{\pi}(2n-1)(8n^2+1)15^n - 9\sqrt{\pi}n(n+1)(8n+1)(2n-1) \\ >& 64n(n+1)(15n-8)15^n - 54(2n-1)(8n^2+1)15^n - 18n(n+1)(8n+1)(2n-1) \\ =& (96n^3+880n^2-620n+54)15^n - 18n(n+1)(8n+1)(2n-1) \\ >& (96n^3+880n^2-620n+54)\cdot 2n^2 - 18n(n+1)(8n+1)(2n-1) \\ =& 2n(96n^4+736n^3-710n^2+117n+9) > 0 \end{aligned}$$

for $n \geq 5$. Hence we get $D_n \geq 0$ for $n \geq 5$ by (2.12).

Therefore, inequality (2.9) holds for each $n \geq 5$ follows easily from (2.10) and (2.11). □

3. PROOFS OF THEOREMS 1.7-1.9

Proof of Theorem 1.7. Logarithmical differentiating H gives

$$\frac{H'(x)}{H(x)} = -\frac{p}{1-x} + \frac{\mathcal{E}(\sqrt{x}) - (1-x)\mathcal{K}(\sqrt{x})}{2x(1-x)\mathcal{K}(\sqrt{x})} = J_p(\sqrt{x}),$$

where J_p is defined by Lemma 2.15.

It follows from Lemma 2.15 that $H'(x)/H(x)$ is strictly increasing if and only if $p \leq 0$ and strictly decreasing if and only if $p \geq 7/32$. Consequently, $H(x)$ is logarithmically convex on $(0,1)$ if and only if $p \leq 0$ and logarithmically concave on $(0,1)$ if and only if $p \geq 7/32$. This completes the proof. □

Proof of Theorem 1.8. By differentiation, one has

$$
\begin{aligned}
G'(x) =& \frac{\frac{\mathcal{E}(\sqrt{x})-(1-x)\mathcal{K}(\sqrt{x})}{2x(1-x)}[\log(1+4/\sqrt{1-x})] - \mathcal{K}(\sqrt{x})\frac{2}{(4+\sqrt{1-x})(1-x)}}{[\log(1+4/\sqrt{1-x})]^2} \\
=& \frac{[\mathcal{E}(\sqrt{x})-(1-x)\mathcal{K}(\sqrt{x})]\log(1+4/\sqrt{1-x}) - \frac{4x\mathcal{K}(\sqrt{x})}{4+\sqrt{1-x}}}{2x(1-x)[\log(1+4/\sqrt{1-x})]^2} \\
=& \frac{1}{2}\left[\frac{\mathcal{E}(\sqrt{x})-(1-x)\mathcal{K}(\sqrt{x})}{x}\right]\left[\frac{1}{\sqrt{1-x}(4+\sqrt{1-x})[\log(1+4/\sqrt{1-x})]^2}\right] \\
&\times \left[\frac{(4+\sqrt{1-x})\log(1+4/\sqrt{1-x}) - \frac{4x\mathcal{K}(\sqrt{x})}{\mathcal{E}(\sqrt{x})-(1-x)\mathcal{K}(\sqrt{x})}}{\sqrt{1-x}}\right].
\end{aligned}
$$

It is not difficult to check that the function $x \mapsto x(4+x)[\log(1+4/x)]^2$ is strictly increasing and positive on $(0,1)$. Applying Lemma 2.12 and Lemma 2.16 we conclude that $G'(x)$ is strictly increasing on $(0,1)$. Therefore, $G(x)$ is convex on $(0,1)$ and the desired inequality follows from the convexity of $G(x)$ immediately. □

Proof of Theorem 1.9. Employing Gaussian hypergeometric series, we get

$$F(x) = \frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2};1;x\right) - \log\left(1+\frac{4}{\sqrt{1-x}}\right)$$

and

$$F'(x) = \frac{\pi}{8}F\left(\frac{3}{2},\frac{3}{2};2;x\right) - \frac{2(4-\sqrt{1-x})}{(15+x)(1-x)}.$$

Let

$$
\begin{aligned}
(3.13)\quad F_1(x) =& F'(x) - F'(0) = \frac{\pi}{8}F\left(\frac{3}{2},\frac{3}{2};2;x\right) - \frac{2(4-\sqrt{1-x})}{(15+x)(1-x)} - \left(\frac{\pi}{8}-\frac{2}{5}\right) \\
=& \sum_{n=1}^{\infty} C_n x^n.
\end{aligned}
$$

Then it suffices to prove that $F_1(x)$ has non-negative Maclaurin series, that is, $C_n \geq 0$ for all $n \geq 1$. Note that

$$\frac{\pi}{8}F\left(\frac{3}{2},\frac{3}{2};2;x\right) = \frac{\pi}{8} + \frac{\pi}{8}\sum_{n=1}^{\infty}\frac{\left[\left(\frac{3}{2}\right)_n\right]^2}{(2)_n n!}x^n$$

and

$$\frac{2(4-\sqrt{1-x})}{(15+x)(1-x)} = \frac{1}{8}(4-\sqrt{1-x})\left(\frac{1}{1-x}+\frac{1}{x+15}\right)$$
$$=\frac{1}{8}\left(4-\sum_{n=0}^{\infty}\frac{(-\frac{1}{2})_n}{n!}x^n\right)\left(\sum_{n=0}^{\infty}x^n+\frac{1}{15}\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{15^n}\right)$$
$$=\frac{1}{8}\left(4-\sum_{n=0}^{\infty}\frac{(-\frac{1}{2})_n}{n!}x^n\right)\sum_{n=0}^{\infty}\left[1+(-1)^n\frac{1}{15^{n+1}}\right]x^n$$
$$=\frac{1}{2}\sum_{n=0}^{\infty}\left[1+(-1)^n\frac{1}{15^{n+1}}\right]x^n-\frac{1}{8}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\left[1+(-1)^k\frac{1}{15^{k+1}}\right]\frac{(-\frac{1}{2})_{n-k}}{(n-k)!}\right)x^n$$
$$=\frac{2}{5}+\sum_{n=1}^{\infty}\left(\frac{1}{2}+(-1)^n\frac{1}{2\cdot 15^{n+1}}-\frac{1}{8}\left(\sum_{k=0}^{n}\left[1+(-1)^k\frac{1}{15^{k+1}}\right]\frac{(-\frac{1}{2})_{n-k}}{(n-k)!}\right)\right)x^n.$$

It follows from (3.13) that

(3.14)
$$C_n=\frac{\pi}{8}\frac{\left[(\frac{3}{2})_n\right]^2}{(2)_n n!}-\frac{1}{2}-(-1)^n\frac{1}{2\cdot 15^{n+1}}+\frac{1}{8}\left(\sum_{k=0}^{n}\left[1+(-1)^k\frac{1}{15^{k+1}}\right]\frac{(-\frac{1}{2})_{n-k}}{(n-k)!}\right)$$
$$=\frac{\pi}{8}\frac{\left[(\frac{1}{2})_n\right]^2}{(n!)^2}\frac{(2n+1)^2}{(n+1)}-\frac{1}{2}-(-1)^n\frac{1}{2\cdot 15^{n+1}}+\frac{1}{8}\sum_{k=0}^{n}\frac{(-\frac{1}{2})_{n-k}}{(n-k)!}$$
$$+\frac{1}{8}\left(\sum_{k=0}^{n-2}(-1)^k\frac{1}{15^{k+1}}\frac{(-\frac{1}{2})_{n-k}}{(n-k)!}+(-1)^n\frac{1}{15^{n+1}}+(-1)^n\frac{1}{2\cdot 15^n}\right)$$
$$=\frac{\pi}{8}\frac{(2n+1)^2}{(n+1)}W_n^2-\frac{1}{2}+(-1)^n\frac{9}{16\cdot 15^{n+1}}+\frac{1}{8}W_n-\frac{1}{16}\sum_{k=0}^{n-2}(-1)^k\frac{1}{15^{k+1}}\frac{(\frac{1}{2})_{n-k-1}}{(n-k)!},$$

where $W_n=(1/2)_n/n!$ is the Wallis ratio defined in Lemma 2.17. Note that

$$\sum_{k=0}^{n}\frac{(-\frac{1}{2})_{n-k}}{(n-k)!}=\sum_{k=0}^{n}\frac{(-\frac{1}{2})_k}{(k)!}=W_n,$$

which can be obtained by comparing the coefficients of two sides of the following identity

$$\frac{1}{1-x}\cdot(1-x)^{1/2}=\sum_{n=0}^{\infty}x^n\sum_{n=0}^{\infty}\frac{(-\frac{1}{2})_n}{n!}x^n=(1-x)^{-1/2}=\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n}{n!}x^n.$$

We claim that

$$\sum_{k=0}^{n-2}(-1)^k\frac{1}{15^{k+1}}\frac{(\frac{1}{2})_{n-k-1}}{(n-k)!}<\frac{1}{15}\frac{(\frac{1}{2})_{n-1}}{n!} \tag{3.15}$$

for $n \geq 2$. Indeed, for fixed $n \geq 2$, if we let $I(k) = (1/2)_{n-k-1}/[15^{k+1}(n-k)!]$ for $k = 0, 1, 2, \cdots, n-2$, then

$$\begin{aligned}\frac{I(k+1)}{I(k)} =& \frac{1}{15}\left(\frac{n-k}{n-k-3/2}\right) = \frac{1}{15}\left(1+\frac{3/2}{n-k-3/2}\right)\\ <& \frac{1}{15}\left(1+\frac{3/2}{2-3/2}\right) = \frac{4}{15} < 1.\end{aligned}$$

So that

$$\sum_{k=0}^{n-2}(-1)^k I(k) < I(0) = \frac{1}{15}\frac{(\frac{1}{2})_{n-1}}{n!}.$$

From (3.14), (3.15) and Lemma 2.17 we clearly see that

$$\begin{aligned}C_n >& \frac{\pi}{8}\frac{(2n+1)^2}{(n+1)}W_n^2 - \frac{1}{2} - \frac{9}{16\cdot 15^{n+1}} + \frac{1}{8}W_n - \frac{1}{240}\frac{(\frac{1}{2})_{n-1}}{n!}\\ =& \frac{\pi}{8}\frac{(2n+1)^2}{(n+1)}{W_n}^2 + \frac{1}{8}\left[1-\frac{1}{15(2n-1)}\right]W_n - \frac{1}{2} - \frac{9}{16\cdot 15^{n+1}} > 0\end{aligned}$$

for $n \geq 5$.

Finally, it is not difficult to verify that $C_1 = 9\pi/64 - 11/25 = 0.00178\cdots > 0$, $C_2 = 75\pi/512 - 227/500 = 0.00619\cdots > 0$, $C_3 = 1225\pi/8192 - 2307/5000 = 0.00838\cdots > 0$, $C_4 = 19845\pi/131073 - 93223/200000 = 0.00953\cdots > 0$. Therefore the proof of Theorem 1.9 is complete. □

Remark 3.18. In [**20, 32, 57**], the authors provided the upper bounds for $\mathcal{K}(r)$ as follows:

$$\mathcal{K}(r) \leq \frac{\pi}{2}\frac{\arctan\frac{\sqrt{1-r'}}{\sqrt{r'}}}{\sqrt{r^2+r'-1}}, \tag{3.16}$$

$$\mathcal{K}(r) < \frac{\pi}{4r}\log\left(\frac{1+r}{1-r}\right), \tag{3.17}$$

$$\mathcal{K}(r) \leq \frac{\pi\sqrt{r^4-32r^2+32}}{8\sqrt{2}\sqrt[4]{(1-r^2)^3}}. \tag{3.18}$$

Computer simulation and experiments show that the upper bound for $\mathcal{K}(r)$ given in (1.3) is better than that given in (3.16), (3.17) and (3.18).

Acknowledgements. The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions. The research was supported by the Natural Science Foundation of China (Grant Nos. 11701176, 11971142, 61673169) and the Natural Science Foundation of Zhejiang Province (Grant No. LY19A010012).

REFERENCES

1. I. Abbas Baloch, Y.-M. Chu: *Petrović-type inequalities for harmonic h-convex functions.* J. Funct. Spaces **2020** (2020), Article ID 3075390, 7 pages.
2. M. Abramowitz, I. A. Stegun: *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables.* U. S. Government Printing Office, Washington, 1964.
3. M. Adil Khan, Y. Khurshid, T.-S. Du, Y.-M. Chu: *Generalization of Hermite-Hadamard type inequalities via conformable fractional integrals.* J. Funct. Spaces **2018** (2018), Article ID 5357463, 12 pages.
4. M. Adil Khan, S.-H. Wu, H. Ullah, Y.-M. Chu: *Discrete majorization type inequalities for convex functions on rectangles.* J. Inequal. Appl. **2019** (2019), Article 16, 18 pages.
5. H. Alzer, S.-L. Qiu: *Monotonicity theorems and inequalities for the complete elliptic integrals.* J. Comput. Appl. Math. **172** (2004), no. 2, 289–312.
6. G. D. Anderson, S.-L. Qiu, M. K. Vamanamurthy, M. Vuorinen: *Generalized elliptic integrals and modular equations.* Pacific J. Math. **192** (2000), no. 1, 1–37.
7. G. D. Anderson, M. K. Vamanamurthy, M. Vuorinen: *Functional inequalities for hypergeometric functions and complete elliptic integrals.* SIAM. J. Math. Anal. **23** (1992), no. 2, 512–524.
8. G. D. Anderson, M. K. Vamanamurthy, M. Vuorinen: *Conformal Invariants, Inequalities, and Quasiconformal Maps.* John Wiley & Sons, New York, 1997.
9. R. W. Barnard, K. Pearce, K. C. Richards: *A monotonicity property involving $_3F_2$ and comparisons of the classical approximations of elliptical arc length.* SIAM J. Math. Anal. **32** (2000), no. 2, 403–419.
10. R. W. Barnard, K. Pearce, K. C. Richards: *An inequality involving the generalized hypergeometric function and the arc length of an ellipse.* SIAM J. Math. Anal. **31** (2000), no. 3, 693–699.
11. B. C. Berndt: *Ramanujan's Notebooks, Part II.* Springer-Verlag, New York, 1989.
12. F. Bowman: *Introduction to Elliptic Functions with Applications.* Dover Publications, New York, 1961.
13. J. M. Borwein, P. B. Borwein: *Pi and the AGM.* John Wiley & Sons, New York, 1987.
14. P. F. Byrd, M. D. Friedman: *Handbook of Elliptic Integrals for Engineers and Scientists.* Springer-Verlag, New York, 1971.
15. Z.-W. Cai, J.-H. Huang, L.-H. Huang: *Periodic orbit analysis for the delayed Filippov system.* Proc. Amer. Math. Soc. **146** (2018), no. 11, 4667–4682.
16. Y.-M. Chu, M.-K. Wang, S.-L. Qiu: *Optimal combinations bounds of root-square and arithmetic means for Toader mean.* Proc. Indian Acad. Sci. Math. Sci. **122** (2012), no. 1, 41–51.
17. P. Duren, J. Pfaltzgraff: *Robin capacity and extremal length.* J. Math. Anal. Appl. **179** (1993), no. 1, 110–119.
18. X.-P. Fang, Y.-J. Deng, J. Li: *Plasmon resonance and heat generation in nanostructures.* Math. Methods Appl. Sci. **38** (2015), no. 18, 4663–4672.

19. A. G. Greenhill: *The Applications of Elliptic Functions*. Dover Publications, New York, 1959.

20. B.-N. Guo, F. Qi: *Some bounds for the complete elliptic integrals of the first and second kinds*. Math. Inequal. Appl. **14** (2011), no. 2, 323-334.

21. H.-J. Hu, L.-Z. Liu: *Weighted inequalities for a general commutator associated to a singular integral operator satisfying a variant of Hörmander's condition*. Math. Notes **101** (2017), no. 5-6, 830-840.

22. X.-M. Hu, J.-F. Tian, Y.-M. Chu, Y.-X. Lu: *On Cauchy-Schwarz inequality for N-tuple diamond-alpha integral*. J. Inequal. Appl. **2020** (2020), Article 8, 15 pages.

23. T.-R. Huang, B.-W. Han, X.-Y. Ma, Y.-M. Chu: *Optimal bounds for the generalized Euler-Mascheroni constant*. J. Inequal. Appl. **2018** (2018), Article 118, 9 pages.

24. T.-R. Huang, S.-Y. Tan, X.-Y. Ma, Y.-M. Chu: *Monotonicity properties and bounds for the complete p-elliptic integrals*. J. Inequal. Appl. **2018** (2018), Article 239, 11 pages.

25. C.-X. Huang, Z.-C. Yang, T.-S. Yi, X.-F. Zou: *On the basins of attraction for a class of delay differential equations with non-monotone bistable nonlinearities*. J. Differential Equations **256** (2014), no. 7, 2101–2114.

26. C.-X. Huang, H. Zhang, J.-D. Cao, H.-J. Hu: *Stability and Hopf bifurcation of a delayed prey-predator model with disease in the predator*. Internat. J. Bifur. Chaos Appl. Sci. Engrg. **29** (2019), no. 7, Article ID 1950091, 23 pages.

27. C.-X. Huang, H. Zhang, L.-H. Huang: *Almost periodicity analysis for a delayed Nicholson's blowflies model with nonlinear density-dependent mortality term*. Commun. Pure Appl. Anal. **18** (2019), no. 6, 3337–3349.

28. S. Khan, M. Adil Khan, Y.-M. Chu: *Converses of the Jensen inequality derived from the Green functions with applications in information theory*. Math. Methods Appl. Sci. https://doi.org/10.1002/mma.6066.

29. M. A. Latif, S. Rashid, S. S. Dragomir, Y.-M. Chu: *Hermite-Hadamard type inequalities for co-ordinated convex and qausi-convex functions and their applications*. J. Inequal. Appl. **2019** (2019), Article 317, 33 pages.

30. D. F. Lawden: *Elliptic Functions and Applications*. Springer-Verlag, New York, 1989.

31. J. Li, J.-Y. Ying, D.-X. Xie: *On the analysis and application of an ion size-modified Poisson-Boltzmann equation*. Nonlinear Anal. Real World Appl. **47** (2019), 188–203.

32. F. Qi, Z. Huang: *Inequalities of the complete elliptic integrals*. Tamkang J. Math. **29** (1998), no. 3, 165–169.

33. W.-M. Qian, Z.-Y. He, Y.-M. Chu: *Approximation for the complete elliptic integral of the first kind*. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. **114** (2020), no. 2, Article 57, 12 pages. DOI: 10.1007/s13398-020-00784-9

34. W.-M. Qian, Z.-Y. He, H.-W. Zhang, Y.-M. Chu: *Sharp bounds for Neuman means in terms of two-parameter contraharmonic and arithmetic mean*. J. Inequal. Appl. **2019** (2019), Article 168, 13 pages.

35. W.-M. Qian, Y.-Y. Yang, H.-W. Zhang, Y.-M. Chu: *Optimal two-parameter geometric and arithmetic mean bounds for the Sándor-Yang mean*. J. Inequal. Appl. **2019** (2019), Article 287, 12 pages.

36. W.-M. Qian, W. Zhang, Y.-M. Chu: *Bounding the convex combination of arithmetic and integral means in terms of one-parameter harmonic and geometric means.* Miskolc Math. Notes **20** (2019), no. 2, 1157–1166.

37. S.-L. Qiu, M.-K. Vamanamurthy, M. Vuorinen: *Some inequalities for the growth of elliptic integrals.* SIAM. J. Math. Anal. **29** (1998), no. 5, 1224–1237.

38. S.-L. Qiu, M. K. Vamanamurthy, M. Vuorinen: *Some inequalities for the Hersch-Pfluger distortion function.* J. Inequal. Appl. **4** (1999), no. 2, 115–139.

39. S. Rafeeq, H. Kalsoom, S. Hussain, S. Rashid, Y.-M. Chu: *Delay dynamic double integral inequalities on time scales with applications.* Adv. Difference Equ. **2020** (2020), Aritcle 40, 32 pages.

40. Y.-Q. Song, M. Adil Khan, S. Zaheer Ullah, Y.-M. Chu: *Integral inequalities involving strongly convex functions.* J. Funct. Spaces **2018** (2018), Aritcle ID 6595921, 8 pages.

41. S. Takeuchi: *Complete p-elliptic integrals and a computation formula of π_p for $p = 4$.* Ramanujan J. **46** (2018), no. 2, 309–321.

42. J.-F. Tian, Y.-R. Zhu, W.-S. Cheung: *n-tuple Diamond-Alpha integral and inequalities on time scales.* Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. **113** (2019), no. 3, 2189–2200.

43. J.-F. Wang, X.-Y. Chen, L.-H. Huang: *The number and stability of limit cycles for planar piecewise linear systems of node-saddle type.* J. Math. Anal. Appl. **469** (2019), no. 1, 405–427.

44. M.-K. Wang, H.-H. Chu, Y.-M. Chu: *Precise bounds for the weighted Hölder mean of the complete p-elliptic integrals.* J. Math. Anal. Appl. **480** (2019), no. 2, Article ID 123388, 9 pages. DOI: 10.1016/j.jmaa.2019.123388

45. M.-K. Wang, Y.-M. Chu, S.-L. Qiu, Y.-P. Jiang: *Convexity of the complete elliptic integrals of the first kind with respect to Hölder means.* J. Math. Anal. Appl. **388** (2012), no. 2, 1141–1146.

46. M.-K. Wang, Y.-M. Chu, S.-L. Qiu, Y.-P. Jiang: *Bounds for the perimeter of an ellipse.* J. Approx. Theory **164** (2012), no. 7, 928–937.

47. M.-K. Wang, Y.-M. Chu, W. Zhang: *Monotonicity and inequalities involving zero-balanced hypergeometric function.* Math. Inequal. Appl. **22** (2019), no. 2, 601–607.

48. M.-K. Wang, Z.-Y. He, Y.-M. Chu: *Sharp power mean inequalities for the generalized elliptic integral of the first kind.* Comput. Methods Funct. Theory, https://doi.org/10.1007/s40315-020-00298-w.

49. M.-K. Wang, M.-Y. Hong, Y.-F. Xu, Z.-H. Shen, Y.-M. Chu: *Inequalities for generalized trigonometric and hyperbolic functions with one parameter.* J. Math. Inequal. **14** (2020), no. 1, 1–21.

50. J.-F. Wang, C.-X. Huang, L.-H. Huang: *Discontinuity-induced limit cycles in a general planar piecewise linear system of saddle-focus type.* Nonlinear Anal. Hybrid Syst. **33** (2019), 162–178.

51. B. Wang, C.-L. Luo, S.-H. Li, Y.-M. Chu: *Sharp one-parameter geometric and quadratic means bounds for the Sándor-Yang means.* Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. **114** (2020), no. 1, Article 7, 10 pages. DOI: 10.1007/s13398-019-00734-0

52. H. Wang, W.-M. Qian, Y.-M. Chu: *Optimal bounds for Gaussian arithmetic-geometric mean with applications to complete elliptic integral.* J. Funct. Spaces **2016** (2016), Article ID 3698463, 6 pages.

53. M.-K. Wang, W. Zhang, Y.-M. Chu: *Monotonicity, convexity and inequalities involving the generalized elliptic integrals.* Acta Math. Sci. **39B** (2019), no. 5, 1440–1450.

54. S.-H. Wu, Y.-M. Chu: *Schur m-power convexity of generalized geometric Bonferroni mean involving three parameters.* J. Inequal. Appl. **2019** (2019), Article 57, 11 pages.

55. Z.-H. Yang, W.-M. Qian, W. Zhang, Y.-M. Chu: *Notes on the complete elliptic integral of the first kind.* Math. Inequal. Appl. **23** (2020), no. 1, 77–93.

56. Z.-H. Yang, J.-F. Tian: *Convexity and monotonicity for elliptic integrals of the first kind and applications.* Appl. Anal. Discrete. Math. **13** (2019), no. 1, 240–260.

57. L. Yin, F. Qi: *Some inequalities for complete elliptic integrals.* Appl. Math. E-Notes **14** (2014), 193–199.

58. W.-T. Ying: *A new approach to prove a two-sided inequality involving Wallis's formula.* J. Taizhou Univ. **30** (2008), no. 3, 1-3, 23. (in Chinese)

59. S. Zaheer Ullah, M. Adil Khan, Y.-M. Chu: *A note on generalized convex functions.* J. Inequal. Appl. **2019** (2019), Article 291, 10 pages.

60. T.-H. Zhao, Y.-M. Chu, H. Wang: *Logarithmically complete monotonicity properties relating to the gamma function.* Abstr. Appl. Anal. **2011** (2011), Article ID 896483, 13 pages.

(Received 24.09.2019)
(Revised 03.03.2020)

Miao-Kun Wang
Department of Mathematics
Huzhou University, Huzhou 313000
China
E-mail: *wmk000@126.com,*
wangmiaokun@zjhu.edu.cn

Hong-Hu Chu
College of Civil Engineering
Hunan University, Changsha 410082
China
E-mail: *chuhonghu2005@126.com*

Yong-Min Li
Department of Mathematics
Huzhou University, Huzhou 313000
China
E-mail: *ymlwww@163.com*

Yu-Ming Chu
College of Science
Hunan City University, Yiyang 413000
China
School of Mathematics and Statistics
Changsha University of Science & Technology,
Changsha 410114
China
E-mail: *chuyuming@zjhu.edu.cn*
chuyuming2005@126.com